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Préparé par

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*Thème*

**TWO-COMPONENT BOSE-EINSTEIN CONDENSATES**

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## **Abstract**

Two-component Bose-Einstein condensate show many interesting phenomena due to the interplay between the inter and intra component interactions. We study the dynamical properties of a highly unbalanced mixture in both trapped and homogeneous cases using two couple Gross-Pitaevskii equation. Effects of inter and intra-component interactions on the collective mode frequencies are discussed in three and one dimensions.



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# General Introduction:

## Brief history of Bose-Einstein condensate:

In classical mechanics it is possible to label identical particles uniquely by their position and the momentum coordinates at any given time, while in the quantum mechanics particles no longer have well defined trajectories, and the fact that identical particles are indistinguishable has profound effects if we arbitrarily form a wave for many-body system by assigning mathematical labels to each particle, an exchange of these labels can lead to different physical predications a situation which is obviously unacceptable.

To obtain un a acceptable physic we must impose certain symmetries on the many-body wave function we find that there are two possibilities of the wave it's must be "symmetric" or "antisymmetric", this is expressed as

$$\Psi(\dots r_i \dots, \dots r_j \dots) = \pm \Psi(\dots r_j \dots, \dots r_i \dots),$$

where particles which have wave functions antisymmetric (-) called fermions and obey the Fermi-Dirac statistics and those which are symmetric (+) called boson and obey the Bose-Einstein statics. It is now possible to deduce that particles within half-integer are fermions while those with integer spin are boson. This requirements on the symmetry of the wave function lead us to a new principle called "the Pauli exclusion principle" where it's said that: the fermions can't sharing the same quantum state", in the other hand the bosons have a tendency to cluster together in the same state; this phenomenon know as Bose-Einstein condensate (BEC).

The phenomena of BEC was first described by Albert Einstein in 1925 [1] basing on the work of Satyendra Nath Bose theory of the quantum statistics of photons. Einstein realized that an ideal Bose gas, cooled below a critical temperature  $T_C$ , would have a macroscopic population of atoms in a single quantum state (the ground state). Stated this way, Einstein's prediction is not surprising; it makes sense that a large ground state occupation would occur if an ideal Bose gas were cooled at temperatures lower than a certain critical value. What is surprising, however, is the fact that the temperature at which this occurs can be many times higher than the energy of the first excited state, and the number of atoms in the ground state, as a function of temperature, was predicted to have a discontinuous derivative at the critical temperature  $T_C$ .

Efforts to realize BEC in a weakly interacting gas were initiated by a paper by Stwalley and Nosanow in 1976 [2]. In this paper the authors pointed out that Hydrogen remains in its gaseous state down to temperatures necessary for BEC, while all other substances with the exception of Helium transform into the solid state. The groups of Silvera and Walraven in the Netherlands and Greytak and Kleppner in the USA started experiments to reach BEC in Hydrogen in the early 80s [3,4]. The American group eventually observed BEC in 1998 [5] three years after, the first condensates in Alkali vapors had been realized. It turned out that ultracold alkali atoms can be stored in a metastable state with a sufficient life time to reach BEC.

Soon after the invention of laser cooling, it was realized that this technique offered a second possible route to BEC apart from the Hydrogen approach. The pioneers of laser cooling were awarded the Nobel Prize in 1997 [6,7,8], two years after BEC (in Sodium, Rubidium and Lithium) was achieved nearly simultaneously by three groups in 1995 [9,10,11]. In 2001 Carl E. Wiemann, Eric Cornell and Wolfgang Ketterle received the Nobel Prize for their contributions to studies on BEC [12,13].

### **Why two-component?**

BEC of mixtures has recently the subject of an intensive experimental and theoretical research because its show many interesting phenomena do not existing in one-component due the interplay between intra- and inter-species BECs, such us interferences, soliton molecules....

### **Experimental realization:**

A mixture of BEC can be produced experimentally by simultaneously trapping atoms in different hyperfine states, or different species. The first experimental realization of a system of two interacting BECs has been obtained at JILA with a double condensate of  $^{87}\text{Rb}$  in two different hyperfine states  $|F, M_F\rangle = |1, -1\rangle$  and  $|2, 2\rangle$  [14]. This mixture was characterized by a partial overlap between the two condensates. Since then several other experiments have been performed with double condensates of rubidium [15,16,17] and with spinor condensates of sodium in optical traps [18]. These systems have been extensively studied also from the theoretical point of view.

On the other hand, the two-component BEC has recently attracted a great deal of interest in theoretical side. Ground state density profiles were first investigated by Ho and Shenoy in 1996 [19] in the context of the Thomas–Fermi (TF) approximation. The boundary between inter-penetrating BECs has been studied in great detail [20]. Metastable states, which do not correspond to the lowest energy for the system, have been obtained numerically [21]. Furthermore, the stability of a two–component system has largely explored in both homogeneous and trapped cases. Indeed, the stability has been investigated when the system is disturbed by an external force. It has been found also that, similarly to a one species condensate, when attractive interactions are present, the binary BEC exhibits a collapse [22, 23]. BEC mixtures have been also studied in non–harmonic trapping potentials such as double well magnetic trap [24], optical lattices (where the potential has spatially periodic wells) [25], and recently ring potentials where the fragility of the system was found to depend on the speed of persistent currents [26]. Other varieties of ultracold atomic mixtures have been intensively investigated namely spinor condensates (see [27] for review), Bose–Fermi mixtures [28, 29, 30] and BEC-impurity mixtures [31, 32]

The aim of this dissertation is to investigate the effects of inter and intra-species interaction on the properties of a two-component BEC at zero temperature, by using two coupled time dependent Gross-Pitaevskii equations (GPE).

### **Outline of the dissertation**

In chapter 1, we start from a second-quantized description of a two-component Bose gas system and derive the coupled GPE for each component. Miscible and immiscible mixtures are deeply discussed in the frame of the Thomas-Fermi (TF) approximation. The elementary excitation will be obtained within the so-called Bogoliubov-de-Gennes equations by considering the small oscillations of the order parameter around the equilibrium solution. Furthermore, we will calculate the superfluid fraction and the depletion of the mixture.

Chapter 2 deals with the dynamics of highly unbalance mixture. We will use the coupled GPE within the hydrodynamic approach. A generalized Stringari’s equation is derived. Linearizing this latter around the equilibrium state, we obtain analytical expressions

## General introduction :

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for the collective modes of both majority and minority components of the mixture for isotropic and anisotropic traps.

Chapter 3 is devoted to study the dynamics of a binary Bose gas in one dimensional geometry. The collective modes are obtained following the same process as in chapter 2. Moreover, we investigate the time evolution of the majority component employing a variational method. The profile in real space reveals sinusoidal width oscillations. Our findings are compared with recent experimental data.

# **Chapter 1**

## **Homogeneous BEC mixture**

# Chapter 1

## Homogeneous BEC mixture

The Bose Einstein Condensate (BEC) containing around one million atoms occupying the ground state so it is impossible to describe the motion of every single particle. It is much better to move from microscopic to macroscopic description. This is done by so called non linear Schrödinger equation (NLSE) known also as Gross-Pitaevskii equation (GPE). This equation has been successfully used to describe the statics and the dynamics of BEC at very low temperature such as the excitation energy and collective mode frequencies.

### 1.1. Gross-Pitaevskii equation:

We consider a two component BEC formed by a mixture of two kinds of atoms (or the same atoms in two different internal states). The Hamiltonian of the system reads

$$H = H_1 + H_2 + H_{12}, \quad (1.1)$$

where

$$H_i = \int dr \hat{\Psi}_i^+(r) \left[ \frac{-\hbar^2}{2m_i} \Delta + V_i(r) + \frac{g_i}{2} \hat{\Psi}_i^+(r) \hat{\Psi}_i(r) \right] \hat{\Psi}_i(r), \quad (1.2)$$

$$H_{ij} = g_{ij} \int dr \hat{\Psi}_i^+(r) \hat{\Psi}_j^+(r) \hat{\Psi}_i(r) \hat{\Psi}_j(r), \quad (1.3)$$

Here,  $i, j = 1, 2$ ,  $V_i(r)$  is the external confining field, and  $\hat{\Psi}_i^+$ ,  $\hat{\Psi}_i$  are the creation and annihilation operators satisfying to the usual commutation relations

$$[\hat{\Psi}(r), \hat{\Psi}(r')] = [\hat{\Psi}^+(r), \hat{\Psi}^+(r')] = 0 \text{ and } [\hat{\Psi}(r), \hat{\Psi}^+(r')] = \delta(r - r'). \quad (1.4)$$

The coupling constants  $g_{ij}$  are given in terms of the  $s$ -wave scattering length  $a_{ij}$  by

$$g_i = \frac{4\pi\hbar^2 a_i}{m_i} \quad \text{and} \quad g_{21} = g_{12} = \frac{4\pi\hbar^2 a_{ij}}{m_{ij}} = 4\pi\hbar^2 a_{ij} \left( \frac{1}{m_i} + \frac{1}{m_j} \right). \quad (1.5)$$

The intercomponent coupling constant  $g_{ij}$  yields new structures and dynamics not found in a single component BEC.

The GP equation can be derived using the Heisenberg picture [33]

$$i\hbar \frac{\partial \hat{\Psi}}{\partial t} = [\hat{\Psi}, H].$$

Then splitting the field operator into the condensate contribution, which corresponds to the macroscopic occupation of a single quantum state, from the remaining part of the Bose field operator. Therefore,

$$\hat{\Psi} = \Phi + \bar{\Psi}, \quad (1.6)$$

where  $\Phi$  corresponding to the condensate wave function (order parameter), and  $\bar{\Psi}$  is the non condensate operator.

