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# Dipolar Bose-Einstein condensates in random potentials

A THESIS SUBMITTED FOR THE DEGREE OF DOCTOR IN PHYSICAL SCIENCES

**Présented by : KELTOUM Redaouia**

**Thesis committee :**

| | | |
|---|---|---|
| Pr Abdelkader ALI BENAMARA | UHB Chlef | CHAIRMAN |
| Dr Abdelali BOUDJEMAA | UHB Chlef | SUPERVISOR |
| Dr Mohamed BELABBAS | UHB Chlef | EXAMINER |
| Dr Afifa YEDJOUR | UST Oran | EXAMINER |
| Pr Sid Ahmed DIAF | UDBKM | EXAMINER |

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# TABLE DES MATIÈRES

**LISTE DES TABLEAUX**

## TABLE DES FIGURES

# ملخص

ان التفاعلات الضعيفة لغازات بوز في محيط مضطرب موضوعاً صعباً للبحث في مجال فيزياء المواد المكثفة بسبب التفاعل المثير للاهتمام بين الميوعة الفائقة والتوطين. في هذه الرسالة نقوم بتطوير توسيع نطاق دراسات نظرية ورقمية لغازات بوز ثنائية القطب في الكمونات العشوائية. تحقيقاً لهذه الغاية ، نقوم باستعمال نظرية لُبِ-سنُ-شذ لمكثفات س-زِنستِن في حالة التفاعلات الضعيفةً في مجال عشوائي ، ثم تخصصها للتفاعلات الثنائية القطبية والاضطرابات الغوسية المترابطة. يوفر هذا النموذج وصفاً معقولاً للتفاعلات الضعيفة لغازات بوز س المتفاعلة في الكمونات العشوائية. كخطوة أولى ، نتحقق من غاز بوز مكثف ثنائي القطب س ثلاثي الأبعاد في وجود تفاعلات ثلاث جسيمات مع كمون إضافي غوسي ُسسِن في كل من درجات الحرارة الصفرية والمحدودة. الأهم من ذلك ، أننا وجدنا أنه في درجة الحرارة المحدودة ، تتواجد المكثفات مع كل من مكونات س-لَسس والمكونات الحرارية. يتم حساب صحيح التصحيحات الناجمة عن التكميم ،الاضطرابات الحرارية و اضطرابات فوضى في نضوب المكثفات ، دالة ارتباط الكثافة لجسيم واحد ، معادلة الحالة وطاقة الحالة الاساسية حسبت بعناية. نظهر أن تفاعلات الفوضى ، تفاعل ثنائي القطب- ثنائي القطب وتفاعل ثلاث جسيمات يلعبان دوراً أساسياً في فيزياء النظام. ومن المثير للاهتمام ، اننجد أن التفاعلات ثلاثية الجسيمات تنتج الذرات المحصورة في المستويات الدنيا للكمون العشوائي. تؤدي زيادة قوة التفاعلات الثلاثية الجسم إلى تقليل عناصر المصفوفة لكثافة الجسم الواحد. علاوة على ذلك ، نقوم بحساب الكمون الكيميائي وإزالة التباعد في الأشعة فوق البنفسجية باستخدام طريقة مناسبة لإعادة التوحيد. ان الآثار مجتمعة للتفاعلات ثنائي القطب ثنائي القطب ، والتفاعلات ثلاثة الجسم ودرجة الحرارة وجدت تؤثر تأثيرا حاسما على الكمون الكيميائي والحالة الاساسية للطاقة . في غياب ي وتفاعلات ثلاث جسيمات ، نعيد ايجاد النتائج الأساسية لهوانغ ومنغ. أخيراً ، نعتبر غازً بوز مكثفاً من بوزونات ثنائية القطب خاضعاً لارتباط غوسي عند درجة حرارة صفر تحمل في شبه ثنائي الطبقات شبه ثنائي الأبعاد اين ثنائيات الاقطاب مصطرة عمودياً على الطبقات وفي نفس \عكس الاتجاه في طبقات مختلفة. نحسب تحليليا ورقميا نضوب المكثفات ، مصفوفة كثافة الجسيم الواحد ، وفائق الميوعة

في إطار نظرية بقليغوف- هوانغ منغ. لا يوفر تحليلنا نتائج جديدة رائعة فقط غير موجودة في الأدبيات ، بل يُظهر أيضاً أن التنافس بين الاضطراب والاقتران بين الطبقات واتجاه الاستقطاب قد يؤدي إلى توطين \ إزالة تركيز الجسيمات المختصرة في الانتقال من زجاج بوز إلى المرحلة الفائقة الميوعة والعكس صحيح. للحصول على تفاعل قصير المدى خالص ومسافة تلاشي الطبقة البينية ، نستعيد النتائج الموجودة لنظام طبقة واحدة. تمهد نتائجنا الطريق لتحقيق تجريبي لغازات بوز ثنائية القطب المضطربة ثلاثية الأبعاد بتفاعلات ثلاثية الجسيم الخالصة ، والبوزونات المشوهة في شبه ثنائي الطبقات ثنائي الأبعاد.

# Abstract

Weakly interacting Bose gases in a disorder environment have long been a challenging topic in the field of condensed matter physics due to the intriguing interplay between superfluidity and localization. In this thesis we perform extensive theoretical and numerical studies of dipolar Bose gases in random potentials. To this end we develop a Bogoliubov-Huang-Meng theory for a weakly interacting Bose-Einstein condensate in a random environment, then specialize it to dipolar interactions and Gaussian correlated disorder. This model provides a reasonable description of weakly interacting Bose gases in random potentials. As a first step, we investigate a three-dimensional dipolar Bose condensed gas in the presence of the three-body interactions with an additional Guassian-correlated disorder potential at both zero and finite temperatures. Importantly, we find that at finite temperature the condensate co-exists with both the Bose-glass and thermal components. Corrections due to quantum, thermal and disorder fluctuations to the condensate depletion, the one-body density correlation function, the equation of state and the ground state energy are properly calculated. We show that the interplay of the disorder, dipole-dipole interaction and three-body interaction play a fundamental role in the physics of the system. Interestingly, we find that the three-body interactions release atoms localized in the respective minima of the random potential. Increasing the strength of the three-body interactions leads to decrease the one-body density matrix. Furthermore, we calculated the chemical potential and ultraviolet divergences are removed using an appropriate renormalization method. The combined effects of the dipole-dipole interactions, three-body interactions, and temperature found to crucially affect the chemical potential and the ground state energy. In the absence of the DDI and the three-body interactions, we reproduce the seminal results of Huang and Meng

Finally, we consider a dilute Bose-condensed gas of dipolar bosons subjected to Gaussian correlation at zero temperature loaded in a quasi-two-dimensional bilayer setup where dipoles are aligned perpendicularly to the layers and in same /opposite directions in different layers. We calculate analytically and numerically the condensate depletion, the one-body density-matrix, and the superfluid fraction in the framework of the Bogoliubov-Huang-Meng theory. Our analysis not only provides fascinating new results do not exist in the literature but also shows that the competition between the disorder, the interlayer coupling and the polarization orientation may lead to localize/delocalize the condensed particles results in the transition from the Bose-glass to the superfluid phase and vice versa. For a pure short-range interaction and vanishing interlayer distance, we recover the results found for a single layer system. Our results pave the way for the experimental realization of three-dimensional disordered dipolar Bose gases with pure three-body interactions, and of dirty bosons in a quasi-two-dimensional bilayer configuration.

# Résumé

Les interactions faibles de gaz de Bose dans un environnement désordonné sont depuis longtemps un sujet difficile dans le domaine de la physique de la matière condensée en raison de l'interaction intrigante entre la superfluidité et la localisation. Dans cette thèse, nous effectuons des études théoriques et numériques approfondies des gaz dipolaires de Bose dans des potentiels aléatoires. À cette fin, nous étudions la théorie de Bogoliubov-Huang-Meng pour un condensat de Bose-Einstein avec des interactions faibles dans un

environnement aléatoire, puis nous la cernons dans les interactions dipolaires et le disordre corrélé gaussien. Ce modèle fournit une description raisonnable des interactions faibles de gaz de Bose dans des potentiels aléatoires. Dans un premier temps, nous étudions un gaz de Bose condensé dipolaire à trois dimensions en présence des interactions à trois corps dans un potentiel désordonné avec corrélé-guassien à températures nulles et finies. Plus important, nous constatons qu'à température finie, le condensat coexiste avec le verre de Bose et les composants thermiques. Les corrections dues aux quantifications, les fluctuations thermiques et désordonnées de la déplétion des condensats, de la fonction de corrélation de densité uni-corps, de l'équation d'état et de l'énergie de l'état fondamental sont correctement calculées. Nous montrons que l'effet du désordre, de l'interaction dipôle-dipôle et de l'interaction à trois corps joue un rôle fondamental dans la physique du système. Le plus intéressant, nous constatons que les interactions à trois corps libèrent des atomes localisés dans les minima respectifs du potentiel aléatoire. L'augmentation de la force des interactions à trois corps conduit a diminution de la matrice de densité à un seul corps. De plus, nous calculons le potentiel chimique en négligeant les divergences ultraviolettes par l'utilisation de la méthode de renormalisation appropriée. Les effets combinés des interactions dipôle-dipôle, des interactions à trois corps et de la température affectent de manière cruciale le potentiel chimique et l'énergie de l'état fondamental. En l'absence de DDI et des interactions à trois corps, nous reproduisons les résultats séminaux de Huang et Meng. Enfin, nous considérons un gaz de Bose-condensé dilué de bosons dipolaires soumis à une corrélation gaussienne à température nulle maintenue dans une configuration bicouche quasi-bidimensionnelle où les dipôles sont alignés perpendiculairement aux couches et dans des directions identiques / opposées dans les autres couches. Nous calculons analytiquement et numériquement l'épuisement des condensats, la matrice de densité à un corps et la fraction superfluide dans le cadre de la théorie de Bogoliubov-Huang-Meng. Notre analyse fournit non seulement de nouveaux résultats fascinants qui n'existent pas dans la littérature, mais il montre également que la compétition entre le désordre, le couplage inter-couche et l'orientation de polarisation peut conduire à localiser / délocaliser les particules condensées, ce qui entraîne la transition du verre de Bose à la phase superfluide et vice versa. Pour une interaction à courte portée pure et une distance inter-couche négligeable, nous récupérons les résultats trouvés pour un système monocouche. Nos résultats ouvrent la voie à la production expérimentale du gaz de Bose dipolaires désordonnés tridimensionnels avec des interactions pures à trois corps, et de bosons sales dans une configuration bicouche quasi bidimensionnelle.

# INTRODUCTION

The experimental realization of a Bose-Einstein condensate (BEC) of alkali atoms at JILA, MIT and Rice University in 1995 [**1**, **2**] initiated new areas of atomic, molecular and optical physics. The BEC, first theorized by A. Einstein in 1925 [**3**], was the coldest matter known in the universe, and was the fruitful result of years of experimental and theoretical progress in the field of optical and magnetic cooling and trapping [**4**]. The BEC did, and still does, offer a tool with which to study a plethora of ultracold phenomena, including superfluidity and superconductivity and their manifestations in quantum matter. Additionally, the realization of ultracold temperatures allows for atoms and molecules to be trapped by purely optical means, which facilitates experimental control over the magnetic substates of these systems and allows for trapping in optical lattice potentials. Atoms and molecules in optical lattices can be used, for example, for quantum computing [**5**, **6**]. Possible applications include metrology and ultra-precise clocks.

The experimental achievement of BEC of $^{52}$Cr [**7**], $^{164}$Dy [**8**], $^{168}$Er [**9**] and recently with a degenerate Fermi gas of $^{161}$Dy [**10**] atoms with large magnetic dipolar interaction (6 $\mu_B$, $10\mu_B$ and $7\mu_B$, respectively) has opened fascinating prospects for the observation of novel quantum phases and many-body phenomena [**6**, **7**, **11**, **12**]. Polar molecules which have much larger electric dipole moments than those of the atomic gases have been also produced in their ground rovibrational state [**13**, **14**]. What render such dipolar systems particularly intriguing is that the atoms interact via a dipole-dipole interactions (DDI) that is both long ranged and anisotropic and it can be also attractive and repulsive. By virtue of this interaction, dipolar gases are expected to open fascinating prospects for the observation of interesting quantum phases in ultracold atomic gases such as Wigner crystal [**15**], unconventional superfluids of fermionic polar molecules [**16-18**], droplet state [**19-32**], supersolidity [**33-35**] and rotonic stripe phase [**36**]. On the other hand, the long range character of the dipolar interaction leads to scattering properties that are radically different from those found on the usual short-ranged potentials of quantum gases and therefore, all of the higher-order partial waves contribute equally to the scattering at low energy [**11**]. At zero temperature, there have been a number of theoretical studies on dipolar BEC in particular on the expansion dynamics [**7**], the ground state [**37-40**], elementary excitations [**41**, **42**], superfluid properties [**43**, **44**], solitons and soliton-molecule [**45-47**] and optical lattices [**48**, **49**]. Additionally, the DDI is partially attractive and exhibits a roton-maxon structure in the spectrum and may enhance the quantum and thermal fluctuations [**50**, **51**]. At finite temperature, the behavior of dipolar BECs have been investigated using the mean field Hartree-Fock-Bogoliubov (HFB) theory [**52-55**], perturbation approach [**56**, **57**], and path-integral Monte Carlo simulations [**58**]. Effects of the three-Body interactions (TBI) on weakly interacting dipolar Bose gases in a pencake trap at finite temperature have been analyzed for the first time by Boudjemaa [**59**].

Disorder is ubiquitous in nature and plays an important role in condensed-matter systems. Usually, our understanding of solid-state physics starts from the concepts of crystalline order, Bloch bands, and metallic conduction. However, Anderson, Mott and others have shown that disorder might be a fundamental aspect to consider, as it may cause a phase transition from metal to insulator in solids because of a localization of the electronic wave functions in space. This effect, known as Anderson localization, was originally put forward for non-interacting electrons. Indeed, the interplay of disorder and interactions is a long-standing question of condensed-matter physics, which has become acute with the discovery that disorder may destroy superconductivity or superfluidity due to localization effects. Ultracold atomic gases offer appealing possibilities to study the physics of disordered quantum systems, as many parameters, including disorder and interactions, may be controlled and well characterized. In addition, the interacting Bose gas in a weak random external potential represents an interesting model for studying the relation between BEC and superfluidity and it has been the subject of many theoretical investigations in the past two decades.

Experimentally, the dirty boson problem was first studied with superfluid helium in aerosol glasses (Vycor) [**60-62**]. Recently, several groups have loaded ultracold atoms into optical potentials and studied BECs in the presence of disorder [**63-64**]. The random potential can be created using different techniques, one of which is the static laser speckle, where the potential felt by atoms is proportional to the speckle intensity with the sign being determined by the detuning from the atomic transition [**65**, **66**]. Laser speckles, produced by passing an expanded laser beam through diffusive plates, are special in that they have (i) exponential, i.e. strongly non-Gaussian, intensity distribution and (ii) finite support of their power spectrum. Recent progress in different experimental realizations of laser speckle disorder is reported in Refs. [**67**, **68**]. Wire traps represent magnetic traps on atomic chips where the roughness and the imperfection of the wire surface generate a disorder potential [**69**, **70**]. Another possibility to create a random potential is to trap one species of atoms randomly in a deep optical lattice, which serves as frozen scatterers for a second atomic species [**71**, **72**]. In addition, incommensurable lattices provide also a useful random environment [**73**, **74**].

From the theoretical side, addressing the problem of disorder in interacting systems represents a considerable challenge. One of the first quantitative studies of the so-called dirty boson problem was introduced by Huang and Meng in 1992 within a Bogoliubov theory [**75**]. Later on this approach was extended by others within either the original framework of second quantization [**76**, **77**] or the replica method [**78**, **79**] and applied to the case of superfluid helium in porous media. For a delta-correlated disorder, it turned out that condensate depletion occurs due to the localization of bosons in the respective minima of the external random potential which is present even at zero temperature. Furthermore, it was found that superfluidity persists despite quenched randomness and that its depletion is generically larger than the condensate depletion as the fragmented BECs in the disorder landscape represent randomly distributed obstacles for the motion of the superfluid. A generalization to the corresponding situation, where the disorder correlation function falls off with a characteristic correlation length as, for instance, a Gauss function [**80**, **77**, **81**] , laser speckles[**82**, **83**], or a Lorentzian [**84**] is straightforward and yields decreasing condensate and superfluid depletions with increasing correlation lengths.

A more complicated situation arises, for BEC with DDI are studied in a random environment. Dipolar bosons moving in a disordered environment may open fascinating prospects for the observation of non-trivial quantum phases, because it connects two central ideas of the condensed matter theory : disorder and dipolar interaction. The interplay of these two ingredients is important for accurately exploring the anisotropy of superfluidity. The anisotropic disorder landscape makes the superfluid density an anisotropic quantity which means that it acquires a characteristic direction dependence, i.e. the number of particles per volume participating in a superfluid motion varies with the chosen direction [**77**, **81**, **82**, **84**]. This peculiar phenomenon, which is not present at zero temperature in the absence of disorder, has the experimentally detectable consequence that also the sound velocity possesses a direction dependence due to disorder. Note that the anisotropy of the DDI has been recently measured experimentally for a pure dipolar BEC by Bismut et al. [**41**]. This shows not only that tuneable disorder allows to control and shape superfluid properties, but also contributes to a deeper understanding of the localization phenomenon as such. Namely for a sufficiently strong disorder anisotropy it turns out that in some particular direction the depletion of the superfluid density becomes even smaller than the condensate depletion. This represents a counterintuitive result, as particles of the fragmented BECs, which are supposed to be localized in the respective minima of the random potential, seem then to contribute to the superfluid motion. Thus, due to the strong disorder anisotropy, the locally condensed particles can only be localized for a certain time scale. For longer time periods, this suggests that an exchange of the localized particles occurs with the nonlocalized particles, thus allowing for a superfluid density in a particular direction that is larger than the condensate density. This supports the finding of Ref. [**85**], in which such a finite localization time for fragmented BECs was calculated within a Hartree-Fock theory of dirty bosons.

For increasing disorder strength the macroscopic occupation of the ground state decreases more and more, so that the fragmented BECs in the minima of the frozen random potential increase. Whereas the coherence of the superfluid is described by a globally fixed phase of the condensate wave function, the respective phases of the tiny BECs fluctuate. For a sufficiently strong disorder the condensate is completely depleted and a phase transition occurs from a superfluid to a Bose-glass, which only consists of fragmented BECs. In the literature this Bose-glass phase is usually characterized by its properties as, for instance, an exponential decay of spatial correlations or a non-vanishing density of states at vanishing energy [**60**]. In order to allow for a quantitative characterization of the emergence of the Bose-glass phase within the general theory of critical phenomena [**86**, **87**], a separate Bose-glass order parameter was introduced in Ref.[**85**] in addition to the superfluid order parameter. To this end, the well-established Edwards-Anderson order parameter for spin glasses [**88**, **89**] was transferred to the Bose glass. The applicability of this new Bose-glass order parameter concept was demonstrated by working out a Hartree-Fock theory for disordered bosonsRef. [**87**]. Within this treatment, and in accordance with other non-perturbative approaches [**90**, **91**], this allows to determine the location of the Bose-glass phase relative to the superfluid and the normal phase, and to investigate in detail its respective thermodynamic properties.

In quasi-2D geometry, it has been found that both BEC and superfluidity are depressed due to the competition between disorder and the rotonization induced by the DDI yielding, the transition to an unusual quantum phase [**92**, **93**]. On the other hand, the impacts of the Lee-Huang-Yang (LHY) quantum corrections on a dirty dipolar Bose gas have been analyzed very recently by Dr. Boudjemaa [**94**] using a perturbative theory. These LHY quantum fluctuations lead to reduce the disorder effects inside the condensate preventing the formation of the Bose glass state.

# This thesis

In this thesis, we examine the properties of weakly interacting dipolar Bose gases with a Gaussian disorder correlation function at both zero and finite temperatures in 3D and 2D geometries, a subject which is currently attracting a great deal of interest. Our study is based on the Bogoliubov-Huang-Meng theory Our study is based on the Bogoliubov-Huang-Meng theory which was first applied to the case of superfluid helium in porous media [**73**], and extended later by others [**66**, **95**]. This theory allows us to go beyond the zero-temperature Gross-Piteavskii (GP) equation solved with perturbative approach [**68**, **70**]. It gives detailed insights into the interplay of thermal fluctuations and disorder effects in the anisotropy of superfluidity which is not the case for GP equation with perturbative treatment. In the Bogoliubov-Huang-Meng approach, the disorder, the quantum and the thermal fluctuations are assumed to be so small.

The first emphasis is set the role of the three-body interactions (TBI) in disordered dipolar Bose gases. The TBI play a key role in a wide variety of interesting physical phenomena, and provide a new physics not existed in systems with two-body interactions. Inelastic three-body processes, including observations of Efimov quantum states and atom loss from recombination have been reported in Refs [**96-100**]. It has been shown that weakly interacting Bose and Fermi gases with competing attractive two-body and large repulsive TBI form a droplet phase [**101**]. Effects of TBI in ultracold bosonic atoms loaded in an optical lattice or a superlattice were also studied in [**102-105**]. The presence of the TBI in Bose condensate may singnificantly modify the collective excitations [**106-108**], the transition temperature, the condensate depletion and the stability of a BEC [**109**, **110**]. In the context of ultracold atoms with DDI, it has been revealed that the combined effects of TBI and DDI lead to the formation of a stable supersolid state [**111**] and a quantum droplet state [**16**, **29**], [**19-21**]. Very recently, we have shown that the TBI may shift the density profiles, the condensed fraction and the collective modes of a dipolar condensate at finite temperature [**59**]. The core questions are therefore : How do the TBI affect disordered dipolar Bose gases ? How does the interplay of DDI, TBI and the disorder potential alter the localization of bosons ?

An interesting 2D arrangement of dipoles is the bilayer dipolar system where the particles are confined in two paralel planes separated by a fixed distance. In this thesis, we investigate the properties of 2D bilayered dipolar Bose condensed gases in a weak random at zero temperature, where the dipoles are oriented perpendicularly to the layers and in parallel/antiparallel configurations. Over the past decade, ultracold dipolar gases in layered structures have attracted considerable attention [**16-18**], [**112-124**]. Unlike single layers, these bilayered configurations in quasi-2D geometry exhibit many interesting phenomena namely : the formation of conventional and unconventional superfluids of polar molecules [**16**, **17**, **94**, **112**], [**114-119**], soliton molecules [**120**] and the enhacement of the roton instability [**113**, **122**] due to the interlayer effects. Here we unveil the intriguing role of the disorder, the interlayer coupling and the polarization orientation in the localization scenario of particles and the superfluidity. Within the Bogoliubov-Huang-Meng theory, we calculate analytically and numerically the condensate depletion, the one-body density-matrix, and the superfluid fraction. The validity criterion of such a theory is also discussed.

# Outline

This thesis is organized as follows.

Chapter 1 shows a review of basic elements of the theory of dipolar BEC. We present the DDI emphasizing its main features, the anisotropy and the long-range character. By means of a mean-field theory a dipolar condensate is described with the non-local Gross-Piteavskii equation. As part of the chapter discusses the general phenomenon of a nondipolar Bose gas. We introduce Bogoliubov theory for dipolar BEC, which is a theory beyond mean-field and takes into account the fluctuations of the order parameter, write the effective Hamiltonian, discuss its diagonalization by means of the Bogoliubov transformation. We derive useful expressions for the noncondensed and anomalous densities in $d$-dimentional case, and calculate the corrections of the ground state energy arising from the quantum fluctuations. The one-body density matrix and superfluid density are also discussed. In the last section of this chapter, the Bogoliubov theory is applied to dipolar Bose gases in 3D case.

Chapter 2 is devoted to an introductory presentation of the weakly interacting Bose gases in random potentials, and provides the technical tools to be used in this thesis. We first present the main statistical properties of disordered potentials and their different forms. We expose the Bogoliubov-Huang-Meng approach for arbitray interaction and disorder potentials. Corrections to the ground state energy and to the condensate fraction due to the external random potential are accurately computed. We also investigate the behavior of the one-body density matrix and the superfluid density. The last section discusses the role of a Gaussian-correlated disorder potential in the 3D dipolar BEC.

Then, chapters 3 and 5, constitute the core of this manuscript, examine the influence of a disorder Gaussian-correlated function on the behavior of dipolar Bose gases in two different cases.

In chapter 3 we dwell on the impacts of a weak disorder potential with a 3D Gaussian-correlated functions on the properties of a homogeneous dipolar Bose gas in the presence of the TBI at finite temperatures. We give a detailed description of the problem and explain how to obtain the excitations spectrum using the Bogoliubov-Huang-Meng approach. Our results show that the TBI are relevant in reducing the influence of the disorder potential in BEC. Corrections to the fluctuations, coherence and the thermodynamics of the condensate due to the disorder and the TBI are also highlighted. We compare our findings against previous theoretical works.

The results presented in this chapter have been the object of the following paper : Keltoum Redaouia and Abdelaali Boudjemaa, Eur. Phys. J. D 73,115 (2019)[**125**].

Chapter 4 is dedicated to the properties of a quasi-2D dipolar Bose gas subjected to a weak random potential with Gaussian correlation by using the Bogoliubov-Huang-Meng theory. In the first part of the chapter, we show that dipolar systems in a quasi-2D geometry are a very active research field from both, theoretical and experimental sides, and review their main features. We then describe the two-body DDI, briefly introduce the Bogoliubov theory, and discuss the appearance of a roton mode in the excitation spectrum. It is shown that the rotonization may strongly enhance quantum and thermal fluctuations as well as the the normal density of the superfluid. The validity criterion of the Bogoliubov approach is well established. In the second part of the chapter, effects of a weak Gaussian-correlated disorder on BEC and on superfluidity in a dilute quasi-2D dipolar Bose gas are studied with a combined numerical and analytical schemes. Our analysis signifies a more pronounced effect of disorder in such a system when the roton is approaching zero with enhancing quantum fluctuations, one-body density-matrix, equation of state, and depleting superfluid density.

Chapter 5 considers the case of a disordered quasi-2D bilayered dipolar BEC with dipoles are oriented perpendicularly to the layers and in same (i.e parallel) /opposite (i.e antiparallel), directions in different layers. The Bogoliubov-Huang-Meng theory is used to quantitatively examine the effects of varying polarization direction and interlayer DDI on the collective excitations, glassy fraction, one-body density matrix and the superfluid fraction. We find that in the parallel configuration, the interlayer DDI causes delocalization of particles enabling the transition to the superfluid phase. Surprisingly, in the antiparallel arrangement, the bosons strongly fill the potential wells formed by disorder fluctuations depressing both the condensate and the superfluidity due to the intriguing interplay of the disorder and the interlayer DDI. The last part deals with a discussion on how the temperature and the polarization direction modify the superfluid fraction.

The most relevant results of this Chapter are published in : Abdelaali Boudjemaa, and Kelthoum Redaouia, Chaos, Solitons, 109543 (2019) [**126**].

The thesis ends with the main conclusions and perspectives for the future.

## CHAPITRE 1

## DIPOLAR BOSE EINSTEIN CONDENSATES

## 1.1 Bose Einstein condensation

Over the past three decades, experimental physicists made tremendous effort to cool and trap neutral atoms [**4**]. They were motivated by Einstein's prediction in 1925 that at ultralow temperatures, dilute gases of bosonic atoms exhibit a novel type of phase transition : below a critical temperature atoms may become indistinguishable by occupying the the same quantum state and form a so-called Bose-Einstein condensate (BEC) [**3**]. The dilute gas BEC was first confirmed experimentally in 1995 by remarkable experiments with rubidium [**1**] (see Fig. 1.1), lithium [**127**] and sodium [**2**], around 70 years after the prediction by Bose and Einstein. The low temperatures required were achieved by a combination of laser cooling and subsequent evaporative cooling [**128**, **129**]. The 1997 Nobel Prize in Physics was awarded to C. Cohen-Tannoudji, S. Chu and W. D. Phillips for their work in the development of laser cooling techniques. Only 4 years later, Carl Wieman, Eric Cornell and Wolfgang Ketterle were awarded the 2001 Nobel prize in physics for their pioneering work on the physics of BEC, a new state of matter. These experiments were followed by demonstrations of some other atomic species to form BEC : hydrogen (H) [**130**], potassium (K) [**131**], helium (He) [**132**], cesium (Cs) [**133**], ytterbium (Yb) [**134**], strontium (Sr) [**135**, **136**], and calcium (Ca) [**137**]. BEC was also achieved by some homonuclear molecules including $Rb_2$, $Cs_2$, $Li_2$[**138**, **139**], K2 [**140**]. Nowadays, ultra-cold atomic and molecular will find highly nontrivial applications in quantum information (quantum simulators) or quantum metrology (see for reviews [**141**, **142**, **136**]).

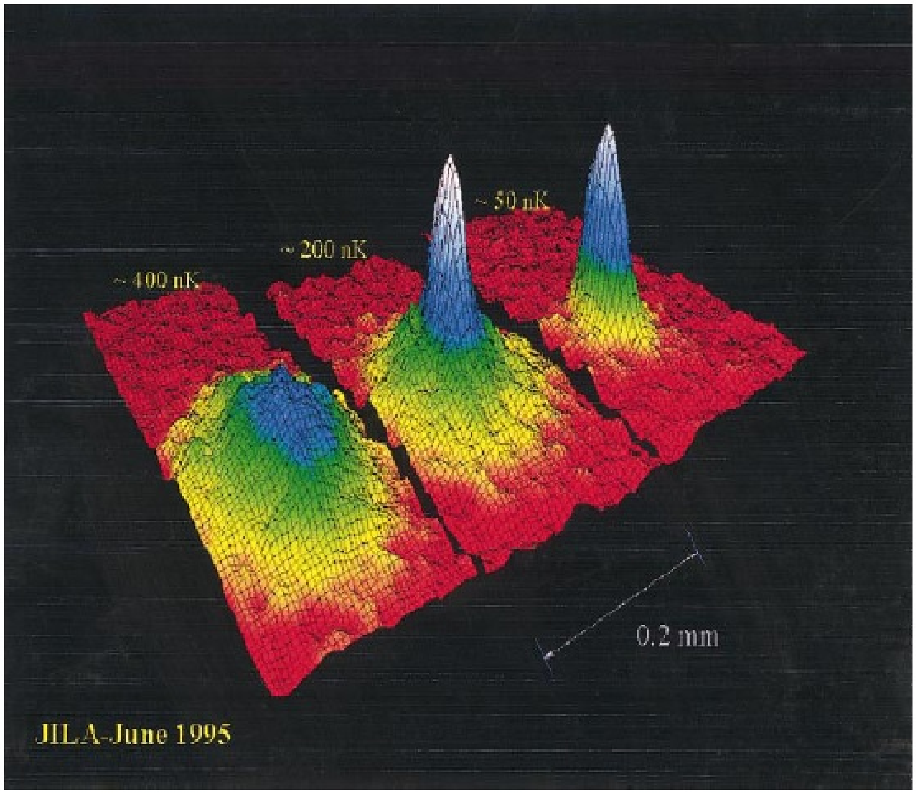


FIGURE 1.1: The first gaseous BEC produced using $^{87}Rb$ atoms at the NIST-JILA lab. The trapping potential was turned off and the BEC allowed to expand before imaging, consequently these figures are representative of the initial velocity distribution. From left to right the temperatures are just above, at and just below the critical BEC temperature $T_c$. The sharp peak to the right provides evidence of BEC. (Copyright (2002) by The American Physical Society [**1**]

At zero and very low temperatures the only interactions considered are those of short-range $s$-wave scattering, which for the aforementioned experiments is sufficient to produce BEC. The interaction potential used to model these scattering effects is given by

$$V_C(\mathbf{r}-\mathbf{r}') = g_2\delta(\mathbf{r}-\mathbf{r}') = \frac{4\pi\hbar^2 a}{m}\delta(\mathbf{r}-\mathbf{r}'), \quad (1.1)$$

where $a$ is the s-wave scattering length, a parameter describing the strength and nature of the effective interactions; $a > 0$ describes repulsive interactions, whereas $a < 0$ describes attractive interactions. In the presence of the repulsive interactions BEC is always stable, whereas attractive interactions give rise to unstable condensate as long as the number of particles is above a critical value, below which the condensate is in a metastable state and $m$ is the particle mass . In 1998 Ketterle's group observed that the $s$-wave scattering length can be tuned in sodium through Feshbach resonances [**143**].