Keeping in mind that at zero temperature all particles are in the lowest state i.e. we can neglect the non-condense field operator. After a straightforward calculation, we obtain the GPE for the two BECs mixture

$$i\hbar \frac{\partial \Phi_1(r,t)}{\partial t} = \left( \frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 |\Phi_1|^2 + g_{12} |\Phi_2|^2 \right) \Phi_1(r,t), \quad (1.7)$$

$$i\hbar \frac{\partial \Phi_2(r,t)}{\partial t} = \left( \frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 |\Phi_2|^2 + g_{12} |\Phi_1|^2 \right) \Phi_2(r,t), \quad (1.8)$$

with the normalization condition

$$\int d\mathbf{r} |\Phi_i(r)|^2 = N_i. \quad (1.9)$$

The static solutions can be found by eliminating the time-dependence via the transformation:

$$\Phi_i(r,t) = \Phi_i(r) e^{-i\mu_i t/\hbar}. \quad (1.10)$$

where  $\mu_i$  the chemical potential

Introducing (1.10) in (1.7) and (1.8) we obtain

$$\mu_1 \Phi_1(r) = \left( \frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 n_1 + g_{12} n_2 \right) \Phi_1(r), \quad (1.11)$$

$$\mu_2 \Phi_2(r) = \left( \frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 n_2 + g_{12} n_1 \right) \Phi_2(r), \quad (1.12)$$

where the density of particles is given by  $n_i = |\Phi_i|^2$ .

## 1.2. Tomas-Fermi (TF) approximation:

For sufficiently large clouds ( $N \gg 1$ ) and for repulsive interaction ( $g_i > 0$ ), an accurate expression for the ground-state energy may be obtained by neglecting the kinetic energy term in the GPE [34]. The densities of both components 1 and 2 take an inverted parabola as

$$n_1 = \frac{\mu_1 - V_1(r) - g_{12} n_2}{g_1}, \quad (1.13)$$

$$n_2 = \frac{\mu_2 - V_2(r) - g_{12} n_1}{g_2}, \quad (1.14)$$

Combining equations (1.14) in (1.13), one gets



$$n_1 = \frac{g_2(\mu_1 - V_1) - g_{12}(\mu_2 - V_2)}{g_1 g_2 - g_{12}^2}, \quad (1.15)$$

$$n_2 = \frac{g_1(\mu_2 - V_2) - g_{12}(\mu_1 - V_1)}{g_1 g_2 - g_{12}^2}, \quad (1.16)$$

The mixture of two BEC can be miscible or immiscible depending on the parameter  $\Delta = g_1 g_2 / g_{12}^2$ . For  $\Delta > 1$  we have a miscible mixture and while for  $\Delta < 1$ , the mixture becomes immiscible [35]

$$n_1 = \frac{1}{g_1} \left( \frac{\Delta}{\Delta - 1} \right) \left( \mu_1 - \frac{g_{12}}{g_2} \mu_2 - V_1 + \frac{g_{12}}{g_2} V_2 \right), \quad (1.17)$$

$$n_2 = \frac{1}{g_1} \left( \frac{\Delta}{\Delta - 1} \right) \left( \mu_2 - \frac{g_{12}}{g_1} \mu_1 - V_2 + \frac{g_{12}}{g_1} V_1 \right), \quad (1.18)$$

By introducing the normalized chemical potential  $\tilde{\mu}_2 = \mu_2 \left( 1 - \frac{g_{12} \mu_1}{g_1 \mu_2} \right)$  and the normalized external potential  $\tilde{V}_2 = V_2 \left( 1 - \frac{g_{12} V_1}{g_1 V_2} \right)$  we obtain

$$n_2 = \frac{1}{g_2} (\tilde{\mu}_2 - \tilde{V}_2). \quad (1.19)$$

### 1.3. Excitation of inhomogeneous mixture BEC:

The study of the excitation BEC can be done by linearizing GPE using the random phase approximation (RPA)

$$\Phi_i(r, t) = [\Phi_{0i}(r) + \delta\Phi_i(r, t)] e^{-i\mu_i t/\hbar}, \quad (1.20)$$

where  $\delta\Phi_i \ll \Phi_{0i}$ ,  $\Phi_{0i} = \sqrt{n_{0i}}$  the equilibrium density of particles.

Inserting (1.20) in the equations (1.7) and (1.8), we obtain

$$\begin{aligned} & \mu_1 \Phi_{01}(r) + \mu_1 \delta\Phi_1(r, t) + i\hbar \frac{\partial}{\partial t} \delta\Phi_1(r, t) \\ &= \left[ \frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 n_{01} + g_1 \sqrt{n_{01}} (\delta\Phi_1 + \delta\Phi_1^*) + g_{12} n_{02} \right. \\ & \quad \left. + g_{12} \sqrt{n_{02}} (\delta\Phi_2 + \delta\Phi_2^*) \right] (\Phi_{01}(r) + \delta\Phi_1(r, t)), \end{aligned} \quad (1.21)$$

$$\begin{aligned} & \mu_2 \Phi_{02}(r) + \mu_2 \delta\Phi_2(r, t) + i\hbar \frac{\partial}{\partial t} \delta\Phi_2(r, t) \\ &= \left[ \frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 n_{02} + g_2 \sqrt{n_{02}} (\delta\Phi_2 + \delta\Phi_2^*) + g_{12} n_{01} \right. \\ & \quad \left. + g_{12} \sqrt{n_{01}} (\delta\Phi_1 + \delta\Phi_1^*) \right] (\Phi_{02}(r) + \delta\Phi_2(r, t)), \end{aligned} \quad (1.22)$$

To the zeroth order, we obtain the usual GPEs

$$\left(\frac{-\hbar^2}{2m_1}\Delta + V_1(r) - \mu_1 + g_1 n_{01} + g_{12} n_{02}\right)\Phi_1(r) = 0 \quad , \quad (1.23)$$

$$\left(\frac{-\hbar^2}{2m_2}\Delta + V_2(r) - \mu_2 + g_2 n_{02} + g_{12} n_{01}\right)\Phi_2(r) = 0 \quad , \quad (1.24)$$

The first order gives

$$i\hbar \frac{\partial}{\partial t} \delta\Phi_1(r, t) = \frac{-\hbar^2}{2m_1} \Delta \delta\Phi_1 + V_1(r) \delta\Phi_1 + g_1 n_{01} (\delta\Phi_1 + \delta\Phi_1^*) + g_{12} \sqrt{n_{01} n_{02}} (\delta\Phi_2 + \delta\Phi_2^*), \quad (1.25)$$

$$i\hbar \frac{\partial}{\partial t} \delta\Phi_2(r, t) = \frac{-\hbar^2}{2m_2} \Delta \delta\Phi_2 + V_2(r) \delta\Phi_2 + g_2 n_{02} (\delta\Phi_2 + \delta\Phi_2^*) + g_{12} \sqrt{n_{01} n_{02}} (\delta\Phi_1 + \delta\Phi_1^*), \quad (1.26)$$

To calculate the energy of elementary excitation we need to use the Bogoliubov transformation:

$$\delta\Phi_1 = u_1 e^{-i\varepsilon_k t/\hbar} + v_1 e^{i\varepsilon_k t/\hbar} \quad , \quad (1.27)$$

$$\delta\Phi_2 = u_2 e^{-i\varepsilon_k t/\hbar} + v_2 e^{i\varepsilon_k t/\hbar} \quad , \quad (1.28)$$

where  $\varepsilon_k$  the excitation energy of the system.  $u_1, v_1, u_2, v_2$  are Bogoliubov parameters.

Inserting these terms in equation (1.25) (1.26) and separate the terms with  $e^{-i\varepsilon_k t/\hbar}$  and with  $e^{i\varepsilon_k t/\hbar}$  so we obtain the following set of equations

$$\begin{aligned} \varepsilon_k u_1 &= \left(\frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 n_{01}\right) u_1 + g_1 n_{01} v_1 + g_{12} \sqrt{n_{01} n_{02}} (u_2 + v_2) \\ -\varepsilon_k v_1 &= \left(\frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 n_{01}\right) v_1 + g_1 n_{01} u_1 + g_{12} \sqrt{n_{01} n_{02}} (u_2 + v_2) \\ \varepsilon_k u_2 &= \left(\frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 n_{02}\right) u_2 + g_2 n_{02} v_2 + g_{12} \sqrt{n_{01} n_{02}} (u_1 + v_1) \quad , \quad (1.29) \\ -\varepsilon_k v_2 &= \left(\frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 n_{02}\right) v_2 + g_2 n_{02} u_2 + g_{12} \sqrt{n_{01} n_{02}} (u_1 + v_1) \end{aligned}$$

In the uniform case where  $V_1(r) = 0$  and  $V_2(r) = 0$ , we obtain the so-called Bogoliubov-de-Gennes equation

$$\begin{aligned} \varepsilon_k u_1 &= \left(\frac{\hbar^2 k^2}{2m_1} + g_1 n_{01}\right) u_1 + g_1 n_{01} v_1 + g_{12} \sqrt{n_{01} n_{02}} (u_2 + v_2) \\ -\varepsilon_k v_1 &= \left(\frac{\hbar^2 k^2}{2m_1} + g_1 n_{01}\right) v_1 + g_1 n_{01} u_1 + g_{12} \sqrt{n_{01} n_{02}} (u_2 + v_2) \quad , \quad (1.30) \\ \varepsilon_k u_2 &= \left(\frac{\hbar^2 k^2}{2m_2} + g_2 n_{02}\right) u_2 + g_2 n_{02} v_2 + g_{12} \sqrt{n_{01} n_{02}} (u_1 + v_1) \\ -\varepsilon_k v_2 &= \left(\frac{\hbar^2 k^2}{2m_2} + g_2 n_{02}\right) v_2 + g_2 n_{02} u_2 + g_{12} \sqrt{n_{01} n_{02}} (u_1 + v_1) \end{aligned}$$

or equivalently in the matrix from

$$\varepsilon_k \begin{pmatrix} u_1 \\ -v_1 \\ u_2 \\ -v_2 \end{pmatrix} = \begin{pmatrix} L_1 & M_1 & A & A \\ M_1 & L_1 & A & A \\ A & A & L_2 & M_2 \\ A & A & M_2 & L_2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}, \quad (1.31)$$

where  $L_i = \frac{\hbar^2 k^2}{2m_i} + M_i$ ,  $M_i = g_i n_{0i}$  and  $A = g_{12} \sqrt{n_{01} n_{02}}$

To calculate the excitation energy, one should first calculate the determinant of this matrix

$$\det \begin{pmatrix} L_1 - \varepsilon_k & M_1 & A & A \\ M_1 & L_1 + \varepsilon_k & A & A \\ A & A & L_2 - \varepsilon_k & M_2 \\ A & A & M_2 & L_2 + \varepsilon_k \end{pmatrix} = 0. \quad (1.32)$$

The excitation energy reads

$$\varepsilon_{k,\pm} = \pm \sqrt{\frac{\varepsilon_1^2 + \varepsilon_2^2}{2} \pm \sqrt{\frac{(\varepsilon_1^2 - \varepsilon_2^2)^2}{4} + \frac{\hbar^4 k^4}{m_1 m_2} g_{12}^2 n_{01} n_{02}}}, \quad (1.33)$$

where

$$\varepsilon_1^2 = \frac{\hbar^2 k^2}{2m_1} \left( \frac{\hbar^2 k^2}{2m_1} + 2g_1 n_{01} \right) \quad \text{and} \quad \varepsilon_2^2 = \frac{\hbar^2 k^2}{2m_2} \left( \frac{\hbar^2 k^2}{2m_2} + 2g_2 n_{02} \right),$$

are the usual single condensate Bogoliubov dispersion relations.

In the long wavelength limit  $k \rightarrow 0$  we have  $\varepsilon_i \approx \hbar k c_i$  with  $c_i = \sqrt{g_i n_i / m_i}$  being the sound velocities of single BEC. The total dispersion is phonon-like

$$\varepsilon_{\pm} \approx \hbar k c_{\pm},$$

with

$$c_{\pm} = \pm \sqrt{\frac{c_1^2 + c_2^2}{2} \pm \sqrt{\frac{(c_1^2 - c_2^2)^2}{4} + \frac{g_{12}^2}{g_1 g_2} c_1^2 c_2^2}}. \quad (1.34)$$

In the immiscible case ( $g_{12}^2 > g_1 g_2$ ), the spectrum (1.33) becomes unstable.

The above excitation energies can be written in dimensionless form as

$$\frac{\varepsilon_i^2}{(n_i g_i)^2} = k \xi_i \left( \frac{k \xi_i}{4} + 1 \right),$$

And

$$\frac{\varepsilon_k}{\mu_1} = \pm \sqrt{\frac{\tilde{\varepsilon}_1^2 + \tilde{\varepsilon}_2^2 \mu_{12}^2}{2} \pm \sqrt{\frac{(\tilde{\varepsilon}_1^2 - \tilde{\varepsilon}_2^2 \mu_{12}^2)^2}{4} + \frac{k^4 \xi_1^4}{\alpha \Delta} \mu_{12}}}, \quad (1.35)$$

where  $\mu_{12} = \frac{g_2 n_{02}}{g_1 n_{01}}$ ,  $\alpha = \frac{m_2}{m_1}$  and  $\xi_1 = \frac{\hbar}{\sqrt{m_1 n_{01} g_1}}$  is the healing length.

The behavior of  $\varepsilon_{k,\pm}$  is displayed in figure.

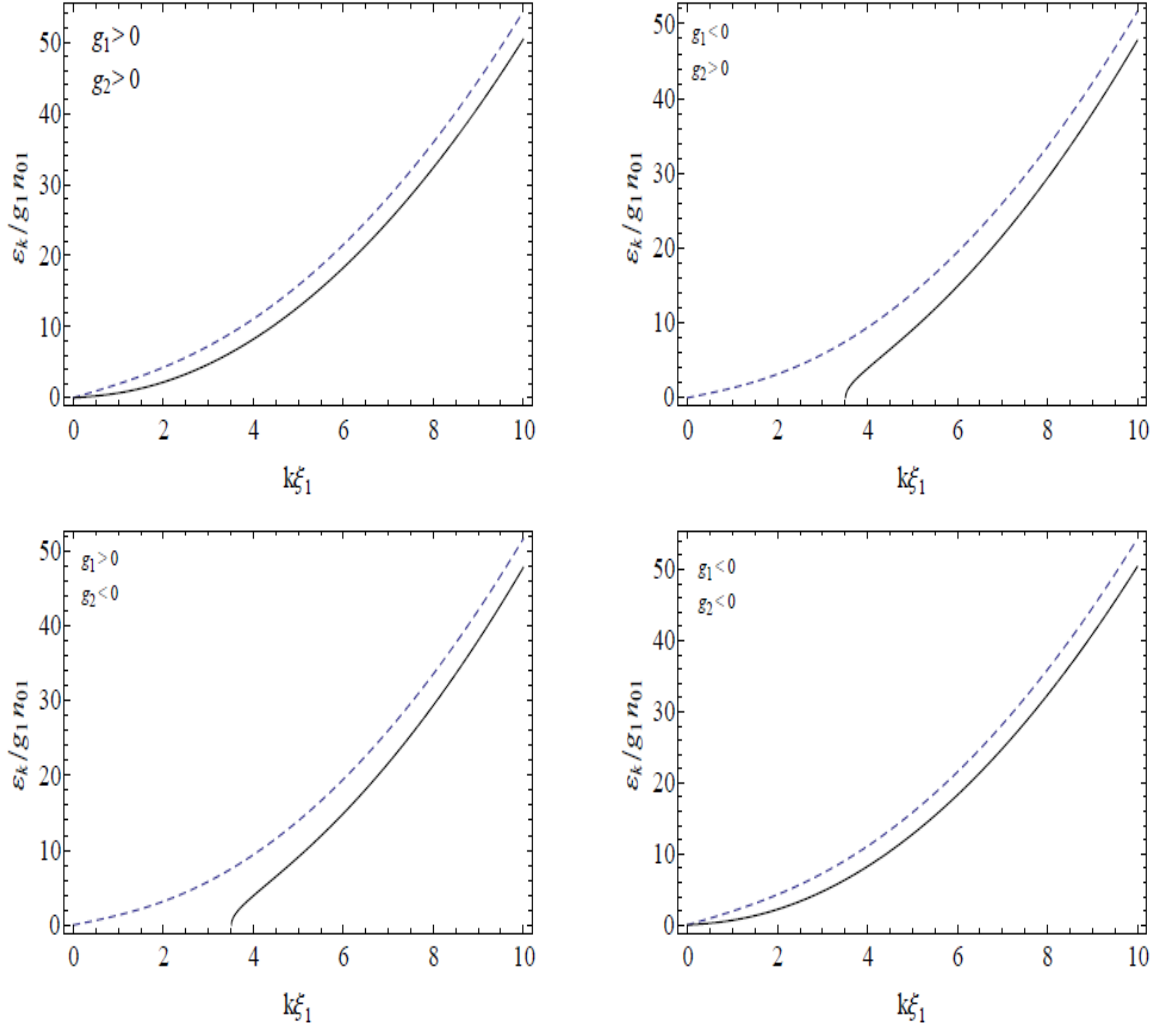


Figure 1.1. The excitation energy for different contact constant in function of  $k\xi_1$ ;  $\varepsilon_{k,-}$  (dashed lines)  $\varepsilon_{k,+}$  (Solid lines)

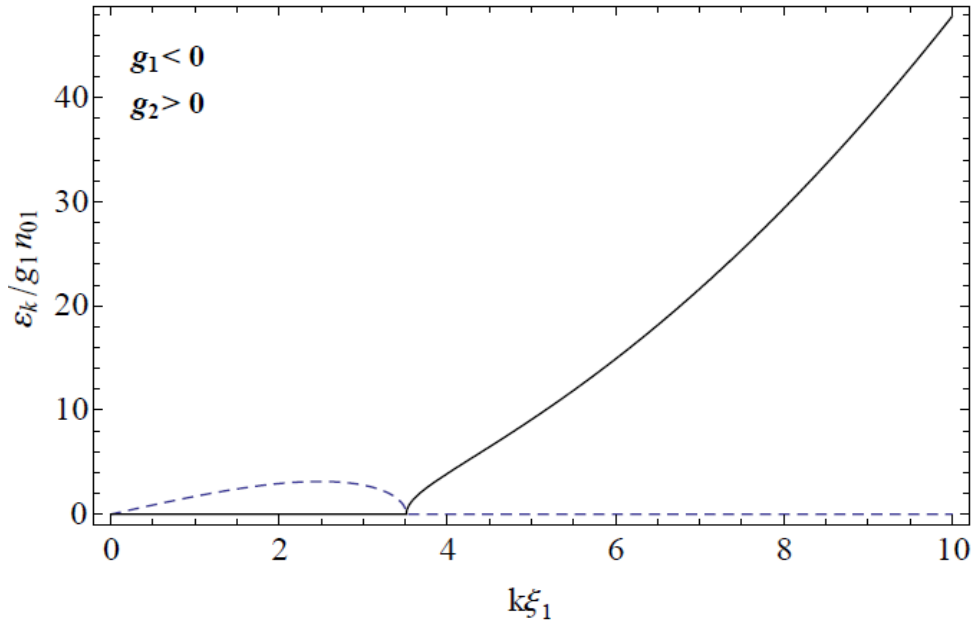


Figure 1.2. Real part (solid lines) and imaginary part (dashed lines) for the excitation energy of two BEC components