# 1.2 Dipolar Bose Einstein condensate

## 1.2.1 The dipole-dipole interaction

Advances in the cooling and trapping of polar molecules have given rise to investigation of dipolar gases [**144**]. The first successful experimental realization in this direction was, due to its large magnetic dipole moment [**7**] , the BEC of chromium ( $^{52}$Cr) in 2005 with a combination of magneto-optical, magnetic, and optical trapping techniques [**145**] and with a different all-optical method in 2008 [**146**]. Recently, condensates of erbium ($^{168}$Er) [**9**] and dysprosium ($^{164}$Dy) [**8**] have been observed which show larger dipole moments. The dipolar interaction has introduced spectacular features via its anisotropic and long range character while its contact counterpart is isotropic and short range. Anisotropy and long-range character of dipolar interaction allows one to control interparticle interactions by means of tuning the external field using Feshbach resonances or readjusting the trap anisotropy.
In contrast to contact interactions, the sign and strength of the dipolar interaction heavily depend on the trap geometry. The advent of quantum degenerate dipolar gases opens the door to a wide range of scientific explorations. Precision measurements, quantum-controlled chemical reactions and novel phases of matter are among a few prominent examples provided by an ultracold gas of polar molecules.

In this section we examine the relevant two-body interactions in a dipolar BEC. We introduce the interactions between two dipoles which, in contrast to the interactions considered in the preceding subsection, cannot be described by the pseudo contact interaction potential due to the long-range nature of the DDI potential.

The DDI potential takes the form :

$$V_d(\mathbf{r}-\mathbf{r}') = \frac{\mu_0 \mu_m^2}{4\pi} \frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3} \tag{1.2}$$

where $\mu_0 = 4\pi.10^{-7} Tm/A$ is the permeability in free space, $\mu_m$ is the permanent magnetic dipole moment and $\theta$ is the angle between the polarization direction and the relative position of the dipoles r, as illustrated in Fig. 1.2 (a). We consider the case $r > 0$ which is important, because $V_d$ diverges when $r$ tends to zero.

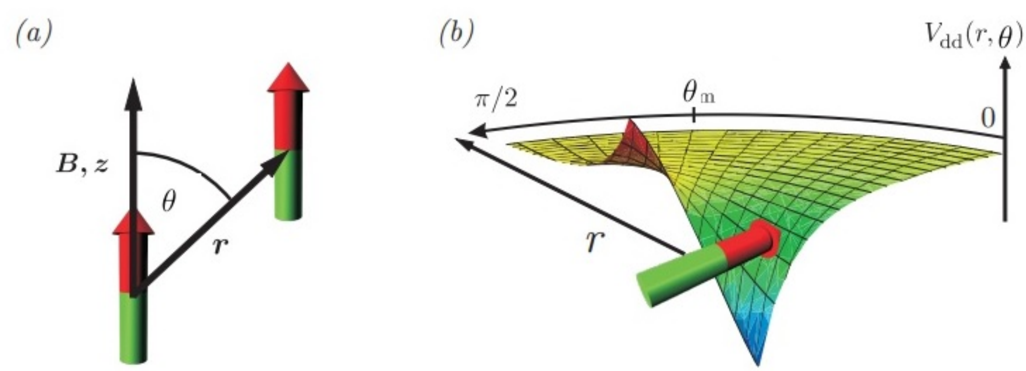


FIGURE 1.2: Dipole-dipole interaction (DDI) : (a) Two dipoles polarized by an external magnetic field B along the $z$direction. The separation $r = |r|$ and the angle $\theta$ enter the DDI potential given by Eq. (1.2). (b) The interaction between two dipoles is attractive in a head-to-tail configuration ($\theta = 0$), repulsive in a side-by-side configuration ($\theta = \pi/2$) and vanishes at the magic angle $\theta = \theta_m$.

As we stated above, the dipolar interaction having either electric or magnetic dipole moment has two properties that are radically different from contact interactions :

— The anisotropy of this interaction implies that it could be either repulsive or attractive depending on the relative orientation of the two dipoles. Side by side particles interact with repulsive DDI, see figure 1.2, while a head-to-tail configuration correspond to an attractive one, see figure 1.2. For the special value $\theta = \arccos(1/\sqrt{3}) \sim 54.7°$, the so-called "magic-angle", the dipole-dipole interaction vanishes. Such an anisotropic interaction leads to a series of interesting phenomena even in classical physics, a fascinating example is the Rosensweig instability. For dipolar quantum gases the DDI is related to the observation of a maxon-roton spectrum and to the stability property of the system.

— In opposite to the short-range interactions, DDI are long-ranged in 3D systems[**145**], which means that in dipolar systems the scattering properties are radically different compared to the other systems.

The full pseudo-potential describing binary contact and dipolar interactions reads

$$V(\mathbf{r}-\mathbf{r}') = \mathbf{g_2}\delta(\mathbf{r}-\mathbf{r}') + \frac{\mu_0\mu_m^2}{4\pi}\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{1.3}$$

To describe the physics of dipolar BECs, we now define some key parameters. In analogy to scattering length $a$, we define a characteristic dipolar length

$$r_* = a_{dd} = \frac{\mu_0\mu_m^2 m}{12\pi\hbar^2}, \tag{1.4}$$

and the dipolar coupling stength

$$g_{dd} = \frac{4\pi\hbar^2}{m}a_{dd} = \frac{\mu_0\mu_m^2}{3}. \tag{1.5}$$

We stress here that the dipolar length $r_*$ does not correspond to a finite interaction radius of the dipolar interactions. We define the relative strength which is the ratio of the dipolar and the contact coupling strengths,

$$\epsilon_{dd} = \frac{g_{dd}}{g_2} = \frac{r_*}{a} = \frac{\mu_0 \mu_m^2 m}{12\pi\hbar^2 a}. \tag{1.6}$$

For a BEC to be dominated by dipolar effects, the dipolar interaction needs to be at least as strong as the contact interaction giving $\epsilon_{dd} \geq 1$. From the angular dependence in Eq.(1.6) it is clear the anisotropic nature of DDI.

In Table 1.1 we show these quantities for all of the bosonic isotopes experimentally created to date. Rubidium features on this table, with a value of $\epsilon_{dd} = 0.007$, highlighting that it is safe to neglect dipolar interactions for modelling this species. The only dominantly dipolar species are the dysprosium isotopes, although through reducing the scattering length of $^{166}Er$ one could access the dominantly dipolar regime, like what has been seen in $^{52}Cr$ [**147**]. It is worth mentioning here that the equivalent value for $\epsilon_{dd}$ of alkali dimers is at least one order of magnitude larger than seen in Table 1.1. For example, the molecule $KRb$ has $\epsilon_{dd} = 37$, whereas$NaRb$ boasts a value of $\epsilon_{dd} = 229$ [**148**].

| Species | $a_s(a_0)$ | $\mu_m(\mu_B)$ | $a_{dd}(a_0)$ | $\epsilon_{dd}$ | References |
|---|---|---|---|---|---|
| $^{87}Rb$ | 100.4(1) | 1 | 0.7 | 0.007 | [**149**] |
| $^{52}Cr$ | 102.5(4) | 6 | 15.1 | 0.15 | [**150**] |
| $^{162}Dy$ | 122(10) | 9.93 | 129.2 | 1.06 | [**151**] |
| $^{164}Dy$ | 92(8) | 9.93 | 130.8 | 1.42 | [**151**, **152**] |
| $^{166}Er$ | 72(13) | 6.98 | 65.5 | 0.91 | [**153**] |
| $^{168}Er$ | 137(1) | 6.98 | 66.2 | 0.48 | [**154**] |
| $^{170}Er$ | -221(22) | 6.98 | 67.0 | -0.3 | [**153**] |

TABLE 1.1: Dipole strengths-darker shade of gray indicates strength.Lengths here are presented in units of the Bohr radius, $a_0 = 5.3\,10^{-10}m$.

### 1.2.2 Non-local Gross-Pitaevskii equation

Let us consider a gas of dipolar bosons. The second-quantized Hamiltonian of the system reads :

$$\hat{H} = \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m}\Delta + V_{ext}(\mathbf{r}) - \mu \right] \hat{\psi}(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') V(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}), \tag{1.7}$$

where $\hat{\psi}^\dagger$ and $\hat{\psi}$ are the particle creation and annihilation operators, which fulfill the usual bosonic commutation relations,

$$[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r}' - \mathbf{r}),\ [\hat{\psi}^\dagger(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r})] = [\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r}')] = 0, \tag{1.8}$$

and $V_{ext}(\mathbf{r})$ is the trapping potential, and $\mu$ is the chemical potential. The interaction potential $V(\mathbf{r})$ may be approximated by the pseudo-potential (1.3), and then our Hamiltonian becomes :

$$\begin{aligned} \hat{H} = & \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m}\Delta + V_{ext}(\mathbf{r}) - \mu + \frac{1}{2} g_2 \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) \\ & + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') V_d(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}). \end{aligned} \tag{1.9}$$

We may then obtain easily the Heisenberg equations for the dynamics of the field operators, by employing the bosonic commutation rules. Since we are interested in the case of BECs far from the critical condensation temperature, we may introduce the usual Bogoliubov approximation $\hat{\psi}(\mathbf{r}) = \psi(\mathbf{r})$. In this way we obtain that the Heisenberg equation transforms into the following equation for the dynamics of the condensate wavefunction :

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m}\Delta + V_{ext}(\mathbf{r}) - \mu + g_2 |\psi(\mathbf{r}, t)|^2 + \frac{C_{dd}}{4\pi} \int d\mathbf{r} \frac{1 - 3\cos^2\theta}{|\mathbf{r}' - \mathbf{r}|^3} |\psi(\mathbf{r}', t)|^2 \right] \psi(\mathbf{r}, t). \tag{1.10}$$

Note that this equation is a modified version of the well-known Gross-Pitaevskii equation (GPE), or equivalently the non-linear Schrödinger equation. In the absence of DDI the nonlinearity is given by the term $g\,|\psi(\mathbf{r},t)|^2$, and hence it is a local nonlinearity similar to that found in many Kerr media in nonlinear optics. On the contrary the nonlinearity introduced by the DDI is non-local, i.e. the wavefunction in $\mathbf{r}$ depends on the wavefunction in $\mathbf{r}'$ through a kernel given by $V_{dd}(\mathbf{r}-\mathbf{r}')$.

## 1.3 Bogoliubov theory

In 1947, Bogoliubov suggested an important theory to compute the excitation spectrum of weakly interacting Bose gases. Such a theory predicts a linear excitation spectrum and provides expressions for the thermodynamic functions which are valid in the dilute limit.

### 1.3.1 Bogoliubov excitations

Let us consider a uniform gas of interacting bosons occupying a $d$-dimensional volume $L^d$. The second quantized Hamiltonian of the system is written as :

$$\hat{H}=\sum_{\mathbf{k}}\frac{\hbar^2k^2}{2m}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}+\frac{1}{2L^d}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}}f(\mathbf{p})\hat{\mathbf{a}}^\dagger_{\mathbf{k}+\mathbf{p}}\hat{\mathbf{a}}^\dagger_{\mathbf{q}-\mathbf{p}}\hat{\mathbf{a}}_{\mathbf{q}}\hat{\mathbf{a}}_{\mathbf{k}}, \tag{1.11}$$

where $\hat{a}^\dagger_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators of particles satifying to the usual Bose commutaion rules $\left[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}\right]=\delta_{\mathbf{k},\mathbf{k}'}$ and $\left[\hat{a}^\dagger_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}\right]=[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}]=0$. In Hamiltonian (1.11), the first term is the single-particle part corresponds to the kinetic energy of particles and the second term describes the two-body interaction Hamiltonian of the dipolar force.

According to the Bogoliubov's idea, the average number of particle in the Bose condensate $N_c$ in the $k=0$ state is close to the total number of particles in the gas $N$ and the zero-momentum operators $\hat{a}_0^\dagger$ and $\hat{a}_0$ operating on the ground state satisfy the relations $\hat{a}_0^\dagger|N_c\rangle = \sqrt{N_c+1}|N_c+1\rangle$ and $\hat{a}_0|N_c\rangle = \sqrt{N_c}|N_c-1\rangle$. Therefore, for $N_c \gg 1$, it is possible to write $\sqrt{N_c+1} \approx \sqrt{N_c}$ and replace each of the operators $\hat{a}_0^\dagger$ and $\hat{a}_0$ by the $c$-number $\sqrt{N_c}$ which leads to $[\hat{a}_0, \hat{a}_0^\dagger] = 0$. The application of perturbation theory means that the last term in (1.11) should be decomposed in powers of the small quantities $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$, with $\mathbf{k} \neq \mathbf{0}$. Therefore, the Hamiltonian (1.11) separates into three distinct parts classified according to the number of the operators $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ in the products $\hat{H} = \sum_{n=0}^{2} \hat{H}^{(n)}$.

The zero-order term does not contain $a_k$ :

$$H^{(0)} = \frac{1}{2V} a_0^\dagger a_0^\dagger a_0 a_0 \tilde{V}(0). \tag{1.12}$$

Using the normalization relation :

$$a_0^\dagger a_0 = N_c = N - \sum_{\mathbf{k}\neq\mathbf{0}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} \Rightarrow (a_0^\dagger a_0)^2 \approx N^2 - 2N \sum_{\mathbf{k}\neq\mathbf{0}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}},$$

the Hamiltonian (1.12) becomes :

$$H^{(0)} = \frac{1}{2} n N \tilde{V}(0) - n\tilde{V}(0) \sum_{\mathbf{k}\neq\mathbf{0}} \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}. \tag{1.13}$$

The first-order term is zero :

$$H^{(1)} = 0. \tag{1.14}$$

The second-order term is given by :

$$\hat{H}^{(2)} = \sum_{\mathbf{k}\neq 0} \left[E_k + n_c \tilde{V}(\mathbf{k}) + n_c \tilde{V}(0)\right] \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2} n_c \sum_{\mathbf{k}\neq 0} \tilde{V}(\mathbf{k}) \left[\hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger\right], \tag{1.15}$$

where $n_c = N_c/V$ is the density of condensed atoms. Here we have used the momentum conservation $\mathbf{k} = -\mathbf{k}$.

The physical interpretation of these components is :

— $\tilde{V}(0)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ represents the Hartree energy which arises from the direct interaction of particle in the state $k$ with $N_c$ atoms in the condensate.

— $\tilde{V}(\mathbf{k})\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ is the exchange term, often known as Fock term, in which an atom in the state $k$ is scattered into the zero momentum state, while at the same time the second is scattered from the condensate to the state $k$. Note that the contact interaction disguises the presence of the direct and exchange contributions (see below).

— $\tilde{V}(\mathbf{k})(\hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}})$ represents the scattering of two atoms in the condensate to states with $\pm k$ in vice versa.

The Bogoliubov approach assumes that the depletion is small i.e. $N_c \approx N$ or $n_c \approx n$ and retains in interaction only the second order terms $H_2$. Summing (1.13) with (1.15), the Hamiltonian (1.11) is reduced to a quadratic one in terms of operators $\hat{a}^{\dagger}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ :

$$\hat{H} = \frac{1}{2}nN\tilde{V}(0) + \sum_{\mathbf{k}\neq 0}\left[\left(E_k + n\tilde{V}(\mathbf{k})\right)\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}n\tilde{V}(\mathbf{k})\left(\hat{a}_{-\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{-\mathbf{k}}\right)\right]. \quad (1.16)$$

In order to calculate the energy levels of the system one has to diagonalize the Hamiltonian (1.16). This can be done by employing the canonical Bogoliubov transformations :

$$\hat{a}^{\dagger}_{\mathbf{k}} = u_k\hat{b}^{\dagger}_{\mathbf{k}} - v_k\hat{b}_{-\mathbf{k}}, \qquad \hat{a}_{\mathbf{k}} = u_k\hat{b}_{\mathbf{k}} - v_k\hat{b}^{\dagger}_{-\mathbf{k}}, \quad (1.17)$$

where $\hat{b}^{\dagger}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are operators of elementary excitations which have to satisfy the same commutation rules as the operators $\hat{a}^{\dagger}_{k}$ and $\hat{a}_{k}$.

$$\left[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}'}\right] = \delta_{\mathbf{k},\mathbf{k}'} = \delta_{\mathbf{k},\mathbf{k}'}, \qquad \left[\hat{b}^{\dagger}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{k}'}\right] = \left[\hat{b}_{\mathbf{k}}, \hat{b}_{\mathbf{k}'}\right] = 0. \quad (1.18)$$

From the commutation rules (1.18) one can show that the coefficients must satisfy the condition : $u_k^2 - v_k^2 = 1$. Setting to zero the coefficient of the term proportional to $\hat{b}_{\mathbf{k}}\hat{b}_{\mathbf{k}}$ or

$\hat{b}^\dagger_{\mathbf{k}}\hat{b}^\dagger_{\mathbf{k}}$, one obtains for he Bogoliubov functions $u_k, v_k$

$$u_k, v_k = (\sqrt{\varepsilon_k/E_k} \pm \sqrt{E_k/\varepsilon_k})/2, \tag{1.19}$$

where $E_k = \hbar^2 k^2/2m$ is the energy of free particle, and the Bogoliubov excitations energy is given by

$$\varepsilon_k = \sqrt{E_k^2 + 2\mu_0(\theta_k)E_k}, \tag{1.20}$$

where $\mu_0 = n_c \lim_{k\to 0} V(\mathbf{k})$ is the chemical potential defined in the first order of perturbation theory [**153**]. The spectrum (1.20) is a gapless. One can check that the Hugenholtz-Pines theorem is well satisfied. For small momenta $k \to 0$, the Bogoliubov dispersion law (1.20) is linear in $k$ and well approximated by the phonon-like linear dispersion form

$$\varepsilon_k = \hbar c_s\, k, \tag{1.21}$$

where $c_s = \sqrt{n\tilde{V}(0)/m}$ is the sound velocity.