The Bogoliubov parameters must satisfy the normalization condition

$$u_i^2 - v_i^2 = 1 \quad (1.36)$$

For sake of simplicity, we proceed by considering the two component system with equal masses.  $m_1 = m_2 = m$ , which is known as the “balanced” system [36]. In such a situation the Bogoliubov amplitudes can be written in a simpler form

$$\begin{cases} u_1 = u_2 = \frac{1}{2} \left( \sqrt{\frac{E_k}{\varepsilon_k}} + \sqrt{\frac{\varepsilon_k}{E_k}} \right) \\ v_1 = v_2 = \frac{1}{2} \left( \sqrt{\frac{E_k}{\varepsilon_k}} - \sqrt{\frac{\varepsilon_k}{E_k}} \right) \end{cases} \quad (1.37)$$

The Bogoliubov parameters are formally identical to that of a single BEC.

#### 1.4. Superfluidity at finite temperature:

At finite temperature the liquid contains excitation which will treat as a gas of quasiparticle. The total momentum per unit volume is thus equal to that carried by the excitations,

$$p_{ex} = \int \frac{d^3p}{(2\pi\hbar)^3} Np, \quad (1.38)$$

where the distribution function equal  $N_p = \frac{1}{e^{(\varepsilon_p - \mu)/T} - 1}$ .

The density of the normal phase is then given by

$$\rho_n = \frac{p_n}{V} = - \int \frac{p^2}{3} \frac{dN_p}{d\varepsilon_p} \frac{d^3p}{(2\pi\hbar)^3}. \quad (1.39)$$

At low temperature, the main contribution to the integral comes from the region of small momentum where  $\varepsilon_p = c_{\pm}p$ . After some algebra, we obtain the following expression for normal component of the superfluid:

$$\rho_n^{\pm} = \frac{2\pi^2 T^4}{45\hbar^3 c_{\pm}^5}, \quad (1.40)$$

where  $c_{\pm}$  is given in equation (1.34).

Equation (1.41) shows that the expression of the normal density is formally identical to that of a single BEC. For  $g_{12}^2 > g_1 g_2$ , the superfluid fraction becomes imaginary yielding the transition to a new quantum phase.

### 1.5.The depletion:

The condensate depletion is defined through

$$n = n_c + \sum_k v_k^2 + \sum_{k=0} (v_k^2 + u_k^2) \langle \hat{b}_k^+ \hat{b}_k \rangle - \sum_{k \neq 0} v_k u_k \langle \hat{b}_k^+ \hat{b}_{-k}^+ + \hat{b}_k \hat{b}_k \rangle, \quad (1.41)$$

The physical interpretation of the expression (1.41) is that the first term is the number of atoms in the condensate. The second term represents the depletion of the condensate by interactions when no real excitations are present.

where  $\hat{b}_k^+$  and  $\hat{b}_k$  are creation and annihilation operator of quasiparticles, often known as ‘‘bogolons’’.

Using the definitions:

$$\begin{aligned} \langle \hat{b}_k^+ \hat{b}_k \rangle &= N_k \\ \langle \hat{b}_k^+ \hat{b}_k^+ \rangle &= \langle \hat{b}_k \hat{b}_k \rangle = 0 \end{aligned} \quad (1.42)$$

we obtain

$$n = n_c + \sum_k v_k^2 + \sum_{k=0} (v_k^2 + u_k^2) N_k. \quad (1.43)$$

At  $T = 0$  almost of atoms are in the ground state i.e.  $N_k = 0$ . Therefore, the condensate depletion turns out to be given as

$$\tilde{n} = n - n_c = \sum_k v_k^2, \quad (1.44)$$

where  $n$  is the total density,  $n_c$  is the condensate density and  $\tilde{n}$  is the non-condensate density.

Replacing the sum over  $k$  by an integral, we get

$$\tilde{n} = \frac{1}{2\pi^2} \int_0^\infty \left( \frac{E_k}{\varepsilon_k} + \frac{\varepsilon_k}{E_k} - 2 \right) k^2 dk , \quad (1.45)$$

here  $\varepsilon_k$  the excitation energy gives in equation (1.33) and  $E_k$  the energy of free particle.

Turning to a new variable  $x = k\xi$ , integral (1.46) gives immediately

$$\tilde{n} = \frac{(mc_\pm)^3}{2\pi^2}, \quad (1.46)$$

Again we see that the depletion of the mixture is formally identical to that of the single BEC and it is imaginary for  $g_{12}^2 > g_1 g_2$ .

## **Chapter 02**

# **Dynamics of highly unbalanced mixture**



## Chapter 02

# Dynamics of highly unbalanced mixture

The time-dependent of BEC clouds, such as collective modes are important sources of information about the physical nature of the condensate [34]. On the experimental side they provide direct in situ information on the system, which are free of the quantitative interpretation of expansion experiments. On the theoretical side the low energy collective modes are the elementary excitations and as such they play an essential role in the physical understanding of these systems [37].

In this chapter we treat the dynamics of two components BEC at zero temperature focusing on the easiest case of trapped highly unbalanced mixture. To this end, we use the time-dependent of the Gross–Pitaevskii equation (1.10) within the hydrodynamic approach to derive an equation allowing us to describe the collective modes of both the majority and the minority components.

### 2.1. Dynamics of highly unbalanced mixture:

#### 2.1.1. Hydrodynamic equation:

We consider a highly unbalance trapped BEC mixture at zero temperature. Assuming that minority component does not affect the majority one ( $g_{12} \approx 0$ ). Thus, the system can be described via the two coupled GP equation

$$i\hbar \frac{\partial \Phi_1(r,t)}{\partial t} = \left( \frac{-\hbar^2}{2m_1} \Delta + V_1(r) + g_1 |\Phi_1|^2 \right) \Phi_1(r, t), \quad (2.1)$$

$$i\hbar \frac{\partial \Phi_2(r,t)}{\partial t} = \left( \frac{-\hbar^2}{2m_2} \Delta + V_2(r) + g_2 |\Phi_2|^2 + g_{21} |\Phi_1|^2 \right) \Phi_2(r, t). \quad (2.2)$$

The wave function of the system  $\Phi_i(r, t)$  is a complex function and can be expanded in terms of amplitude  $\sqrt{n_i(r, t)}$  and phase  $\theta_i(r, t)$  variables through the Madelung transformation [38],

$$\Phi_i(r, t) = \sqrt{n_i(r, t)} e^{i\theta_i(r, t)} . \quad (2.3)$$

where  $n_i(r, t) = |\Phi_i(r, t)|^2$  represents the density of the system.

Plugging (2.3) into the GPEs (2.1) and (2.2), one finds

$$\begin{aligned} i\hbar \frac{\partial \Phi_1(r, t)}{\partial t} &= i\hbar \left( \frac{\partial \sqrt{n_1}}{\partial t} \right) - \hbar \sqrt{n_1} \left( \frac{\partial \theta_1}{\partial t} \right) \\ &= \left( \frac{-\hbar^2}{2m_1} \left( \frac{\partial^2 \sqrt{n_1}}{\partial r^2} + 2i \frac{\partial \sqrt{n_1}}{\partial r} \frac{\partial \theta_1}{\partial r} + i \frac{\partial^2 \theta_1}{\partial r^2} \sqrt{n_1} - \sqrt{n_1} \left( \frac{\partial \theta_1}{\partial r} \right)^2 \right) + V_1(r) + g_1 n_1 \right) \sqrt{n_1}, \end{aligned} \quad (2.4)$$

$$\begin{aligned} i\hbar \frac{\partial \Phi_2(r, t)}{\partial t} &= i\hbar \left( \frac{\partial \sqrt{n_2}}{\partial t} \right) - \hbar \sqrt{n_2} \left( \frac{\partial \theta_2}{\partial t} \right) \\ &= \left( \frac{-\hbar^2}{2m_2} \left( \frac{\partial^2 \sqrt{n_2}}{\partial r^2} + 2i \frac{\partial \sqrt{n_2}}{\partial r} \frac{\partial \theta_2}{\partial r} + i \frac{\partial^2 \theta_2}{\partial r^2} \sqrt{n_2} - \sqrt{n_2} \left( \frac{\partial \theta_2}{\partial r} \right)^2 \right) + V_2(r) + g_2 n_2 + g_{21} n_1 \right) \sqrt{n_2}, \end{aligned} \quad (2.5)$$

Separating the real and the imaginary parts, we get

$$\hbar \frac{\partial \sqrt{n_1}}{\partial t} = -\frac{\hbar^2}{m_1} \frac{\partial \sqrt{n_1}}{\partial r} \frac{\partial \theta_1}{\partial r} - \frac{\hbar^2}{2m_1} \frac{\partial^2 \theta_1}{\partial r^2} \sqrt{n_1}, \quad (2.6)$$

$$\hbar \frac{\partial \sqrt{n_2}}{\partial t} = -\frac{\hbar^2}{m_2} \frac{\partial \sqrt{n_2}}{\partial r} \frac{\partial \theta_2}{\partial r} - \frac{\hbar^2}{2m_2} \frac{\partial^2 \theta_2}{\partial r^2} \sqrt{n_2}, \quad (2.7)$$

and

$$-\hbar \sqrt{n_1} \left( \frac{\partial \theta_1}{\partial t} \right) = \left( \frac{-\hbar^2}{2m_1} \left( \frac{\partial^2 \sqrt{n_1}}{\partial r^2} - \sqrt{n_1} \left( \frac{\partial \theta_1}{\partial r} \right)^2 \right) + V_1(r) + g_1 n_1 \right) \sqrt{n_1}, \quad (2.8)$$

$$-\hbar \sqrt{n_2} \left( \frac{\partial \theta_2}{\partial t} \right) = \left( \frac{-\hbar^2}{2m_2} \left( \frac{\partial^2 \sqrt{n_2}}{\partial r^2} - \sqrt{n_2} \left( \frac{\partial \theta_2}{\partial r} \right)^2 \right) + V_2(r) + g_2 n_2 + g_{21} n_1 \right) \sqrt{n_2}. \quad (2.9)$$

Using the fact that the superfluid velocity is defined as  $\vartheta_i = \frac{\hbar}{m_i} \nabla \theta_i$ , the imaginary terms

lead to the following continuity equations

$$\frac{\partial}{\partial t} n_1 + \nabla(\vartheta_1 \cdot n_1) = 0, \quad (2.10)$$

$$\frac{\partial}{\partial t} n_2 + \nabla(\vartheta_2 \cdot n_2) = 0. \quad (2.11)$$

Similarly, the real parts give us the Euler-like equations

$$m_1 \frac{\partial \vartheta_1}{\partial t} = -\nabla \left( \frac{-\hbar^2}{2m_1} \frac{\Delta \sqrt{n_1}}{\sqrt{n_1}} + \frac{1}{2} m_1 \vartheta_1^2 + V_1(r) + g_1 n_1 \right), \quad (2.12)$$

$$m_2 \frac{\partial \vartheta_2}{\partial t} = -\nabla \left( \frac{-\hbar^2}{2m_2} \frac{\Delta \sqrt{n_2}}{\sqrt{n_2}} + \frac{1}{2} m_2 \vartheta_2^2 + V_2(r) + g_2 n_2 + g_{21} n_1 \right). \quad (2.13)$$

The first one expresses the mass conservation while the second is the Euler like equation, with  $\theta$  playing the role of the velocity potential. The term  $(-\hbar^2/2m_i)(\Delta \sqrt{n_i}/\sqrt{n_i})$  is called

the quantum pressure. In the semi-classical limit  $\hbar \rightarrow 0$ , it can be neglected so Euler's equations become similar to the Newton law [39].

### 2.1.2. Stringari's equation:

The properties of elementary excitations may be investigated by considering small deviations around the equilibrium state. Writing the density as

$$n_i = (n_{0i} + \delta n_i) e^{-i\mu_i/\hbar}, \quad (2.14)$$

where  $n_{0i}$  is the equilibrium density and  $\delta n_i$  is the fluctuation of the density around its equilibrium value. Linearizing equations (2.10) and (2.12) by treating the velocity  $\vartheta$  and  $\delta n$  as small quantities, one finds to the zeroth order

$$\mu_1 = \frac{-\hbar^2 \Delta \sqrt{n_1}}{2m_1 \sqrt{n_1}} + V_1(r) + g_1 n_1. \quad (2.15)$$

In the TF approximation we can neglect the kinetic term so we obtain

$$\mu_1 = V_1(r) + g_1 n_1. \quad (2.16)$$

The first order terms give

$$\frac{\partial \delta n_1}{\partial t} + \nabla(n_{01} \cdot \vartheta_1) = 0, \quad (2.17)$$

$$m_1 \frac{\partial \vartheta_1}{\partial t} = -\nabla \delta \mu_{01}. \quad (2.18)$$

Taking the time derivative of equation (2.17) then inserting (2.18) in the resulting equation, we get

$$\frac{\partial^2 \delta n_1}{\partial t^2} - \nabla \left( \frac{n_{01}}{m_1} \nabla \delta \mu_1 \right) = 0, \quad (2.19)$$

where  $\delta \mu_1 = g_1 \delta n_1$ .

Therefore, equation (2.19) takes the form

$$\frac{\partial^2 \delta n_1}{\partial t^2} - \nabla (c_1^2(r) \nabla \delta n_1) = 0. \quad (2.20)$$

Following the same procedures, we find for the minority component

$$\frac{\partial^2 \delta n_2}{\partial t^2} - \nabla (c_2^2(r) \nabla \delta n_2) = \frac{g_{21}}{g_2} \nabla (c_2^2(r) \nabla \delta n_1), \quad (2.21)$$

with  $c_i^2(r) = \frac{g_i n_{0i}(r)}{m_i}$  being the local sound velocity.

Equations (2.20) and (2.21) describe, respectively the collective modes of the majority and the minority components of the condensate in any arbitrary potential. They were first derived by

Stringari in 1996[40]. Recently, these equations have been extended to deal with finite temperature BEC-impurity mixture [41, 42, 43].

## 2.2. Collective mode:

### 2.2.1. Anisotropic trap:

In this section, we consider an anisotropic trap which is most suitable for experimental setups.

We write the potential in the form:

$$V_i(x, y, z) = \frac{1}{2} m_i \omega_{\perp i}^2 \rho^2 + \frac{1}{2} m_i \omega_{zi}^2 z^2 = \frac{1}{2} m_i \omega_{\perp i}^2 (\rho^2 + \lambda^2 z^2), \quad (2.22)$$

where  $\rho^2 = x^2 + y^2$  and  $\lambda = \omega_z/\omega_{\perp}$  is the anisotropy parameter. For  $\lambda > 1$ , the atomic cloud resembles a pancake trap and for  $\lambda < 1$ , it is cigar shaped.

In the TF approximation the chemical potential can be written in the classical turning point as:

$$\mu_i = \frac{1}{2} m_i \omega_{\perp i}^2 R_{\perp i}^2. \quad (2.23)$$

The densities and their first derivatives read

$$n_{01} = \frac{m_1 \omega_{\perp 1}^2}{2g_1} (R_{\perp 1}^2 - \rho^2 - \lambda_1^2 z^2) \Rightarrow \nabla n_{01} = -\frac{m_1 \omega_{\perp 1}^2}{g_1} (\rho + \lambda_1^2 z), \quad (2.24)$$

$$n_{02} = \frac{m_2 \tilde{\omega}_{\perp 2}^2}{2g_2} (\tilde{R}_{\perp 2}^2 - \rho^2 - \tilde{\lambda}_2^2 z^2) \Rightarrow \nabla n_{02} = -\frac{m_2 \tilde{\omega}_{\perp 2}^2}{g_2} (\rho + \tilde{\lambda}_2^2 z), \quad (2.25)$$

here  $R_{\perp i}$  is the TF radius.

Inserting equations (2.24), (2.25) into (2.20) and (2.21), we obtain

$$\frac{\omega_{m1}^2}{\omega_{\perp 1}^2} \delta n_1 = \left( \rho \frac{\partial}{\partial \rho} + \lambda_1^2 z \frac{\partial}{\partial z} \right) \delta n_1 - \frac{1}{2} (R_{\perp 1}^2 - \rho^2 - \lambda_1^2 z^2) \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right] \delta n_1, \quad (2.26)$$

$$\frac{\omega_{m2}^2}{\tilde{\omega}_{\perp 2}^2} \delta n_2 = \left( \rho \frac{\partial}{\partial \rho} + \tilde{\lambda}_2^2 z \frac{\partial}{\partial z} \right) \delta n_2 - \frac{1}{2} (\tilde{R}_{\perp 2}^2 - \rho^2 - \tilde{\lambda}_2^2 z^2) \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right] \delta n_2. \quad (2.27)$$

The analytic solutions to these equations are obtained by expressing the density fluctuations in the form [34]:

$$\delta n_i \propto \rho^l \exp(im\varphi) = (x \pm iy)^l \propto \rho^l Y_l^m(\theta, \varphi). \quad (2.28)$$

where  $m = \pm l$ .