In the limit of large momenta, the Bogoliubov dispersion law (3.3) reduces to the free-particle form :

$$\varepsilon_k = E_k, \tag{1.22}$$

which corresponds to $v_k = 0$, $u_k = 1$, and the distribution function $N_k$ reduces to the ordinary Bose distribution.

Thus, the Hamiltonian (1.11) reduces to the diagonal form :

$$\hat{H} = E_0 + \sum_{\mathbf{k}} \varepsilon_k \hat{b}^\dagger_{\mathbf{k}}\hat{b}_{\mathbf{k}}, \tag{1.23}$$

where

$$E_0 = \frac{1}{2}n\tilde{V}(0)N + \frac{1}{2}\sum_{\mathbf{k}\neq 0}\left(\varepsilon_k - E_k - n\tilde{V}(\mathbf{k})\right), \tag{1.24}$$

is the ground-state energy.

From the Hamiltonian (1.23 ) and the commutation rules (1.18) one can identify $\hat{b}_{\mathbf{k}}^{\dagger}$ and $\hat{b}_{\mathbf{k}}$ as the creation and annihilation operators of quasiparticles with energy $\varepsilon_k$. The ground state energy is given by $E_0$ which is the energy of the "vacuum" of quasiparticles $\hat{H}|0\rangle = E_0|0\rangle$, where the "vacuum" state is defined as $\hat{b}_{\mathbf{k}}|0\rangle = 0$ for any value of $\mathbf{k} \neq \mathbf{0}$. The excited states are given by $|\mathbf{k}\rangle = \hat{\mathbf{b}}_{\mathbf{k}}|\mathbf{0}\rangle$ and have energy $E(k)$ and momentum $\mathbf{k}$.

### 1.3.2 Noncondensed and anomalous densities

The noncondensed and the anomalous densities are defined, respectively as

$$\tilde{n} = \frac{1}{\vee}\sum_{\mathbf{k}} \langle \hat{a}_{\mathbf{k}}^{\dagger}\hat{a}_{\mathbf{k}}\rangle, \tag{1.25}$$

and

$$\tilde{m} = \frac{1}{\vee}\sum_{\mathbf{k}} \langle \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}}\rangle. \tag{1.26}$$

where $\vee$ is a quantization volume. Then invoking for the operators $a_k$ the transformation (1.17), setting $\langle \hat{b}_{\mathbf{k}}^{\dagger}\hat{b}_{\mathbf{k}}\rangle = \delta_{\mathbf{k}'\mathbf{k}}N_k$ and putting the rest of the expectation values equal to zero, where $N_k = [\exp(\varepsilon_k/T) - 1]^{-1}$ are occupation numbers for the excitations. As we work in the thermodynamic limit, the sum over $\mathbf{k}$ can be replaced by the integral $\sum_{\mathbf{k}} \equiv L^d \int d^dk/(2\pi)^d$ and using the fact that $2N(x) + 1 = \coth(x/2)$, we obtain :

$$\tilde{n} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left[\frac{E_k + \tilde{V}(k)n_c}{\varepsilon_k} - 1\right] + \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{E_k + \tilde{V}(k)n_c}{\varepsilon_k}\left[\coth\left(\frac{\varepsilon_k}{2T}\right) - 1\right], \tag{1.27}$$

and

$$\tilde{m} = -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{\tilde{V}(k)}{\varepsilon_k} - \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{\tilde{V}(k)}{\varepsilon_k}\coth\left(\frac{\varepsilon_k}{2T}\right). \tag{1.28}$$

First terms in Eqs.(1.27) and (1.28) are the zero-temperature contribution to the noncondensed $\tilde{n}_0$ and anomalous $\tilde{m}_0$ densities, respectively. Second terms represent the contribution of the so-called thermal fluctuations and we denote them as $\tilde{n}_T$ and $\tilde{m}_T$, respectively.

Expressions (1.27) and (1.28) must satisfy the equality [**156-158**]

$$I_k = (2\tilde{n}_k + 1)^2 - |2\tilde{m}_k|^2 = \coth^2\left(\frac{\varepsilon_k}{2T}\right). \tag{1.29}$$

Equation (1.29) clearly shows that $\tilde{m}$ is larger than $\tilde{n}$ at low temperature, so the omission of the anomalous density in this situation is principally unjustified approximation and wrong from the mathematical point of view.

The expression of $I$ allows us to calculate in a very useful way the dissipated heat $Q = (1/n)\int E_k I_k d^d k/(2\pi)^d$ for $d$-dimensional Bose gas [**158**, **159**], where $n = n_c + \tilde{n}$ is the total density. Indeed, the dissipated heat or the superfluid fraction (see below) are defined through the dispersion of the total momentum operator of the whole system. This definition is valid for any system, including nonequilibrium and nonuniform systems of arbitrary statistics. In an equilibrium system, the average total momentum is zero. Hence, the corresponding heat becomes just the average total kinetic energy per particle.

### 1.3.3 One body correlation function

Another interesting result that can be obtained from Bogoliubov theory is the one-body correlation function which is defined as :

$$\tilde{n}(\mathbf{r}, \mathbf{r}', t, t') = g^{(1)}(\mathbf{r}, \mathbf{r}', t, t') = \langle \hat{\Psi}^\dagger(\mathbf{r}, t)\hat{\Psi}(\mathbf{r}', t')\rangle. \tag{1.30}$$

At equal times, $t = t'$, we have $g^{(1)}$ is independent of $t$ and we may put, $t = t' = 0$. In uniform case $g^{(1)}$ depends only on the difference $|\mathbf{r} - \mathbf{r}'| = \mathbf{r}$ and we may set $\mathbf{r}' = 0$. Using the decomposition, expressing the noncondensed field operator as $\hat{\tilde{\psi}} = (1/V)\sum_k [u_k \hat{b}_k e^{i\mathbf{k}.\mathbf{r}} - v_k \hat{b}_k^\dagger e^{-i\mathbf{k}.\mathbf{r}}]$, and then taking into account that $|\Phi(\mathbf{r}, t)| = \sqrt{n_c}$. We thus, obtain

$$g^{(1)}(r) = g_0^{(1)}(r) + g_{th}^{(1)}(r) \quad = \int \frac{d\mathbf{k}}{(2\pi)^3} v_k^2\, \mathrm{e}^{i\mathbf{k}.\mathbf{r}} + \int \frac{d\mathbf{k}}{(2\pi)^3}(u_k^2 + v_k^2) N_k\, \mathrm{e}^{i\mathbf{k}.\mathbf{r}}. \tag{1.31}$$

The first term of (1.31) represents the zero temperature contribution to the correlation function, and the second term accounts for the thermal contribution to one-body density matrix.

### 1.3.4 Superfluid fraction

We consider a fluid at zero temperature, in which all particles are in the ground state and flowing along a capillary at constant velocity. If the fluid is viscous, the motion will produce dissipation of energy via friction with the capillary wall and decrease of the kinetic energy. We assume that such dissipative processes take place through the creation of an elementary excitation, which is the Bogoliubov quasi-particle $\varepsilon_k$ for the case of an interacting Bose gas. The Landau's criteria of superfluidity satisfies the following condition [**160**, **161**]

$$v_c = min_p \frac{\varepsilon_P}{|P|}, \tag{1.32}$$

where $\overrightarrow{P} = \hbar \overrightarrow{k}$ the minimum is calculated over all the values of $P$. If instead the fluid velocity $v$ is smaller than (1.32), then no elementary excitation will be spontaneously formed. Thus, the Landau's criterion of superfluidity is summarized as the relative velocity between the fluid and the capillary is smaller than the critical value $v_c \leq v$.

By a general definition, the superfluid density is the partial density appearing as a response to a velocity boost,

$$n_s = \frac{1}{3mv} lim_{v\to 0} \frac{\partial}{\partial V} \left\langle \hat{P}_v \right\rangle_v, \tag{1.33}$$

where the average of the system momentum $\hat{P}_v = \hat{P} + NmV$. In order to calculate the superfluid fraction we follow the following steps : The superfluid fraction can be given in

$d$-dimensional case as [**57**] :

$$\frac{n_s}{n} = 1 - \frac{2Q}{dT}, \tag{1.34}$$

where $n_s$ is the superfluid density, $n$ is the total density and $Q$ is the dissipated heat, having for an equilibrium system. It is given :

$$Q = \frac{\langle P^2 \rangle}{2mN}, \tag{1.35}$$

passing to Fourier transform, we have $\langle P^2 \rangle = \sum_{k,p} \mathbf{kp} \, \langle \hat{n}_k \hat{n}_p \rangle$, where $\hat{n}_k = \hat{a}_k^\dagger \hat{a}_k$ and $\hat{n}_p = \hat{a}_p^\dagger \hat{a}_p$. In the Hatree-Fock-Bogoliubov approximation

$$\langle \hat{n}_k \hat{n}_p \rangle = \langle \hat{a}_k^+ \hat{a}_k \hat{a}_p^+ \hat{a}_p \rangle = \langle \hat{a}_k^+ \hat{a}_k \rangle \langle \hat{a}_p^+ \hat{a}_p \rangle + \langle \hat{a}_k^\dagger \hat{a}_p \rangle \langle \hat{a}_k \hat{a}_p^\dagger \rangle + \langle \hat{a}_k \hat{a}_p \rangle \langle \hat{a}_k^\dagger \hat{a}_p^\dagger \rangle. \tag{1.36}$$

Using the fact that $\hat{a}_k \hat{a}_p^\dagger = \hat{a}_p^\dagger \hat{a}_k + \delta_{kp}$ and $\hat{a}_k \hat{a}_p = \tilde{m}_{kp} \left\langle \hat{a}_k^\dagger \hat{a}_p \right\rangle = \tilde{n}_{kp}$, we obtain

$$\langle \hat{n}_k \hat{n}_p \rangle = \tilde{n}_k \tilde{n}_p + \tilde{m}_{kp^2} \delta_{kp} + \tilde{n}_k (1 + \tilde{n}_k) \delta_{kp}. \tag{1.37}$$

Thus,

$$\begin{aligned} \langle P^2 \rangle &= \sum_{k,p} \mathbf{kp} [\tilde{n}_k \tilde{n}_p + \tilde{m}_{kp^2} \delta_{kp} + \tilde{n}_k (1 + \tilde{n}_k) \delta_{kp}] \\ &= \frac{1}{\vee} \sum_k \hbar^2 k^2 [\tilde{n}_k (1 + \tilde{n}_k) - \tilde{m}_k^2], \end{aligned} \tag{1.38}$$

puting (1.38) into (1.35), the dissipated heat becomes

$$Q = \frac{1}{n} \int_0^\infty \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} E_k (\tilde{n}_k^2 + \tilde{n}_k - \tilde{m}_k^2). \tag{1.39}$$

From the equality (1.29)

$$Q = \frac{1}{n}\int_0^\infty \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{2\hbar^2}{Tmn}\left[\frac{k_i k_j}{4\sinh^2(\varepsilon_k/2T)}\right], \tag{1.40}$$

where the quantity represents the normal fraction of the Bose-condensed gas (liquid). Thus, we finally arrive at the following expression for the normal density $2Q/dT$ which has a tensor structure [**82**]

$$\frac{n_s^{ij}}{n} = \delta_{ij} - \frac{2}{Tn}\int \frac{d^3k}{(2\pi)^3}\left[\frac{\hbar^2}{2m}\frac{k_i k_j}{4\sinh^2(\varepsilon_k/2T)}\right]. \tag{1.41}$$

It is worth stressing that if in expression (1.41) $\tilde{m}$ were omitted, then the related integral would be divergent leading to the meaningless value $n_s \to \infty$. This indicates that the presence of the anomalous density is crucial for the occurrence of the superfluidity in Bose gases [**50**, **162**] which is in fact understandable since both quantities are caused by atomic correlations.

## 1.4 Three-dimensional homogeneous dipolar Bose gases

The Fourier transform of the DDI simplifies the calculation of the integral appearing in the long-range interaction term of the energy. In 3D case, it can be written as :

$$\tilde{V}(\mathbf{k}) = g_2[1 + \epsilon_{dd}(3\cos^2\theta_k - 1)], \tag{1.42}$$

where $\epsilon_{dd} = C_{dd}/3g$ is the dimensionless relative strength which describes the interplay between the DDI and the short-range interaction.

The Bogoliubov excitations energy is given by

$$\varepsilon_k = \sqrt{E_k^2 + 2\mu_{0d}(\theta)E_k}, \tag{1.43}$$

where $\mu_{0d}(\theta) = n \lim_{k\to 0} \tilde{V}(\mathbf{k})$ is the zeroth order chemical potential. For $k \to 0$, the excitations are sound waves $\varepsilon_k = \hbar c_{sd}(\theta)k$, where $c_{sd}(\theta) = c_s\sqrt{1+\epsilon_{dd}(3\cos^2\theta - 1)}$ with $c_s = \sqrt{g_2 n/m}$ is the sound velocity without DDI. Due to the anisotropy of the dipolar interaction, the sound velocity acquires a dependence on the propagation direction, which is fixed by the angle $\theta$ between the propagation direction and the dipolar orientation. This angular dependence of the sound velocity has been confirmed experimentally [**41**].

### 1.4.1 Condensate depletion and anomalous density

At zero temperature, the quantum depletion can be obtained from the integral (1.27)

$$\frac{\tilde{n}}{n_c} = \frac{8}{3}\sqrt{\frac{n_c a^3}{\pi}}\mathcal{Q}_3(\epsilon_{dd}). \tag{1.44}$$

The contribution of the DDI is expressed by the function $\mathcal{Q}_3(\epsilon_{dd})$, which is special case $j = 3$ of $\mathcal{Q}_j(\epsilon_{dd}) = (1-\epsilon_{dd})^{j/2}{}_2F_1\left(-\frac{j}{2}, \frac{1}{2}; \frac{3}{2}; \frac{3\epsilon_{dd}}{\epsilon_{dd}-1}\right)$, where ${}_2F_1$ is the hypergeometric function. Note that functions $\mathcal{Q}_j(\epsilon_{dd})$ attain their maximal values for $\epsilon_{dd} \approx 1$ and become imaginary for $\epsilon_{dd} > 1$.

Equation (1.44) is formally similar to the that obtained from the zeroth order of perturbation theory [**163**]. The density $n_c$ of condensed particles which constitutes our corrections, appears as a key parameter instead of the total density $n$.

Now if we use the integral in Eq.(1.28) directly by summing over all states, we find that the expression for $\tilde{m}$ diverges as we take the sum over higher and higher states i.e. the so called ultraviolet divergence. The price to be paid to circumvent this divergence is to introduce the Beliaev-type second order coupling constant [**163**, **164**]

$$g_R(\mathbf{k}) = V(\mathbf{k}) - \frac{m}{\hbar^2}\int \frac{d^3q}{(2\pi)^3}\frac{V(-\mathbf{q})V(\mathbf{q})}{2E_q}. \tag{1.45}$$

After the subtraction of the ultraviolet divergent part, the renormalized anomalous density is given [**165**]

$$\tilde{m}_R = -n_c \int \frac{d^3k}{(2\pi)^3} \tilde{V}(\mathbf{k}) \left[ \frac{1}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) - \frac{1}{2E_k} \right]. \tag{1.46}$$

In contrast to $\tilde{m}$ in (1.28), $\tilde{m}_R$ has no ultraviolet divergence from large $k$ contributions. The authors of [**166**] have pointed out that the self-consistent ladder diagram approximation for the $T$-matrix can be expressed in terms of $\tilde{m}_R$.

To obtain an estimate value of $\tilde{m}$, we note that the quasi-particle energy goes over to the free particle energy for $\varepsilon_k > gn_c$. At zero temperature ($\tilde{m} = \tilde{m}_0$), we find

$$\frac{\tilde{m}}{n_c} = 8\sqrt{\frac{n_c a^3}{\pi}} \mathcal{Q}_3(\epsilon_{dd}). \tag{1.47}$$

One should mention at this level that this expression has never been obtained before in the literature.

Equation (1.47) is important in several respects : first of all, it shows that the anomalous density is three times larger than the noncondensed density whatever the type of the interaction. Second, $\tilde{m}$ has a positive value in argreement with the case of uniform Bose gas with pure contact interaction [**158**, **159**]. Likewise, the anomalous density obtained in Eq.(1.47) leads us to reproduce exactly the Lee-Huang-Yang (LHY) corrected equation of state [**167**].

Remarkably, we see from expressions (2.28) and (2.29) that the noncondensed and the anomalous densities increase monotocally with $\epsilon_{dd}$. For a condensate with pure contact interactions ($\mathcal{Q}_3(\epsilon_{dd} = 0) = 1$), $\tilde{n}$ and $\tilde{m}$ reduce to their usual expressions. While, for maximal value of DDI i.e. $\epsilon_{dd} \approx 1$, they are 1.3 larger than their values of pure contact interactions which means that the DDI may enhance fluctuations of the condensate at zero temperature.

### 1.4.2 Thermodynamics quantities

The presence of quantum fluctuations leads also to corrections of the chemical potential which are given by [**55**, **158**, **165**].

$$\delta\mu = \sum_{\mathbf{k}} f(\mathbf{k})[v_k(v_k - u_k)] = \sum_{\mathbf{k}} V(\mathbf{k})(\tilde{n} + \tilde{m}). \tag{1.48}$$

Inserting the definitions (1.27) and (1.28) into the expression of $\delta\mu$, we find after integration :

$$\delta\mu = \frac{32}{3} g n_c \sqrt{\frac{n_c a^3}{\pi}} \mathcal{Q}_5(\epsilon_{dd}). \tag{1.49}$$

The total chemical potential is then written as $\mu = \mu_0(\theta_k) + \delta\mu$. For $n_c \approx n$ and for a condensate with pure contact interaction ($\mathcal{Q}_5(\epsilon_{dd} = 0) = 1$), the obtained chemical potential excellently agrees with the famous LHY quantum corrected equation of state [**167**].

By integrating the chemical potential correction with respect to the density, one obtains beyond mean field the ground state energy as

$$E = E_0(\theta_k) + \frac{64}{15} \vee g n_c^2 \sqrt{\frac{n_c a^3}{\pi}} \mathcal{Q}_5(\epsilon_{dd}), \tag{1.50}$$

where $E_0(\theta_k) = \mu_0(\theta_k) N_c/2$ with $N_c$ is the number of condensed particles.
Note that our formulas of the equation of state (1.49) and the ground state energy (1.50) constitute a natural extension of those obtained in Ref [**163**].

At $T = 0$, the inverse compressibility is equal to $\kappa^{-1} = n^2 \partial\mu/\partial n$. Then, using Eq.(1.49), we get

$$\frac{\kappa^{-1}}{n^2} = \frac{\mu_0(\theta_k)}{n_c} + 16g \sqrt{\frac{n_c a^3}{\pi}} \mathcal{Q}_5(\epsilon_{dd}). \tag{1.51}$$

One can also show that the shift of the sound velocity is $16g\sqrt{n_c a^3/\pi}\mathcal{Q}_5(\epsilon_{dd})$, which is consistent with the change in the compressibility $mc_s^2 = n\partial\mu/\partial n$ [**166**] associated with the

LHY correction in the equation of state (1.49). Expanding the square root of the obtained formula with $\epsilon_{dd} = 0$ in powers of the gas parameter $n_c a^3$, we recover easily the Beliaev sound velocity of Bose gas with pure contact interaction $\delta c_s/c_s \approx 8\sqrt{n_c a^3/\pi}$ [**155**, **166**].

What is noticeable is that the chemical potential, the energy and the compressibility are increasing with dipole interaction parameter. For $\epsilon_{dd} \approx 1$, these quantities are 2.6 larger than their values of pure contact interaction which means that DDI effects are more significant for thermodynamic quantities than for the condensate depletion and the anomalous density.

The Bogoliubov approach assumes that fluctuations should be small. We thus conclude from Eqs. (2.28) and (2.29) that at $T = 0$, the validity of the Bogoliubov theory requires the inequality

$$\sqrt{n_c a^3}\mathcal{Q}_3(\epsilon_{dd}) \ll 1. \tag{1.52}$$

For $n_c = n$, this parameter differs only by the factor $\mathcal{Q}_3(\epsilon_{dd})$ from the universal small parameter of the theory, $\sqrt{na^3} \ll 1$, in the absence of DDI.

We now generalize the above obtained results for the case of a spatially homogeneous dipolar Bose-condensed gas at finite temperature.

At temperatures $T \ll gn_c$, the main contribution to integrals (1.27) and (1.28) comes from the region of small momentum where $\varepsilon_k = \hbar c_{sd} k$. After some algebra, we obtain the following expressions for the thermal contribution of the noncondensed and anomalous densities[**51**] :

$$\frac{\tilde{n}_T}{n_c} = -\frac{\tilde{m}_T}{n_c} = \frac{2}{3}\sqrt{\frac{n_c a^3}{\pi}} \left(\frac{\pi T}{g n_c}\right)^2 \mathcal{Q}_{-1}(\epsilon_{dd}). \tag{1.53}$$

Equation (1.53) shows clearly that $\tilde{n}$ and $\tilde{m}$ are of the same order of magnitude at low temperature and only their signs are opposite.

Comparing the result of Eq. (1.53) with the zero-temperature noncondensed $\tilde{n}_0$ and anomalous $\tilde{m}_0$ densities following from Eqs. (2.28) and (2.29) we see that at temperatures

$T \ll gn_c$, thermal contributions $\tilde{n}_T$ and $\tilde{m}_T$ are small and can be omitted when calculating the total fractions. The situation is quite different at temperatures $T \gg gn_c$, where the main contribution to integrals (1.27) and (1.28) comes from the single particle excitations. Hence, $\tilde{n}_T \approx (mT/2\pi\hbar^2)^{3/2}\zeta(3/2)$, where $\zeta(3/2)$ is the Riemann Zeta function. The obtained $\tilde{n}_T$ is nothing else than the density of noncondensed atoms in ideal Bose gas. Moreover, the anomalous density being proportional to the condensed density, tend to zero together and hence their contribution becomes automatically small.
Another important remark is that for $\epsilon_{dd} \approx 1$, thermal fluctuations (1.53) are 10.7 greater than their values of pure short range interaction. This reflects that the DDIs may strongly enhance fluctuations of the condensate at finite temperature than at zero temperature (see figure.1.3).

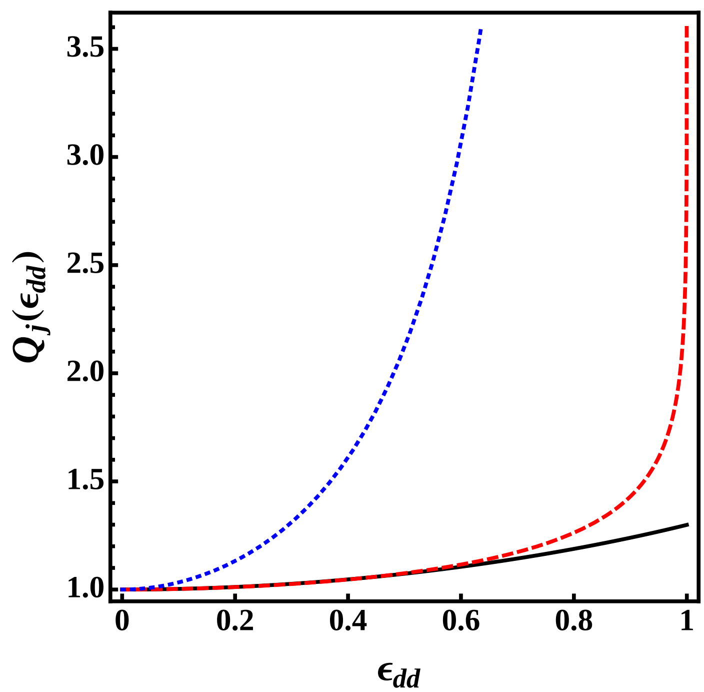


FIGURE 1.3: Functions $\mathcal{Q}_3$ (solide line), $\mathcal{Q}_{-1}$ (red dashed line) and $\mathcal{Q}_{-5}$ (blue dotted line), which govern the dependence of the condensate depletion, the anomalous fraction correction and superfluid fraction vs. the dipolar interaction parameter $\epsilon_{dd}$ [**56**].

Thermal fluctuations corrections to the chemical potential and the energy can be also obtained easily through expressions (1.53).

The Bogoliubov approach requires the conditions $\tilde{n}_T \ll n_c$ and $\tilde{m}_T \ll n_c$. Therefore, at temperatures $T \ll gn_c$, the small parameter of the theory turns out to be given as

$$\frac{T}{gn_c}\sqrt{n_c a^3}\mathcal{Q}_{-1}(\epsilon_{dd}) \ll 1. \tag{1.54}$$

The appearance of the extra factor $(T/gn_c)$ originates from the thermal fluctuations corrections.

### 1.4.3 Superfluid fraction

Depending on the boost direction, we have two different superfluid fractions in the directions parallel or perpendicular to the dipole polarization.

In the parallel direction, the superfluid fraction reads

$$\frac{n_s^{\|}}{n} = 1 - \frac{2\pi^2\hbar}{45mnc_s}\left(\frac{T}{\hbar c_s}\right)^4 \mathcal{Q}^{\|}_{-5}(\epsilon_{dd}), \tag{1.55}$$

where the function $\mathcal{Q}^{\|}_j(\epsilon_{dd}) = \frac{1}{3}(1-\epsilon_{dd})^{j/2}{}_2F_1\left(-\frac{j}{2}, \frac{5}{2}; \frac{3}{2}; \frac{3\epsilon_{dd}}{\epsilon_{dd}-1}\right)$, have the following properties : $\mathcal{Q}^{\|}_j(\epsilon_{dd}=0) = 1/3$ and imaginary for $\epsilon_{dd} > 1$ [**56**].

In the perpendicular direction, the superfluid fraction (4.23) takes the form

$$\frac{n_s^{\perp}}{n} = 1 - \frac{\pi^2\hbar}{45mnc_s}\left(\frac{T}{\hbar c_s}\right)^4 \mathcal{Q}^{\perp}_{-5}(\epsilon_{dd}), \tag{1.56}$$

where the function $\mathcal{Q}^{\perp}_j(\epsilon_{dd}) = \mathcal{Q}_j(\epsilon_{dd}) - \mathcal{Q}^{\|}_j(\epsilon_{dd})$.

The second terms in (1.55) and (1.56) represent the thermal contribution of $n_s^{\perp}$ and $n_s^{\|}$. These thermal terms are calculated at low temperatures $T \ll ng$. Whereas, at $T \gg ng$, there is copious evidence that both thermal terms of $n_s$ concide with the noncondensed density of an ideal Bose gas. Furthermore, we read off from Eqs.(1.55) and (1.56) that for $\epsilon_{dd} \leq 0.5$, the thermal contribution of $n_s^{\perp}$ is smaller than that of $n_s^{\|}$, whereas the situations is inverted for $\epsilon_{dd} > 0.5$ [**51**].

# CHAPITRE 2
# DIPOLAR BOSE GAS WITH WEAK DISORDER : BOGOLIUBOV-HUANG-MENG THEORY

In real systems, disorder is always present to some extent. It has been shown that even a small amount of disorder can dramatically alter the physics. Disordered BEC is nowadays a field of intense research both theoretically and experimentally. What happens to a homogeneous BEC if a weak random external potential is switched on ? Indeed, the presence of a disordered potential may lead to decrease both BEC and superfluidity. Furthermore, one of the intriguing features of disordered Bose gas is the appearance of the so-called Anderson localization [**168**] in the non-interacting case. This phenomenon which originally put forward for non-interacting electrons, can be understood as the effect of multiple reflections of a plane wave by random scatterers or random potential barriers, has recently attracted a great deal of interest [**74**, **64**].

Random potentials find areas of application even far from its physical origins. For example, the transport in random media and diffusion-controlled reactions can be modeled by random walks in random trapping environments [**64**]. The dynamics of stock markets have also been modeled as a tracer in a Gaussian random field [**169**]. Furthermore, the behavior of polymer chains in random media is strongly connected to this field [**170**]. Disorder appears either naturally as, e.g., in magnetic wire traps [**171**], where imperfections of the wire itself can induce local disorder, or it may be created artificially and controllably as, e.g., by the use of laser speckle fields [**172**]. The speckle effect is an interesting disorder problem. It is a result of the interference of many waves of the same frequency, and different phases and amplitudes, which add together to give a resultant wave whose amplitude and intensity is constant over time, but varies randomly in space [**173**], see Fig. 2.1.

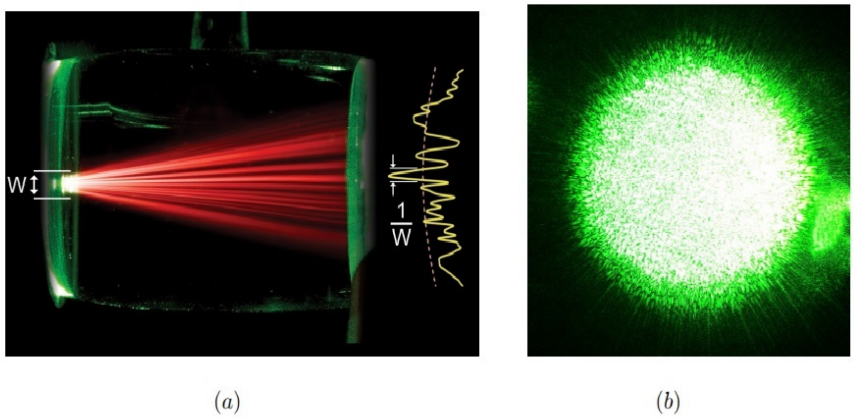


FIGURE 2.1: (a) Schematic representation of speckle formation and (b) typical speckle pattern [**176**]

Recent progress in different experimental realizations of laser speckle disorder is reported in Refs.[**174**, **175**]. The random external potential is characterized by a strength, which is the average height of its maxima and depth of its minima, as well as a correlation length, which represents the average width of its maxima and minima, see Fig.2.2.

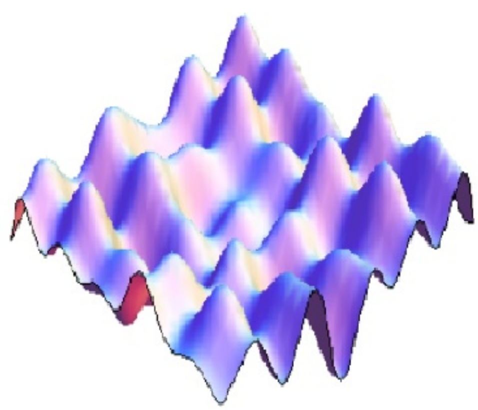

FIGURE 2.2: One realization of a disorder potential, which shows a random distribution of maxima and minima [**176**]

In this chapter, we first introduce fundamental statistical properties of the random potentials which we use in this thesis with ultracold atoms. We then study the impact of a weak disorder potential with Gaussian autocorrelation function on the properties of a homogeneous dipolar Bose gas in 2D and 3D geometries. Within the Bogoliubov-Huang-Meng theory we calculate in particular the corrections to the condensed depletion, the anomalous fraction, the ground state energy, the equation of state and the superfluid fraction due to the external random potential. In noninteracting systems, this model is important to fully understand the interplay of disorder and interactions. In 3D case, we show that the anisotropy of the DDI may enhance quantum, thermal and disorder fluctuations as well as the superfluid fraction.