Introducing (2.28) in equation (2.26) and (2.27),

$$\begin{cases} \rho \frac{\partial}{\partial \rho} \delta n_1 = l \delta n_1 \\ z \frac{\partial}{\partial z} \delta n_1 = 0 \\ \Delta \delta n_1 = 0 \end{cases}, \quad \begin{cases} \rho \frac{\partial}{\partial \rho} \delta n_2 = l \delta n_2 \\ z \frac{\partial}{\partial z} \delta n_2 = 0 \\ \Delta \delta n_2 = 0 \end{cases}$$

Thus, the frequencies for the majority component are given by

$$\omega_{m1} = \sqrt{l}\omega_{\perp 1}. \quad (2.29)$$

Whereas, the frequencies of the minority component read

$$\omega_{m2} = \sqrt{l}\tilde{\omega}_{\perp 2} = \sqrt{l}\sqrt{\left(1 - \frac{g_{21}m_1\omega_{\perp 1}^2}{g_1m_2\omega_{\perp 2}^2}\right)}\omega_{\perp 2}. \quad (2.30)$$

There exist a second class of solution which has the form

$$\delta n_{i,j} \propto z(x \pm iy)^{l-1} \propto r^l Y_l^{\pm(l-1)}(\theta, \varphi). \quad (2.31)$$

Again inserting (2.31) in equations (2.26) and (2.27), we obtain

$$\left\{ \begin{array}{l} \rho \frac{\partial}{\partial \rho} \delta n_1 = (l-1)\delta n_1 \\ z \frac{\partial}{\partial z} \delta n_1 = \delta n_1 \\ \Delta \delta n_1 = 0 \end{array} \right. , \left\{ \begin{array}{l} \rho \frac{\partial}{\partial \rho} \delta n_2 = (l-1)\delta n_2 \\ z \frac{\partial}{\partial z} \delta n_2 = \delta n_2 \\ \Delta \delta n_2 = 0 \end{array} \right.$$

which yields for the frequencies of the majority and minority components, respectively

$$\omega_{m1} = \sqrt{l-1 + \lambda_1^2}\omega_{\perp 1}, \quad (2.32)$$

and

$$\omega_{m2} = \sqrt{l-1 + \lambda_2^2}\tilde{\omega}_{\perp 2} = \sqrt{l-1 + \lambda_2^2}\sqrt{\left(1 - \frac{g_{21}m_1\omega_{\perp 1}^2}{g_1m_2\omega_{\perp 2}^2}\right)}\omega_{\perp 2}. \quad (2.33)$$

### 2.2.2. Isotropic trap:

We consider an isotropic harmonic trap ( $\lambda = 1$ )

$$V_i(r) = \frac{1}{2}m_i\omega_i^2 r^2. \quad (2.34)$$

Because of the spherical symmetry, the general solution for the density deviation is a sum of terms of the form

$$\delta n_i = R_l Y_l^m(\theta, \varphi). \quad (2.35)$$

Here  $Y_l^m$  is a spherical harmonic function and  $R_l$  is a radial function. In a quantum-mechanical description,  $l$  is the quantum number for the magnitude of the total angular momentum and  $m$  that for its projection on the polar axis [34]. To investigate the mode corresponding with isotropic trap, we should take only the radial function in (2.35) and unserting in equation (2.21),(2.20) we obtain:

$$\frac{\omega_{m1}^2}{\omega_{\perp 1}^2} R_l = r \frac{d}{dr} R_l(r) - \frac{1}{2}(R_{TF}^2 - r^2) \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) R_l(r), \quad (2.36)$$

$$\frac{\omega_{m2}^2}{\tilde{\omega}_{\perp 2}^2} R_l = r \frac{d}{dr} R_l(r) - \frac{1}{2}(\tilde{R}_{TF}^2 - r^2) \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) R_l(r), \quad (2.37)$$

Defining new radial function

$$G(r) = \frac{R(r)}{r^l} \rightarrow R_l(r) = G(r)r^l.$$

Thus,

$$R'(r) = lr^{l-1}G(r) + r^lG'(r)$$

$$R''(r) = l(l-1)r^{l-2}G(r) + 2lr^{l-1}G'(r) + r^lG''(r),$$

we obtain

$$\frac{\omega_{m1}^2}{\omega_{l1}^2} G(r) = lG(r) + rG'(r) - \frac{1}{2}(R_{TF}^2 - r^2)(G''(r) + \frac{2(l+1)}{r}G'(r)), \quad (2.38)$$

$$\frac{\omega_{m1}^2}{\tilde{\omega}_{l2}^2} G(r) = lG(r) + rG'(r) - \frac{1}{2}(\tilde{R}_{TF}^2 - r^2)(G''(r) + \frac{2(l+1)}{r}G'(r)), \quad (2.39)$$

To solve these equations we introduce a new variable  $u = r^2/R_{TF}^2$ , where

$$G'(r) = \frac{2r}{R_{TF}^2} G'(u)$$

$$G''(r) = \frac{2}{R_{TF}^2} G'(u) + 4 \frac{r^2}{R_{TF}^4} G''(u).$$

and putting  $\omega_{m1}^2/\omega_{l1}^2 = \Omega_1$  and  $\omega_{m2}^2/\tilde{\omega}_{l2}^2 = \tilde{\Omega}_2$ .

Hence, the differential equations can be written as function of  $G(u)$  as

$$u(1-u)G''(u) + \left(\frac{2l+3}{2} - \frac{2l+5}{2}u\right)G'(u) + \frac{(\Omega_1-l)}{2}G(u) = 0, \quad (2.40)$$

$$u(1-u)G''(u) + \left(\frac{2l+3}{2} - \frac{2l+5}{2}u\right)G'(u) + \frac{(\tilde{\Omega}_2-l)}{2}G(u) = 0. \quad (2.41)$$

The solutions of these equations are the standard hypergeometric function  $F(\alpha, \beta, \gamma, y)$

$$y(1-y)F''(y) + [\gamma - (\alpha + \beta + 1)y]F'(y) - \alpha\beta F(y) = 0. \quad (2.42)$$

Identifying with equation (2.42) and (2.40) (2.41) we obtain:

$$\begin{cases} \alpha + \beta + 1 = \frac{2l+5}{2} \\ \gamma = \frac{2l+3}{2} \\ \alpha\beta = \frac{(\Omega_1-l)}{2} \end{cases},$$

and

$$\begin{cases} \alpha + \beta + 1 = \frac{2l+5}{2} \\ \gamma = \frac{2l+3}{2} \\ \alpha\beta = \frac{(\tilde{\Omega}_2-l)}{2} \end{cases}.$$

Let us put  $\alpha = -j$  or inversely  $\beta = -j$  (where  $j$  is a quantum number)

$$\alpha = -j, \beta = \frac{2l+3+2j}{2}$$

$$\Omega_1 = 2j^2 + 3j + 2lj + l$$

$$\tilde{\Omega}_2 = 2j^2 + 3j + 2lj + l$$

So the frequencies of the majority and the minority components are written respectively as:

$$\omega_{m1} = \omega_{\perp 1} \sqrt{2j^2 + 3j + 2lj + l}. \quad (2.43)$$

and

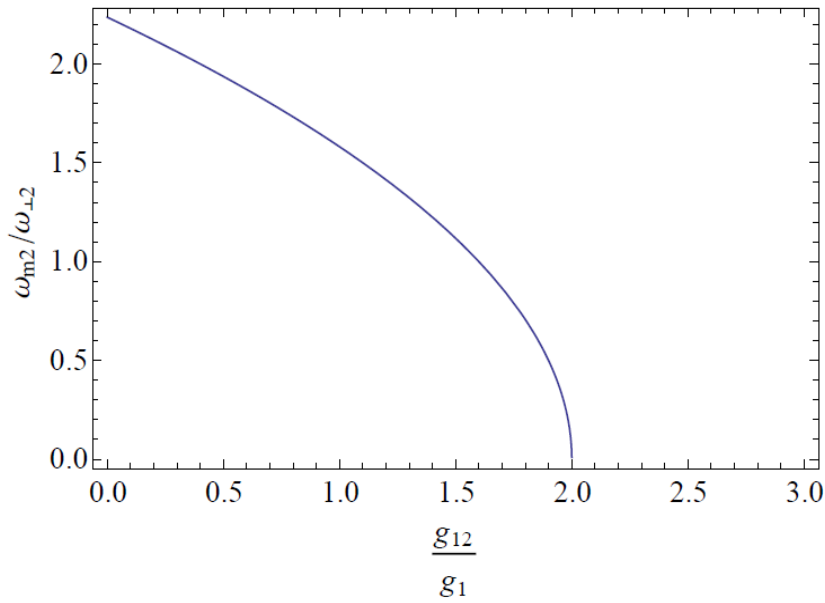
$$\omega_{m2} = \tilde{\omega}_{\perp 2} \sqrt{2j^2 + 3j + 2lj + l} = \sqrt{2j^2 + 3j + 2lj + l} \sqrt{\left(1 - \frac{g_{21} m_1 \omega_{\perp 1}^2}{g_1 m_2 \omega_{\perp 2}^2}\right)} \omega_{\perp 2}. \quad (2.44)$$

For  $j = 1$  and  $l = 0$ , these values correspond the monopole oscillation which is often known as "breathing mode"

$$\omega_{m1} = \omega_{\perp 1} \sqrt{5}. \quad (2.45)$$

$$\omega_{m2} = \tilde{\omega}_{\perp 2} \sqrt{5} = \sqrt{5} \sqrt{\left(1 - \frac{g_{21} m_1 \omega_{\perp 1}^2}{g_1 m_2 \omega_{\perp 2}^2}\right)} \omega_{\perp 2}. \quad (2.46)$$

Equation (2.46) shows the effects of the interspecies interaction strength  $g_{12}$  on the breathing modes of minority component.



**Figure 2.1. Breathing modes of the minority component in 3D. Parameter are  $m1 = m2/2 = 87 \text{ uma}$ ,  $\omega_1 = \omega_2 = 2\pi 200 \text{ Hz}$  and  $g_1 = g_2$ .**

Figure 2.1. shows the results of the breathing modes of minority component by varying  $g_{12}$  and taking the intraspecies interaction strengths  $g_1 = g_2$ . One can clearly see that the frequency of the minority component decreases with increasing  $g_{21}$  and vanishes near the phase separation region where  $g_{21} = 2g_1$ . After that the mixed system becomes unstable.