## 2.1 Statistical properties of random potentials

One studies bosons moving in a one-particle potential $U(\mathbf{r})$ we consider a different physical situation, where the one-particle potential $U(\mathbf{r})$ is fluctuating at each space point $\mathbf{r}$ i.e. for each point $\mathbf{r}$, $U(\mathbf{r})$ is a random variable. A random potential is characterized by n-point probability distributions $P(\mathbf{r})$. To completely describe a random potential, it is however sufficient to know all the n-point correlation functions :

$$R_n(\mathbf{r_1}, \mathbf{r_2}, \cdots, \mathbf{r_n}) = \langle U(\mathbf{r_1}) U(\mathbf{r_2}) \cdots U(\mathbf{r_n}) \rangle \,, \tag{2.1}$$

where $\langle \cdots \rangle$ denotes statistical averaging (i.e. ensemble averaging over all realizations of the potential).

### 2.1.1 General properties

#### Homogeneity

A random potential is assumed to be spatially homogeneous, which means that its statistical properties are translation-invariant i.e the one-point probability distribution is independent of the position). As a consequence, one has

$$R_n(\mathbf{r_1}+\rho, \mathbf{r_2}+\rho, \cdots, \mathbf{r_n}+\rho) = \langle U(\mathbf{r_1})U(\mathbf{r_2})\cdots U(\mathbf{r_n})\rangle\,, \tag{2.2}$$

and depend only on $n-1$ relative coordinates.

#### Correlation functions

One of the key assumption of the random potential is the disappearance of statistical correlations between values of the potential at points with infinitely large separation.

In the following we assume for the disorder potential that it is homogeneous after the disorder ensemble average, i.e., after having performed the average $\langle\rangle$ over all possible realizations. Thus, the average value of the disorder potential, without loss of generality, will be assumed to vanish

$$\langle U(\mathbf{r})\rangle = 0. \tag{2.3}$$

Due to the homogeneity, the disorder ensemble average $\langle U(x)\rangle$ represents a constant, which can be absorbed into the chemical potential within a grand-canonical description. Furthermore, a homogeneous disorder potential has a correlation function, which depends on the difference of the space points :

$$R(\mathbf{r}, \mathbf{r}') = \langle U(\mathbf{r})U(\mathbf{r}')\rangle\,. \tag{2.4}$$

The assumption of homogeneity implies the symmetries :

$$R(\mathbf{r}) = R(-\mathbf{r})$$

and

$$R(\mathbf{k}) = R(-\mathbf{k}),$$

where the Fourier transform of the two-point correlation function, also referred to as the power spectrum of the disorder, is defined by $R(\mathbf{k}) = \int d^d r R(\mathbf{r}) e^{-i\mathbf{k}.\mathbf{r}}$.

In many cases, the disorder is further assumed to be isotropic, which means that $R(\mathbf{r}) = R(r)$ is a radial function (and so is its Fourier transform).

### 2.1.2 Standard forms

Disorder may take different shapes. Below, we present some examples of random potentials which are the most relevant.

#### White-noise disorder

A white-noise (or uncorrelated) disorder is a Gaussian disorder whose autocorrelation function is delta-correlated,

$$R(\mathbf{r} - \mathbf{r}') = R\delta(\mathbf{r}, \mathbf{r}'), \tag{2.5}$$

where $\delta$ is the $d$-dimensional Dirac function, and parametrizes the strength of the potential. Note that $R$ has dimension (energy)$^2$(length)$^d$, where $d$ is the spatial dimension. White-noise disorders are widely used in the case of the "weak" disorder regime which is of interest here, mostly because at low-energy, many continuous random potentials can be replaced by a white-noise potential.

### Speckle disorder

Speckle fields typically arise as a result of the reflection or transmission of a coherent wave on a rough surface (see Fig.(2.1)). Experimentally, an isotropic 3D speckle, can be produced as the interference pattern of many wavevectors inside a closed optical cavity [**177**]. Another realization of 3D disordered speckle configuration was proposed in Ref [**83**], where the speckle is formed in the focal point of an empty ellipsoidal optic cavity.

The autocorrelation function of the 3D isotropic laser speckle is given by $R(\mathbf{r}) = R|C_A(\mathbf{r})|^2$ [**83**], where

$$C_A(y) = \left|\left(3/y^3\right)(\sin y - y\cos y)\right|^2, \tag{2.6}$$

where $y = \pi r/\sigma$.

Another form of the autocorrelation function for the 3D isotropic speckle has been proposed in [**177**]

$$C_A(\mathbf{r}) = \mathrm{sinc}(r/\sigma). \tag{2.7}$$

Speckle potentials are easily tuned in amplitude, geometry and correlation length.

### Gaussian correlated disorder

The statistical properties of a Gaussian random potential are hence entirely determined by the two-point correlator. In $d$ spatial dimensions, it takes the form

$$R(\mathbf{r}, \mathbf{r}') = R\frac{e^{-(r-r')^2\sigma^2/2}}{(2\pi\sigma^2)^{d/2}}, \tag{2.8}$$

where its correlation length $\sigma$ can be identified with the average extension [**80**]. In the limit of a vanishing correlation length $\sigma$, we obtain a qualitative model for disordered bosons with a delta correlation of Eq.(2.5).

Throughout this thesis we use the Gaussian correlated disorder. This model of disorder is particularly convenient since it can be easily treated analytically within the Bogoliubov theory of a dilute Bose gas.

## 2.2 Interaction and disorder

The question of disorder in interacting systems is one of the most challenging problems. It is important to stress that in the presence of a random potential, there are different quantum states can be occured namely : the Lifshits glass, the Bose glass and the disordered BEC phase. In absence of interactions, bosons in disordered environments will condense into the lowest-energy state, which is a localized state. This phase is known as the Lifshits glass which is an insulating and non-compressible phase. A repulsive interaction among the particles causes them to delocalize and hence, leads to the disordered BEC phase [**60**]. The Bose glass is a highly inhomogeneous phase of matter. It characterizes by a finite compressibility, the absence of a gap in the single particle spectrum, and a nonvanishing density of states at zero energy. Note that the non-interacting Bose gas is extremely sensitive to external random potentials.

## 2.3 Bogoliubov-Huang-Meng theory

In order to study the properties of the so-called dirty boson problem, Huang and Meng in 1992 generalized the Bogoliubov theory of ultracold Bose gases in random potentials. Here, we use the Bogoliubov-Huang-Meng theory [**75**] to investigate disordered dipolar Bose gases at finite temperature in both 3D and 2D geometries. This theory allows us to go beyond the zero-temperature GP equation solved with perturbative approach [**81**, **84**]. This extension gives detailed insights into the interplay of thermal fluctuations and disorder effects in the anisotropy of superfluidity which not the case for GP equation with perturbative treatment.

We consider the effects of an external random field on a dilute 3D dipolar Bose gas with dipoles oriented perpendicularly to the plane. Let us write the Hamiltonian (1.11) of chapter.1 in the form

$$\hat{H}=\sum_{\mathbf{k}}\frac{\hbar^2k^2}{2m}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}+\frac{1}{V}\sum_{\mathbf{k},\mathbf{p}}U_{\mathbf{k-p}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{p}}+\frac{1}{2V}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}}V(\mathbf{p})\hat{a}^\dagger_{\mathbf{k+q}}\hat{a}^\dagger_{\mathbf{k-q}}\hat{a}_{\mathbf{k+p}}\hat{a}_{\mathbf{k-p}}, \tag{2.9}$$

where we have included the external potential $U$, and the interaction potential in momentum space is given by [**55**]

$$V(\mathbf{k})=g[1+\epsilon_{dd}(3\cos^2\theta_k-1)], \tag{2.10}$$

here $\epsilon_{dd}=C_{dd}/3g$ is the dimensionless relative strength which describes the interplay between the DDI and the short-range interaction.

Assuming the weakly interacting regime where $r_*\ll\xi$ with $\xi=\hbar/\sqrt{mgn}$ being the healing length and $n$ is the total density, we may use the Bogoliubov approach. Recalling that the Bogoliubov prescription assumes $\hat{a}^\dagger_0=\hat{a}_0=\sqrt{N_c}$, and $\hat{a}_{\mathbf{k}}$ for $\mathbf{k}\neq\mathbf{0}$ as small perturbations. To second order in $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$, the external potential term can be evaluated as

$$\sum_{\mathbf{k},\mathbf{p}}U_{\mathbf{k-p}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{p}}=N_cU_0+\sqrt{N_c}\sum_{\mathbf{k}}\left(\hat{a}^\dagger_{\mathbf{k}}U_{\mathbf{k}}+\hat{a}_{\mathbf{k}}U_{-\mathbf{k}}\right). \tag{2.11}$$

The term $N_cU_0$ must be calculated in the second Born approximation. In the absence of the external potential, the Hamiltonian (2.9) can be diagonalized by the standard Bogoliubov transformation (1.17).

In order to diagonalize the full Hamiltonian (2.9), we apply the Huang-Meng transformations [**75**] :

$$\hat{a}_{\mathbf{k}}=u_k\hat{b}_{\mathbf{k}}-v_k\hat{b}^\dagger_{-\mathbf{k}}-\beta_{\mathbf{k}},\qquad \hat{a}^\dagger_{\mathbf{k}}=u_k\hat{b}^\dagger_{\mathbf{k}}-v_k\hat{b}_{-\mathbf{k}}-\beta^*_{\mathbf{k}}, \tag{2.12}$$

where $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are operators of elementary excitations. The transformation (2.12) does not change the commutation rules and the quasiparticle operators $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ satisfy the usual bosonic commutation relations.

The Bogoliubov functions $u_k, v_k$ are expressed in a standard way : $u_k, v_k = (\sqrt{\varepsilon_k/E_k} \pm \sqrt{E_k/\varepsilon_k})/2$ where $E_k = \hbar^2 k^2/2m$ is the energy of a free particle, and

$$\beta_{\mathbf{k}} = \sqrt{\frac{n}{\mathcal{V}}} \frac{E_k}{\varepsilon_k^2} U_k. \tag{2.13}$$

The Bogoliubov excitations energy is given by

$$\varepsilon_k = \sqrt{E_k^2 + 2\mu_{0d}(\theta) E_k}, \tag{2.14}$$

where $\mu_{0d} = n \lim_{k\to 0} f(\mathbf{k})$ is the zeroth order chemical potential.

Importantaly, the spectrum (2.14) is independent of the random potential. This independence holds in fact only in zeroth order in perturbation theory ; conversely, higher order calculations render the spectrum dependent on the random potential due to the contribution of the anomalous terms (see below). For $k \to 0$, the excitations are sound waves $\varepsilon_k = \hbar c_{sd}(\theta) k$, where $c_{sd}(\theta) = c_s \sqrt{1 + \epsilon_{dd}(3\cos^2\theta - 1)}$ with $c_s = \sqrt{gn/m}$ is the sound velocity without DDI. Due to the anisotropy of the dipolar interaction, the sound velocity acquires a dependence on the propagation direction, which is fixed by the angle $\theta$ between the propagation direction and the dipolar orientation. This angular dependence of the sound velocity has been confirmed experimentally [**41**]. Therefore, the diagonal form of the Hamiltonian of the dirty dipolar Bose gas (2.9) can be written as

$$\hat{H} = E + \sum_{\vec{k}} \varepsilon_k \hat{b}^\dagger_{\mathbf{k}} \hat{b}_{\mathbf{k}}, \tag{2.15}$$

where $E = E_{0d} + \delta E + E_R$,

$E_{0d}(\theta) = \mu_{0d}(\theta)N/2$ with $N$ being the total number of particles.

$$\delta E = \frac{1}{2}\sum_{\mathbf{k}}[\varepsilon_k - E_k - nf(\mathbf{k})], \tag{2.16}$$

is the ground-state energy correction due to quantum fluctuations.

$$E_R = -\sum_{\mathbf{k}} n\langle|U_k|^2\rangle \frac{E_k}{\varepsilon_k^2} = -\sum_{\mathbf{k}} nR_k \frac{E_k}{\varepsilon_k^2}, \tag{2.17}$$

gives the correction to the ground-state energy due to the external random potential.

The condensate depletion can be obtained utilizing the definition (1.25)

$$\tilde{n} = \tilde{n}_0 + \tilde{n}_{th} + n_R, \tag{2.18a}$$

$$= \frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\left[\frac{E_k + \bar{V}(\mathbf{k})n_c}{\varepsilon_k} - 1\right] \tag{2.18b}$$

$$+ \frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{E_k + \bar{V}(\mathbf{k})n_c}{\varepsilon_k}\left[\coth\left(\frac{\varepsilon_k}{2T}\right) - 1\right] \tag{2.18c}$$

$$+ n_c\int \frac{d\mathbf{k}}{(2\pi)^3} R_k \frac{E_k^2}{\varepsilon_k^4}. \tag{2.18d}$$

The leading term (2.18b) denotes the zero temperature contribution to the noncondensed density. The subleading term (2.18c) stands for thermal fluctuation corrections to the noncondensed density. Whereas the third term (2.18d) represents the contribution of the random potential known also as ***glassy fraction*** is analog to the Edwards-Anderson order parameter of a spin glass [**84**, **89**, **159**]. It originates from the accumulation of density near the potential minima and density depletion around the maxima. The anomalous density reads

$$\tilde{m} = \tilde{m}_0 + \tilde{m}_{th} + n_R, \tag{2.19a}$$

$$-\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\bar{V}(\mathbf{k})n_c}{\varepsilon_k} \tag{2.19b}$$

$$-\frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^3}\frac{\bar{V}(\mathbf{k})n_c}{\varepsilon_k}\coth\left(\frac{\varepsilon_k}{2T}\right) \tag{2.19c}$$

$$+\, n_c \int \frac{d\mathbf{k}}{(2\pi)^3} R_k \frac{E_k^2}{\varepsilon_k^4}. \tag{2.19d}$$

The zero temperature term (2.19b) in the anomalous density is ultraviolet divergent. This divergency comes from the contact interactions. To overcome such a problem one should use the dimensional regularization which is valid for very dilute gases, and gives for the integral $\int_0^\infty dx(x/\sqrt{1+x^2}) = -1$ [**50**, **160**, **168**]. The second term (2.19c) accounts for thermal contributions to the anomalous density.

## 2.4 Superfluid fraction

In the presence of an external disorder potential, the normal density of the sueprfluid which only exists in the zero-temperature limit, reads as [**84**]

$$n_R^{ij} = n\frac{2\hbar^2}{m}\int \frac{d^dk}{(2\pi)^d}\frac{R_k k_i k_j}{E_k[E_k - 2nV(\mathbf{k})]^2}. \tag{2.20}$$

Equation (2.20) is valid for arbitrary disorder correlation function $R(\mathbf{k})$, and effective two-particle interaction $V(\mathbf{k})$. For a system possessing cylindrical symmetry, say around the $z$-axis, hence the integral of Eq.(4.22) over the azimuthal angle $\varphi$ gives

$$\sin\theta\int_0^{2\pi} d\varphi\, \mathbf{e_k}\mathbf{e_k^T} = \pi\begin{pmatrix} \sin\theta\sin^2\theta & 0 & 0 \\ 0 & \sin\theta\sin^2\theta & 0 \\ 0 & 0 & 2\sin\theta\cos^2\theta \end{pmatrix}, \tag{2.21}$$

where $\theta$ is the polar angle.

The superfluid fraction (1.41) evaluated in chapter 1 is then written

$$\frac{n_s^{ij}}{n} = \delta_{ij} - 4\int \frac{d^3k}{(2\pi)^3}\frac{\hbar^2}{2m}\frac{nR_k k_i k_j}{E_k[E_k-2nf(\mathbf{k})]^2} - \frac{2}{Tn}\int \frac{d^3k}{(2\pi)^3}\left[\frac{\hbar^2}{2m}\frac{k_i k_j}{4\sinh^2(\varepsilon_k/2T)}\right]. \tag{2.22}$$

The tensorial superfluid fraction separates into a parallel and a perpendicular part defined respectively, as

$$\frac{n_s^{\parallel}}{n} = 1 - 4\int \frac{dk\, d\theta}{(2\pi)^2}\frac{k^4 R(\mathbf{k})}{\left[E_k + 2\bar{V}(\mathbf{k})n\right]^2}\sin\theta\cos^2\theta, \tag{2.23}$$

and

$$\frac{n_s^{\perp}}{n} = 1 - 4\int \frac{dk\, d\theta}{8\pi^2}\frac{k^4 R(\mathbf{k})}{\left[E_k + 2\bar{V}(\mathbf{k})n\right]^2}\sin\theta\sin^2\theta. \tag{2.24}$$

In the absence of the dipolar interaction, the superfluid density and the disorder correlation become isotropic i.e. $V(\mathbf{k})$ and $R(\mathbf{k})$ are $\theta$ independent. signaling that the superfluid fraction in both directions reduce to the standard result of Huang and Meng $n_s/n = 1 - 4n_R/3n$ [**75**]. This indicates that the normal component of the superfluid is 4/3 times larger than the condensate fluctuations due to the disorder effects $n_R$.