## **Chapter 03**

# **Dynamics of a trapped highly unbalanced one-dimensional Bose mixture**



## Chapter 03

# Dynamics of a trapped highly unbalanced one-dimensional Bose mixture

One-dimensional (1D) systems constitute an ideal platform for the study of out-of-equilibrium phenomena because of their intrinsic strong correlations and the possibility to realize integrable models. The theoretical investigation of their quantum dynamics is of particular relevance in clarifying questions of collective modes at both zero and finite temperatures. The sum-rule approach [44], the time-dependent modified nonlinear Schrodinger equation (MNLSE) [45] and the numerical Monte carlo simulations [46,47], Hatree-Fock-Bogliubov-Popov [48] and Time dependent Hatree-Fock-Bogliubov [49] are among the theoretical studies which have successfully predicted the collective modes of both weakly and strongly interacting 1D Bose gas. Experimental investigations of the breathing oscillations in 1D ultracold gas have been reported by several groups [50,51].

The purpose of this chapter is to calculate the collective mode frequencies of a harmonically trapped highly unbalanced Bose mixture in 1D geometry in weakly interacting regime. In addition, by means of a variational method, we show also that the time evolution of the majority component reveals sinusoidal variation in agreement with experimental data.

### 3.1. Collective modes

We consider a highly unbalanced mixture in a one dimensional harmonically trapped case ( $\lambda = 1$ ). The trapping potential is

$$V_i(x) = \frac{1}{2} m_i \omega_i^2 x^2. \quad (3.1)$$

In order to calculate the collective mode oscillations, we follow the same scenario as in the previous chapter.

Assuming that the general solution of the GP equation (2.10) has the form

$$\delta n_i(x) = R(x). \quad (3.2)$$

Inserting (3.2) in equations (2.20) (2.21), we get

$$\frac{\omega_{m1}^2}{\omega_{11}^2} R(x) = x \frac{d}{dx} R(x) - \frac{1}{2} (R_{TF}^2 - x^2) \frac{d^2}{dx^2} R(x), \quad (3.3)$$

$$\frac{\omega_{m2}^2}{\tilde{\omega}_2^2} R(x) = x \frac{d}{dx} R(x) - \frac{1}{2} (\tilde{R}_{TF}^2 - x^2) \frac{d^2}{dx^2} R(x), \quad (3.4)$$

To solve these equations, we first introduce a new variable  $u = x^2/R_{TF}^2$  and using the fact that

$$R'(x) = \frac{2x}{R_{TF}^2} R'(u),$$

$$R''(x) = \frac{2}{R_{TF}^2} R'(u) + 4 \frac{x^2}{R_{TF}^4} R''(u).$$

The resulting equations can be reduced to the following hypergeometrical equations

$$u(1-u)R''(u) + \left(\frac{1}{2} - \frac{3}{2}u\right)R'(u) + \frac{\Omega_1}{2}R(u) = 0, \quad (3.5)$$

and

$$u(1-u)R''(u) + \left(\frac{1}{2} - \frac{3}{2}u\right)R'(u) + \frac{\tilde{\Omega}_2}{2}R(u) = 0, \quad (3.6)$$

where  $\omega_{m1}^2/\omega_{11}^2 = \Omega_1$  and  $\omega_{m2}^2/\tilde{\omega}_{12}^2 = \tilde{\Omega}_2$ .

Comparing with the standard hypergeometric function  $F(\alpha, \beta, \gamma, y)$

$$F(\alpha, \beta, \gamma, y) = y(1-y)F''(y) + [\gamma - (\alpha + \beta + 1)y]F'(y) - \alpha\beta F(y) = 0, \quad (3.7)$$

we obtain

$$\begin{cases} \alpha + \beta = \gamma = \frac{1}{2} \\ -\alpha\beta = \frac{\Omega_1}{2} \end{cases}, \quad \begin{cases} \alpha + \beta = \gamma = \frac{1}{2} \\ -\alpha\beta = \frac{\tilde{\Omega}_2}{2} \end{cases}, \quad (3.8)$$

For the function to be well behaved, either  $\alpha$  or  $\beta$  must be a negative integer, so  $\beta = -k$  and  $\alpha = k + \frac{1}{2}$ . This gives for the spectrum

$$\Omega_1 = k(2k + 1) \quad \text{and} \quad \tilde{\Omega}_2 = k(2k + 1). \quad (3.9)$$

Employing the property

$$u_{2k}(y) = F\left(k + \frac{1}{2}, -k, \frac{1}{2}, y\right) \propto P_n(y), \quad (3.10)$$

with  $P_n$  being the Jacobi polynomials.

Setting  $2k = n$ , we obtain [44]

$$\Omega_1 = \frac{n}{2}(n+1),$$

and

$$\tilde{\Omega}_2 = \frac{n}{2}(n+1),$$

Therefore, the frequency of the majority and the minority components can be written in the following form, respectively:

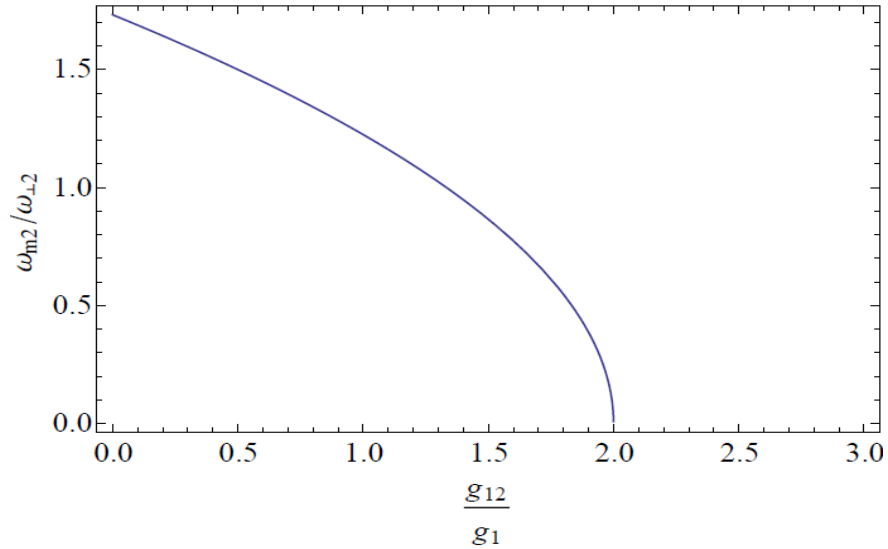
$$\omega_{m1} = \omega_{\perp 1} \sqrt{\frac{n(n+1)}{2}}, \quad (3.11)$$

and

$$\omega_{m2} = \tilde{\omega}_{\perp 2} \sqrt{\frac{n(n+1)}{2}} = \sqrt{\frac{n(n+1)}{2}} \sqrt{\left(1 - \frac{g_{12} m_1 \omega_{\perp 1}^2}{g_1 m_2 \omega_{\perp 2}^2}\right)} \omega_{\perp 2}, \quad (3.12)$$

where  $n$  depends on the mode one considers.

For the dipole (center of mass) mode ( $n = 1$ ), the majority and the minority components have, respectively  $\omega_{m1} = \omega_{\perp 1}$  and  $\omega_{m2} = \tilde{\omega}_{\perp 2}$ . Whereas for the breathing modes ( $n = 2$ ),  $\omega_{m1} = \sqrt{3}\omega_{\perp 1}$  and  $\omega_{m2} = \sqrt{3}\tilde{\omega}_{\perp 2}$ .



**Figure 3.1. Breathing modes of the minority component in 1D. Where The parameter are  $m_1 = m_2/2 = 87 \text{ uma}$ ,  $\omega_1 = \omega_2 = 2\pi 200 \text{ Hz}$  and  $g_1 = g_2$ .**

### 3.2. Time variation of the width for the majority component

The variational approach is proved to be efficient for the analysis of non-integrable systems. It is based on restating the governing equation in terms of variational problem, where the Lagrangian generating the original equation is minimized for a particular trial function. In our case Gaussian function can be employed to describe accurately BEC in harmonic trap see [52].

$$\Phi(x, t) = A(t) \exp \left( -\frac{x^2}{2\sigma^2} - i \gamma(t) x^2 \right), \quad (3.13)$$

where  $A(t)$  is the normalization constant,  $\gamma(t)$  is the phase and  $\sigma(t)$  is the width of the condensate.

Lagrangian density corresponding to the GPE (2.10) of the majority component is defined as:

$$\mathcal{L} = i\hbar \Phi^* \frac{\partial}{\partial t} \Phi + \frac{\hbar^2}{2m} \left( \Phi^* \frac{\partial^2}{\partial x^2} \Phi \right) - \Phi^* V \Phi + \frac{g}{2} |\Phi|^4, \quad (3.14)$$

where the asterisk denotes complex conjugation (here we put for sake of simplicity  $g_1 = g$ ,  $m_1 = m$ ).

Inserting (3.13) into the density Lagrangian, one finds

$$\begin{aligned} \mathcal{L} = & \left[ i\hbar A^* \dot{A} + \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right) \right] \exp \left( \frac{-x^2}{\sigma^2} \right) + \left( i\hbar |A|^2 \left( \frac{\dot{\sigma}(t)}{\sigma(t)^3} - i \dot{\gamma}(t) \right) + \right. \\ & \left. \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right)^2 - \frac{1}{2} m \omega^2 |A|^2 \right) x^2 \exp \left( \frac{-x^2}{\sigma^2} \right) + \frac{g}{2} |A|^4 \exp \left( \frac{-2x^2}{\sigma^2} \right). \end{aligned} \quad (3.15)$$

The dots denote derivatives with respect to  $t$ .

It is possible to calculate the ordinary Lagrangian by integrating over all space

$$L = \int_{-\infty}^{\infty} \mathcal{L} dx.$$

$$\begin{aligned}
L = & \left[ i\hbar A^* \dot{A} + \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right) \right] \int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{\sigma^2}\right) + \left[ i\hbar |A|^2 \left( \frac{\dot{\sigma}(t)}{\sigma(t)^3} - i\dot{\gamma}(t) \right) + \right. \\
& \left. \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right)^2 - \frac{1}{2} m \omega^2 |A|^2 \right] \int_{-\infty}^{+\infty} x^2 \exp\left(\frac{-x^2}{\sigma^2}\right) + \frac{g}{2} |A|^4 \int_{-\infty}^{+\infty} \exp\left(\frac{-2x^2}{\sigma^2}\right) ,
\end{aligned} \tag{3.16}$$

To calculate (3.16) we use

$$\begin{aligned}
\int_{-\infty}^{+\infty} \exp\left(\frac{-x^2}{\sigma^2}\right) &= \sigma(t) \sqrt{\pi} , \\
\int_{-\infty}^{+\infty} x^2 \exp\left(\frac{-x^2}{\sigma^2}\right) &= \frac{1}{2} \sigma(t)^3 \sqrt{\pi} .
\end{aligned}$$

After straightforward calculation, we find

$$\begin{aligned}
L = & \left[ i\hbar A^* \dot{A} + \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right) \right] \sigma(t) \sqrt{\pi} + \left[ i\hbar |A|^2 \left( \frac{\dot{\sigma}(t)}{\sigma(t)^3} - i\dot{\gamma}(t) \right) + \right. \\
& \left. \frac{\hbar^2}{2m} |A|^2 \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right)^2 - \frac{1}{2} m \omega^2 |A|^2 \right] \frac{1}{2} \sigma(t)^3 \sqrt{\pi} + \sqrt{\frac{\pi}{2}} \frac{g}{2} |A|^4 \sigma(t) ,
\end{aligned} \tag{3.17}$$

The normalization condition  $\int_{-\infty}^{+\infty} \Phi^* \Phi dx = 1$  yields  $A = (1/\sigma(t)\sqrt{\pi})^{1/2}$ .

Then, we obtain for Lagrangian

$$\begin{aligned}
L = & -\frac{i\hbar}{2\pi^{1/4}} \frac{\dot{\sigma}(t)}{\sigma(t)} + \frac{\hbar^2}{2m} \left( -\frac{1}{\sigma(t)^2} - 2i \gamma(t) \right) + \frac{i\hbar \sigma(t)^2}{2} \left( \frac{\dot{\sigma}(t)}{\sigma(t)^3} - i\dot{\gamma}(t) \right) + \frac{\hbar^2 \sigma(t)^2}{4m} \left( \frac{1}{\sigma(t)^4} - \right. \\
& \left. 4 \gamma(t)^2 + i \frac{4 \gamma(t)}{\sigma(t)^2} \right) - \frac{1}{4} m \omega^2 \sigma(t)^2 + \frac{g}{4\sqrt{\pi} \sigma(t)} .
\end{aligned} \tag{3.18}$$

### 3.2.1. Equations of motion

Variational equations of motion can be derived from the Euler-Lagrange equations

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\varphi}} \right) - \frac{\partial L}{\partial \varphi} = 0, \tag{3.19}$$

where  $\varphi$  stands for the two variational parameters.

$$\begin{cases} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}} \right) = \hbar \dot{\sigma}(t) \sigma(t) \\ \frac{\partial L}{\partial \gamma} = -\frac{\hbar^2}{m} \sigma(t)^2 (2 \gamma(t)) \end{cases}$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}} \right) - \frac{\partial L}{\partial \gamma} = \hbar \dot{\sigma}(t) \sigma(t) + \frac{\hbar^2}{m} 2 \sigma(t)^2 \gamma(t) = 0. \quad (3.20)$$

This gives a direct relation between the phase and the width

$$\gamma(t) = -\frac{m \dot{\sigma}(t)}{2 \hbar \sigma(t)}. \quad (3.21)$$

After having eliminating  $\gamma(t)$  via (3.21), equation (3.18) takes the form

$$L = i\hbar \left( -\frac{1}{2 \pi^{1/4}} + 1 \right) \frac{\dot{\sigma}(t)}{\sigma(t)} - \frac{\hbar^2}{4 m \sigma(t)^2} - \frac{m_2}{4} \ddot{\sigma}(t) \sigma(t) - \frac{1}{4} m \omega^2 \sigma(t)^2 + \frac{g}{4 \sqrt{\pi} \sigma(t)}. \quad (3.22)$$

Equation (3.22) provides a useful expression for the width

$$\begin{cases} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\sigma}} \right) = -i\hbar \left( -\frac{1}{2 \pi^{1/4}} + 1 \right) \frac{\dot{\sigma}(t)}{\sigma(t)^2} \\ \frac{\partial L}{\partial \sigma} = -i\hbar \left( -\frac{1}{2 \pi^{1/4}} + 1 \right) \frac{\dot{\sigma}(t)}{\sigma(t)^2} + \frac{\hbar^2}{2 m \sigma(t)^3} - \frac{m}{4} \ddot{\sigma}(t) - \frac{1}{2} m \omega^2 \sigma(t) - \frac{g}{4 \sqrt{\pi} \sigma(t)^2} \end{cases},$$

$$m \ddot{\sigma}(t) = F(\sigma), \quad (3.23)$$

where

$$F(\sigma) = \frac{2 \hbar}{m \sigma(t)^3} - 2 m \omega^2 \sigma(t) - \frac{g}{4 \sqrt{\pi} \sigma(t)^2}, \quad (3.24)$$

is the force interaction among atoms in the condensate.

Equation (3.24) is a non trivial differential equation describing the time evolution of the condensate width. Its numerical solution is shown in figure. 3.2.

Figure 3.2. Depicts that the condensate width in 1D follows a sinusoidal variation over time due to the harmonic confining potential. We clearly see that for  $t < 150$ , our variational treatment well agrees with the experimental data of Bouchoule et al [51]. Whereas for larger periods, the variational results differ from the experimental data. This discrepancy is, in fact, due to the finite temperature effects which cannot properly be described by the usual GP equation.

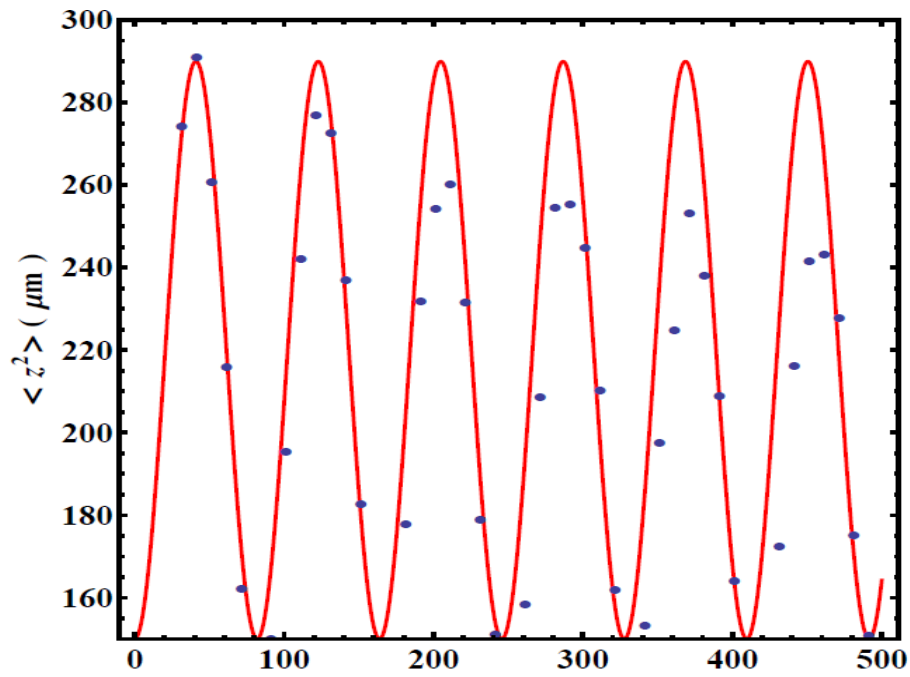


Figure 3.2. Evolution of the condensate width in 1 D harmonic trap. Solid line: variational calculation. Circles: experimental data of Bouchoule *et al* [51].

# General Conclusion

In this dissertation, we have studied the dynamical properties of two components Bose-Einstein condensate.

In the homogeneous case, by using the Bogoliubov theory, we have derived a useful expression for the excitation energy. The stability of such a spectrum has been discussed in terms of  $g_{12}^2 > g_1 g_2$ , in the miscible and immiscible mixtures. The behavior of the depletion and the superfluid fraction of the mixture has been also discussed in terms of the inter and intra-component interaction.

In a trapped mixture, we have calculated the collective modes of a highly unbalanced mixture in anisotropic and isotropic traps utilizing the hydrodynamic approach. We have shown in particular that the inter-species interaction plays a crucial role on the breathing mode frequencies of the minority component.

On the other hand, we have generalized our results to the one-dimensional case. Useful expressions for the breathing modes have been obtained. Moreover, time variation of the width of the majority component has been analyzed using a variational method. We found that the condensate width exhibits a sinusoidal variation over time. Our predictions are found to be in good agreement with recent experiments.

An important extension of this work concerns Bose mixtures at finite temperature. Another interesting future work is to investigate the binary BECs in the presence of dipole-dipole interaction.



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