## 2.5 Three-dimentional dipolar Bose gas in a random potential

In the context of the present chapter, we wish to understand the ground-state properties and superfluidity of disordered dipolar Bose gas in 3D case.

To proceed further in practical calculations, we must specify the type of random potential. For this purpose, we take the case of a spatially decaying disorder correlation $R(\mathbf{r})$. Therefore, in what follows, we restrict ourselves to the case of a Gaussian correlation with the Fourier transform [**80**, **81**]

$$R(\mathbf{k}) = Re^{-\sigma^2 k^2/2}, \tag{2.25}$$

where $R$ with dimension (energy) $^2$ (length)$^3$ and $\sigma$ characterize the strength and the correlation length of the disorder, respectively. Equation (2.25) makes the macroscopic wave function of BEC not sensitive to disorder in and between pores, but instead depends on the disorder averaged over the coherence length. Hence the ensemble-averaged system can become nearly uniform [**80**].

### 2.5.1 Fluctuations and thermodynamic quantities

Substituting (2.25) in Eq.(2.18d), we obtain for the condensate fluctuation due to the disordered potential

$$n_R = \frac{m^2 R}{8\pi^{3/2}\hbar^4}\sqrt{\frac{n}{a}}h(\epsilon_{dd},\alpha), \tag{2.26}$$

where

$$h(\epsilon_{dd},\alpha) = \int_0^{\pi} \frac{d\theta \sin\theta \mathcal{F}(\alpha)}{2\sqrt{1+\epsilon_{dd}(3\cos^2\theta_k - 1)}}, \tag{2.27}$$

is depicted in Fig.2.3.

The function $\mathcal{F}(\alpha) = e^{2\alpha}(4\alpha + 1)\left[1 - \text{erf}(\sqrt{2\alpha})\right] - 2\sqrt{2\alpha/\pi}$ with $\alpha = \sigma^2[1 + \epsilon_{dd}(3\cos^2\theta - 1)]/\xi^2$, has the following asymptotics for small $\alpha$ : $\mathcal{F}(\alpha) = 1 - 4\sqrt{2\alpha/\pi} + 6\alpha - (32/3)\alpha\sqrt{2\alpha/\pi} + 10\alpha^2 + O\left(\alpha^{5/2}\right)$. Equation (2.26) is in good agreement with that obtained using the mean field theory [**81**].

We observe from Fig.2.3 that for a small disorder correlation length $\sigma < \xi$, the contribution of the DDI on the disorder fluctuation is not important whereas in the case of $\sigma > \xi$, the DDIs tend to enhance the disorder fluctuation.

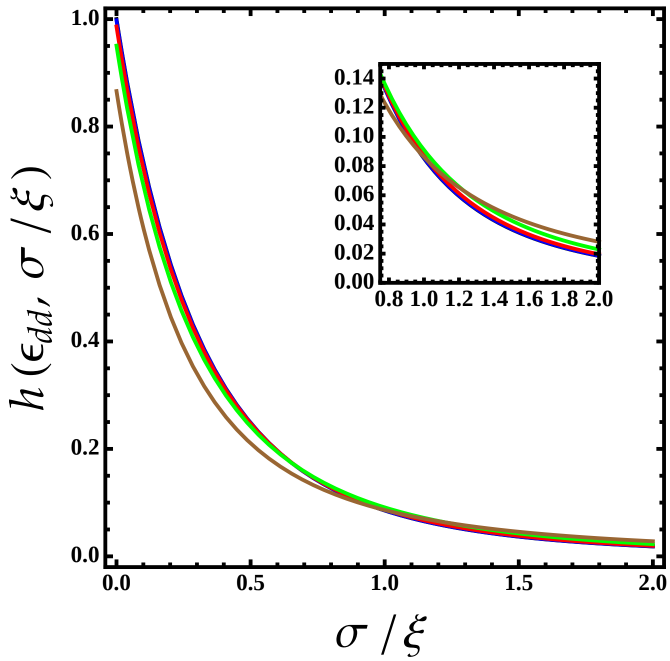


FIGURE 2.3: Behavior of the disorder function $h(\epsilon_{dd}, \sigma/\xi)$ from Eq.(2.27), as a function of $\sigma/\xi$. Black line : $\epsilon_{dd} = 0$ (pure contact interaction), blue line : $\epsilon_{dd} = 0.15$ (Cr atoms), red line : $\epsilon_{dd} = 0.38$ (Er atoms), green line : $\epsilon_{dd} = 0.7$ and brown line : $\epsilon_{dd} = 1$ [**77**].

On the other hand, for $\sigma/\xi \to 0$, we get from Eq.(2.27) that $h(\epsilon_{dd}, 0) = \mathcal{Q}_{-1}(\epsilon_{dd})$. Thus, the disorder fluctuation (2.26) becomes identical to that obtained in 3D dipolar BEC with delta-correlated disorder [**178**] $n_R = (m^2 R/8\pi^{3/2}\hbar^4)\sqrt{n/a}\mathcal{Q}_{-1}(\epsilon_{dd})$, where the contribution of the DDI is expressed by the functions $\mathcal{Q}_j(\epsilon_{dd}) = (1 - \epsilon_{dd})^{j/2} {}_2F_1\left(-\frac{j}{2}, \frac{1}{2}; \frac{3}{2}; \frac{3\epsilon_{dd}}{\epsilon_{dd}-1}\right)$, where ${}_2F_1$ is the hypergeometric function. Note that functions $\mathcal{Q}_j(\epsilon_{dd})$ attain their maximal values for $\epsilon_{dd} \approx 1$ and become imaginary for $\epsilon_{dd} > 1$ [**41**, **55**].

For $\sigma/\xi \to 0$ and $\epsilon_{dd} = 0$, we read off from Eq.(2.27) that one obtains $h(\epsilon_{dd}, \alpha) \to 1$. Therefore, we should reproduce the Huang and Meng result [**75**] for the disorder fluctuation in this limit.

Upon calculating the integral in Eq.(2.18), we get for the noncondensate depletion

$$\frac{\tilde{n}}{n} = \frac{8}{3}\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_3(\epsilon_{dd}) + \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{\pi T}{gn}\right)^2 \mathcal{Q}_{-1}(\epsilon_{dd}) + 2\pi R'\sqrt{\frac{na^3}{\pi}}h(\epsilon_{dd}, \alpha), \tag{2.28}$$

where $R' = R/g^2n$ is a dimensionless disorder strength. The condensed fraction can be calculated employing $n_c/n = 1 - \tilde{n}/n$.

The integral in Eq.(2.19) is ultraviolet divergent. A general way of treating such integrals is to introduce the Beliaev-type second order coupling constant (1.45). This gives for the anomalous fraction

$$\frac{\tilde{m}}{n} = 8\sqrt{\frac{na^3}{\pi}}\mathcal{Q}_3(\epsilon_{dd}) - \frac{2}{3}\sqrt{\frac{na^3}{\pi}}\left(\frac{\pi T}{gn}\right)^2 \mathcal{Q}_{-1}(\epsilon_{dd}) + 2\pi R'\sqrt{\frac{na^3}{\pi}}h(\epsilon_{dd}, \alpha). \tag{2.29}$$

The leading terms in Eqs.(2.28) and (2.29) represent the qunatum fluctuation[**55**]. The subleading terms which represent the thermal fluctuation[**55**], are calculated at temperatures $T \ll gn$, where the main contribution to integrals (1.27) and (1.28) comes from the region of small momenta ($\varepsilon_k = \hbar c_{sd} k$). The situation is quite different at higher temperatures i.e. $T \gg gn$, where the main contribution to integrals (1.27) and (1.28) comes from the single particle excitations. Hence, the thermal contribution of $\tilde{n}$ becomes identical to the density of noncondensed atoms in an ideal Bose gas [**55**], while the thermal contribution of $\tilde{m}$ tends to zero since the gas is completely thermalized in this range of temperature [**55**, **159**, **167**]. The last terms in (2.28) and (2.29) describe the effect of disorder on the noncondensed and on the anomalous densities.

Equation (2.29) clearly shows that at zero temperature, the anomalous density is three times larger than the noncondensed density for any range of the dipolar interaction as well as for any value of the strength and the correlation length of the disorder as it has been anticipated above. Moreover, $\tilde{m}$ changes its sign with increasing temperature in agreement with uniform Bose gas with a pure contact interaction [**55**]. For $\epsilon_{dd} = 0$, $\mathcal{Q}_j(\epsilon_{dd}) = 1$ and thus, Eqs.(2.28) and (2.29) reproduce the short-range interaction results. Furthermore, the DDI enhances the condensate depletion and the anomalous fraction for increasing $\epsilon_{dd}$.

The energy shift due to the interaction and the quantum fluctuations (2.16) are ultraviolet divergent. The difficulty is overcome if one takes into account the second-order correction to the coupling constant (1.45). A straightforward calculation yields[**42**, **55**]

$$\delta E = \frac{64}{15} \vee gn^2 \sqrt{\frac{na^3}{\pi}} \mathcal{Q}_5(\epsilon_{dd}). \tag{2.30}$$

However, the energy shift (2.17) due to the external random potential (2.25) is not divergent and it can be evaluated as

$$\frac{E_R}{E_0} = 16\pi R' \sqrt{\frac{na^3}{\pi}} h_1(\epsilon_{dd}, \alpha), \tag{2.31}$$

where $E_0 = Ngn/2$, and

$$h_1(\epsilon_{dd}, \alpha) = \frac{1}{2} \int_0^{\pi} d\theta \sin \sqrt{1 + \epsilon_{dd}(3\cos^2\theta_k - 1)} \mathcal{F}_1(\alpha), \tag{2.32}$$

is displayed in Fig.2.4.

The function $\mathcal{F}_1(\alpha) = e^{2\alpha}[1 - \text{erf}(\sqrt{2\alpha})] - \sqrt{1/2\pi\alpha}$ has the asymptotics behavior for small $\alpha$ : $\mathcal{F}_1(\alpha) = 1 - (4+\pi)\sqrt{\alpha/2\pi} + 2\alpha - (8/3)\sqrt{2/\pi}\alpha^{3/2} + 2\alpha^2 + O\left(\alpha^{5/2}\right)$.

As is seen from Fig.2.4 that for $\sigma < 2\xi$, the energy correction due to the disorder effect (2.31) is negative which leads to lower the total energy of the system. Note that this result still valid for any value of $\epsilon_{dd} < 1$. Another important remark is that the energy decreases with increasing $\epsilon_{dd}$.

For a condensate with a pure contact interaction ($\mathcal{Q}_5(\epsilon_{dd} = 0) = 1$) and in the absence of disordered potential ($R = 0$), the obtained energy excellently agrees with the seminal Lee-Huang-Yang quantum corrected ground state energy [**167**].

For, $\sigma/\xi \to 0$, the energy shift due to the external random potential (2.25) becomes ultraviolet divergent. Again, by introducing the renormalized coupling constant (1.45) one gets : $E_R/E_0 = 16\pi R' \sqrt{na^3/\pi} \mathcal{Q}_1(\epsilon_{dd})$ which well coincides with the result obtained with delta-correlated disorder of Ref [**178**].

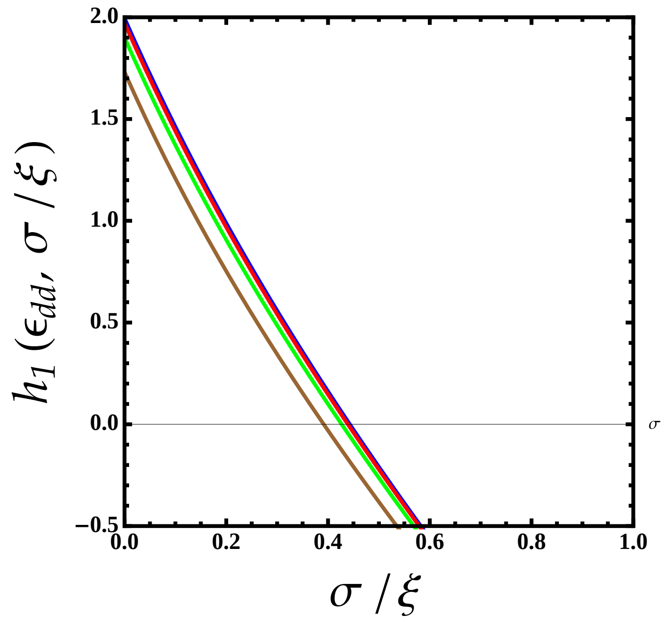


FIGURE 2.4: Behavior of the disorder energy function $h_1(\epsilon_{dd}, \sigma/\xi)$, from Eq. (2.32) as a function of $\sigma/\xi$ for same values of $\epsilon_{dd}$ as in Fig.2.3 [**77**].

### 2.5.2 Validity condition of the Bogoliubov theory

The Bogoliubov approach assumes that fluctuations should be small. We thus conclude from Eqs. (2.28) and (2.29) that at $T = 0$, the validity of the Bogoliubov theory requires the following inequalities

$$\sqrt{na^3}\mathcal{Q}_3(\epsilon_{dd}) \ll 1, \qquad R'\sqrt{na^3}h(\epsilon_{dd}, \alpha) \ll 1. \tag{2.33}$$

For $R' = 0$, this parameter differs only by the factor $\mathcal{Q}_3(\epsilon_{dd})$ from the universal small parameter of the theory, $\sqrt{na^3} \ll 1$, in the absence of DDI. At $T \ll gn$, the Bogoliubov theory requires the condition $(T/gn)\sqrt{na^3}\mathcal{Q}_{-1}(\epsilon_{dd}) \ll 1$. The appearance of the extra factor $(T/gn)$ originates from the thermal fluctuations corrections.

### 2.5.3 Superfluid fraction

Equation. (2.22) yields a superfluid density that depends on the direction of the superfluid motion with respect to the orientation of the dipoles. In the parallel direction and at low temperatures where $\varepsilon_k = \hbar c_{sd} k$, the superfluid fraction can be obtained from Eq.(2.23).

$$\frac{n_s^{\parallel}}{n} = 1 - 4\pi R' \sqrt{\frac{na^3}{\pi}} h^{\parallel}(\epsilon_{dd}, \alpha) - \frac{2\pi^2 T^4}{45 m n \hbar^3 c_s^5} \mathcal{Q}^{\parallel}_{-5}(\epsilon_{dd}), \tag{2.34}$$

where the function

$$h^{\parallel}(\epsilon_{dd}, \alpha) = \int_0^{\pi} d\theta \frac{\sin\theta \cos^2\theta \mathcal{F}(\alpha)}{2\sqrt{1 + \epsilon_{dd}(3\cos^2\theta - 1)}}, \tag{2.35}$$

is decreasing with increasing $\epsilon_{dd}$ and vanishing for large $\sigma/\xi$ as is depicted in Fig.2.5.a. And the functions $\mathcal{Q}_j^{\parallel}(\epsilon_{dd}) = \frac{1}{3}(1-\epsilon_{dd})^{j/2} {}_2F_1\left(-\frac{j}{2}, \frac{5}{2}; \frac{3}{2}; \frac{3\epsilon_{dd}}{\epsilon_{dd}-1}\right)$, have the properties $\mathcal{Q}_j^{\parallel}(\epsilon_{dd} = 0) = 1/3$ and become imaginary for $\epsilon_{dd} > 1$ (see Fig.2.6). Therefore, Eq.(2.34) reveals that DDI effects are more significant for condensate fracion (2.28) than for the parallel superfluid fraction.

Again at low temperatures, the perpendicular direction of the superfluid fraction (2.24) takes the form

$$\frac{n_s^{\perp}}{n} = 1 - 2\pi R' \sqrt{\frac{na^3}{\pi}} h^{\perp}(\epsilon_{dd}, \alpha) - \frac{\pi^2 T^4}{45 m n \hbar^3 c_s^5} \mathcal{Q}^{\perp}_{-5}(\epsilon_{dd}), \tag{2.36}$$

where the functions

$$h^{\perp}(\epsilon_{dd}, \alpha) = \int_0^{\pi} d\theta \frac{\sin\theta (1-\cos^2\theta) \mathcal{F}(\alpha)}{2\sqrt{1 + \epsilon_{dd}(3\cos^2\theta - 1)}} = h(\epsilon_{dd}, \alpha) - h^{\parallel}(\epsilon_{dd}, \alpha),$$

and $\mathcal{Q}_j^{\perp}(\epsilon_{dd}) = \mathcal{Q}_j(\epsilon_{dd}) - \mathcal{Q}_j^{\parallel}(\epsilon_{dd})$, are displayed in Figs.(2.5.b), and (2.6), respectively. Expressions (2.34) and (2.36) constitute a natural extension of those obtained in [**81**]

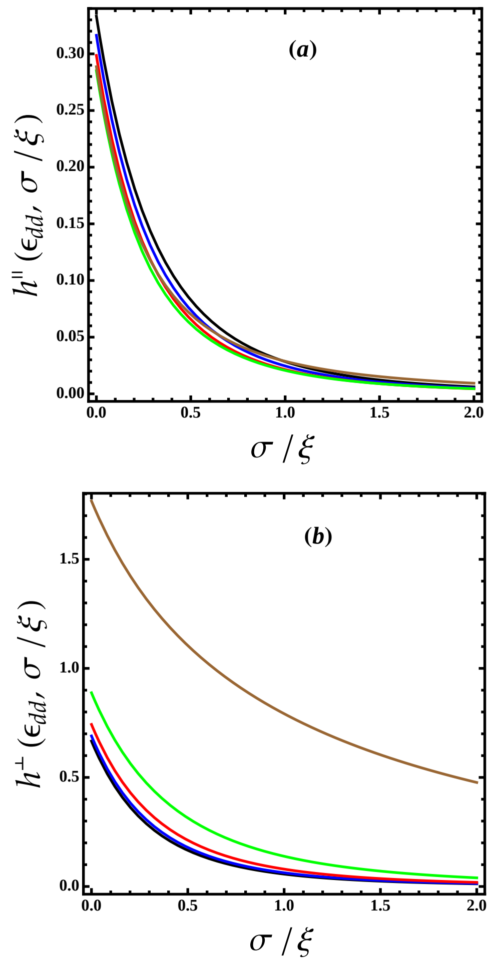


FIGURE 2.5: Behavior of the disorder functions $h^{\parallel}(\epsilon_{dd}, \sigma/\xi)$ (a) and $h^{\perp}(\epsilon_{dd}, \sigma/\xi)$ (b) as a function of $\sigma/\xi$ for same values of $\epsilon_{dd}$ as in Fig.2.3 [**77**].

since they contain the temperature correction (third terms). At $T \gg ng$, it is evident that both thermal terms of $n_s$ coincide with the noncondensed density of an ideal Bose gas. Figure (2.6) shows that the thermal contribution of $n_s^{\perp}$ is smaller than that of $n_s^{\parallel}$ for $\epsilon_{dd} \leq 0.5$, while the situation is inverted for $\epsilon_{dd} > 0.5$.

For $\sigma/\xi \to 0$ and $\epsilon_{dd} = 0$, both components of the superfluid fraction (2.34) and (2.36) reduce to $n_s/n = 1 - 4n_R/3n$, which well recove earlier results of Refs [**75**, **76**, **79**] for isotropic contact interaction. For $\sigma/\xi \to 0$, we have $h^{\parallel}(\epsilon_{dd}, 0) = \mathcal{Q}_{-1}^{\parallel}(\epsilon_{dd})$ and $h^{\perp}(\epsilon_{dd}, 0) = \mathcal{Q}_{-1}^{\perp}(\epsilon_{dd})$. Consequently, the disorder correction to superfluid fraction

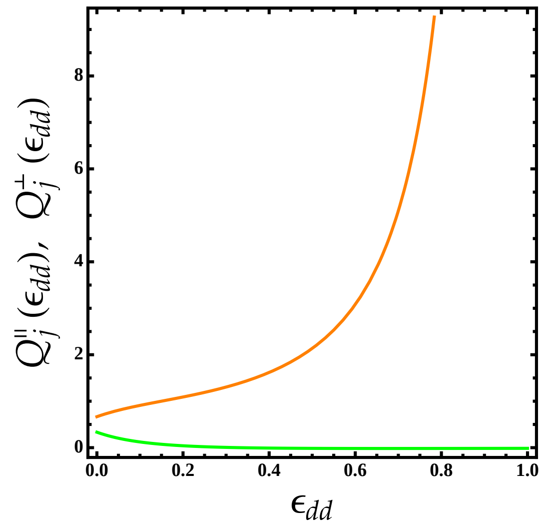


FIGURE 2.6: Behavior of thermal functions $\mathcal{Q}_j^{\|}$ (Green line) and $\mathcal{Q}_j^{\perp}$ (Orange line), as a function of $\epsilon_{dd}$ [**77**].

(2.26) becomes identical to that obtained in 3D dipolar BEC with delta-correlated disorder [**178**]. We should stress also that for increasing $\epsilon_{dd}$, $h^{\|}(\epsilon_{dd},\alpha)$ decreases, whereas $h(\epsilon_{dd},\alpha)$ increases for fixed $\sigma/\xi$. Therefore, this reveals that there exists a critical value $\epsilon_{dd}^c$ beyond which the system has the surprising property that the disorder-induced depletion of the parallel superfluid density is smaller than the condensate depletion even at $T=0$. This can be attributed to the fact that the localized particles cannot contribute to superfluidity and, hence, form obstacles for the superfluid flow. For a large disorder correlation length i.e. $\sigma \gg \xi$, $\epsilon_{dd}^c$ decreases indicating that the locally condensed particles are localized in the respective minima for the disorder potential for a finite localization time [**86**].

The superfluid fraction can be either larger or smaller than the condensate fraction $n_c/n$, depending on temperature, the strength of interactions, and on the strength of disorder. Increasing $R'$ leads to the simultaneous disappearance of the superfluid and condensate fractions. So for any value of $na^3$ and of $\epsilon_{dd}$ there exists a critical strength of disorder

$$R_c^{\|} = \frac{4}{\pi}\frac{\mathcal{Q}_3(\epsilon_{dd})}{h^{\|}(\epsilon_{dd},\alpha)}, \quad R_c^{\perp} = \frac{2}{\pi}\frac{\mathcal{Q}_3(\epsilon_{dd})}{h^{\perp}(\epsilon_{dd},\alpha)}, \tag{2.37}$$

for which $n_s^{\|}/n < n_c/n$ and $n_s^{\perp}/n < n_c/n$.

When $\sigma/\xi \to 0$, $R_c^{\|} = (4/\pi)[\mathcal{Q}_3(\epsilon_{dd})/\mathcal{Q}_{-1}^{\|}(\epsilon_{dd})]$ and $R_c^{\perp} = (2/\pi)[\mathcal{Q}_3(\epsilon_{dd})/\mathcal{Q}_{-1}^{\perp}(\epsilon_{dd})]$ . In the case of Er atoms ($\epsilon_{dd} = 0.38$), $R_c^{\|} \approx 6.74$ and $R_c^{\perp} \approx 0.78$. For Cr atoms ($\epsilon_{dd} = 0.15$), $R_c^{\|} \approx 4.96$ and $R_c^{\perp} \approx 0.86$. This clearly shows that $R_c^{\|}$ is decreasing with $\epsilon_{dd}$, while $R_c^{\perp}$ is increasing with $\epsilon_{dd}$.

Therefore, the Bogoliubov approach should satisfy the condition :$R' < R_c$. However, it is not clear whether these results are still valid for $R' > R_c$ in a range of densities where the difference between $n_s/n$ and $n_c/n$ can be significant and hence, the system yields a transition to a new quantum regime. The response to these questions requires either a non-perturbative scheme or numerical Quantum Monte Carlo simulation, which are out of the scope of the present work.

# CHAPITRE 3

# DIPOLAR BOSE GAS WITH THREE-BODY INTERACTIONS IN WEAK DISORDER

The interplay of disorder and interactions may trigger transitions to new quantum phases which cannot exist in clean systems. Because of to the complexity of the problem, there are still many open questions. In the following sections, we examine the role of both statistical correlations of the disorder potential and interactions on the ground state properties and superfluidity of dipolar ultracold Bose gases in the presence of the three-body interactions (TBI) These latter play a key role in a wide variety of interesting physical phenomena, and provide a new physics not existed in systems with two-body interactions. Inelastic three-body processes, including observations of Efimov quantum states and atom loss from recombination have been reported in Refs [**96-100**]. Weakly interacting Bose and Fermi gases with competing attractive two-body and large repulsive TBI may form droplets [**101**]. Effects of TBI in ultracold bosonic atoms loaded in an optical lattice or a superlattice were also studied in [**10-105**]. It was shown also that the TBI in Bose condensate may singnificantly modify the collective excitations [**106-108**], the transition temperature, the condensate depletion and the stability of a BEC [**109**, **110**]. In the context of ultracold atoms with DDI, it has been revealed that the combined effects of TBI and DDI may lead to the formation of a stable supersolid state [**111**] and a quantum droplet state [**16-29** ], [**19-21**]. Very recently, we have shown that the TBI may shift the density profiles, the condensed fraction and the collective modes of a dipolar condensate at finite temperature [**98**].

Disordered dipolar Bose gases with TBI present a different physical picture and may open prospects to achieve a stable superfluid. In the present chapter we study, for the first time to the best of our knowledge, effects of a weak disorder potential with Gaussian correlation function (2.25) on the properties of BEC with two-body interactions and TBI. To this end, we use the Bogoliubov-Huang-Meng theory which includes an additional TBI term in the momentum space. Our results show that the TBI are relevant in reducing the influence of the disorder potential in BEC. We deeply discuss impacts of the disorder potential and the TBI on the fluctuations, coherence and the thermodynamics of the condensate.

## 3.1 Hatree Fock Bogoliubov with TBI

We consider the effects of an external random potential $U(\mathbf{r})$ on a dilute 3D dipolar Bose gas with contact two- and three-body interactions. Assuming that dipoles are oriented along $z$-axis. The Hamiltonian of the system reads :

$$\hat{H} = \int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r}) \left[-\frac{\hbar^2}{2m}\Delta + U(\mathbf{r})\right] \hat{\psi}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r} \int d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')\hat{\psi}(\mathbf{r})$$
$$+ \frac{g_3}{6}\int d\mathbf{r}\, \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}), \tag{3.1}$$

where $\hat{\psi}^\dagger$ and $\hat{\psi}$ denote, respectiveiy the usual creation and annihilation field operators, $m$ is the particle mass. The two-body interactions is described by the potential $V(\mathbf{r}-\mathbf{r}') = g_2\delta(\mathbf{r}-\mathbf{r}')+V_d(\mathbf{r}-\mathbf{r}')$, where $g_2 = 4\pi\hbar^2 a/m$ with $a$ being the $s$-wave scattering length is assumed to be positive. The DDI term $V_d(\mathbf{r}) = \mu_0\mu_m^2(1-3\cos^2\theta)/4\pi r^3$. The three-body coupling constant $g_3$ is in general a complex number with $Im(g_3)$ describing the three-body recombination loss and $Re(g_3)$ quantifying the three-body scattering parameter. Here, we will assume that the imaginary part of $g_3$ is negligible [**21**, **98**, **107**, **179**, **180**] which means that the loss rate is sufficiently small and hence, the system is stable. This well coincides with the experimental conditions reported in Ref.[**19**]. Note that the strength of the three-body coupling $g_3$ is related to the atomic species and can be adjusted by Feshbach resonance [**20**, **181**]. It is therefore, hard to predict the exact value of $g_3$ (see e.g. [**101**, **179**, **182**]). The disorder potential must satisfy the conditions (2.3) and (2.4).

In the frame of the Boguliubov formalism, the Bose-field operator can be written as

$$\hat{\psi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \hat{\bar{\psi}}(\mathbf{r},t), \tag{3.2}$$

where $\Phi$ is the condensate wavefunction, and $\hat{\bar{\psi}}$ stands for the field of the noncondensed thermal atoms. Working in Fourier space, the condensate wavefunction is taken as $\Phi(\mathbf{r},t) = \sqrt{n_c}$ with $n_c$ being the condensate density, and the field operator of noncondensed atoms can be expanded in terms of plane waves $\hat{\bar{\psi}} = (1/\sqrt{V}) \sum_k a_k e^{i\mathbf{k}.\mathbf{r}}$. The DDI potential in momentum space is given by : $V_d(\mathbf{k}) = (\mu_0\mu_m^2/12\pi)(3\cos^2\theta_{\mathbf{k}} - 1)$, where the vector $\mathbf{k}$ represents the momentum transfer imparted by the collision.

Now we deal with a weakly interacting system and assume that the disorder is sufficiently weak. Then it is possible to use the Bogoliubov-Huang-Meng approach [**75**] which suggests the transformations (2.12). This gives the Bogoliubov excitations energy

$$\varepsilon_k = \sqrt{E_k^2 + 2n_c E_k \bar{V}(\mathbf{k})}, \tag{3.3}$$

where

$$\bar{V}(\mathbf{k}) = g_2(1 + g_3 n_c/g_2)[1 + \gamma(3\cos^2\theta_{\mathbf{k}} - 1)]$$

with $\gamma = \epsilon_{dd}/(1 + g_3 n_c/g_2)$.

For $k \rightarrow 0$, the excitations are sound waves $\varepsilon_k = \hbar c_s(\theta_{\mathbf{k}})k$, where $c_s(\theta_{\mathbf{k}}) = c_0\sqrt{(1 + g_3 n_c/g_2)[1 + \gamma(3\cos^2\theta_{\mathbf{k}} - 1)]}$ with $c_0 = \sqrt{g_2 n_c/m}$ being the sound velocity without DDI and TBI.

The diagonal form of the Hamiltonian (3.1) can be written as $\hat{H} = E + \sum\limits_k \varepsilon_k \hat{b}_k^\dagger \hat{b}_k$.

The total energy $E = E_0(\theta) + \delta E + E_R$, where the zeroth order term

$$E_0(\theta) = \bar{V}(\theta) n_c N_c/2, \tag{3.4}$$

which should be computed in the limit $k \to 0$ since it accounts for the condensate (lowest state). The ground-state energy shift due to quantum fluctuations is

$$\delta E = \frac{1}{2} \sum_k [\varepsilon_k - E_k - n_c \bar{V}(\mathbf{k})], \tag{3.5}$$

and $E_R = -\sum_k n_c R_k \frac{E_k}{\varepsilon_k^2}$ gives the correction to the ground-state energy due to the external random potential.

## 3.2 Condensate fluctuations

After having inserting the expression (2.25) of Gaussian disorder into Eq.(2.18d), one obtains for the glassy fraction [**125**]

$$n_R = n_{\rm HM} (1 + g_3 n_c/g_2)^{-1/2} h(\gamma, \sigma/\xi), \tag{3.6}$$

where $n_{\rm HM} = [m^2 R_0 / 8\pi^{3/2} \hbar^4] \sqrt{n_c/a}$ is the usual Huang-Meng result [**75**]. The anisotropic disorder function is given as

$$h(\gamma, \sigma/\xi) = \int_0^\pi d\theta \frac{\sin\theta S(\alpha)}{2\sqrt{1 + \gamma(3\cos^2\theta - 1)}}, \tag{3.7}$$

where the function $S(\alpha) = e^{2\alpha}(4\alpha + 1) \left[1 - \text{erf}(\sqrt{2\alpha})\right] - 2\sqrt{2\alpha/\pi}$, and $\alpha = \sigma^2 [\epsilon_{dd}/\gamma(1 + \gamma(3\cos^2\theta - 1))]/\xi^2$ with $\xi = \hbar/\sqrt{m n_c g_2}$ being the healing length. In the absence of the DDI ($\epsilon_{dd} = 0$), and in the limit $\sigma/\xi \to 0$ and $g_3 = 0$, one has $h(\gamma, \alpha) \to 1$, thus, one recovers the well-known Hang and Meng result ($n_R = n_{\rm HM}$)

[75].

The effects of both correlation length and effective interaction parameter $\gamma$ on the behavior of the disorder function are presented in Fig.3.1. We observe that the function $h(\gamma, \alpha)$ is decreasing with $g_3 n_c/g_2$ indicating that the TBI lead to reduce the disorder fluctuations (glassy fraction) inside the condensate even in the limit $\sigma < \xi$. As is expected, the disorder fraction becomes significant for large DDI in contrast to the case of a disordered dipolar BEC with Lee-Huang-Yang (LHY) quantum corrections [50]. The main difference between the TBI and the LHY corrections is that these latter are valid only in the regime of weak disorder, since they are computed within the local density approximation which assumes that the external random potential should vary smoothly in space on a length scale comparable to the healing length or the characteristic correlation length of the disorder [18]. Whereas, the TBI still remain applicable for both weak and strong disorder potentials. For $\sigma > \xi$, the disorder effects is not important (see Fig.3.1.b).

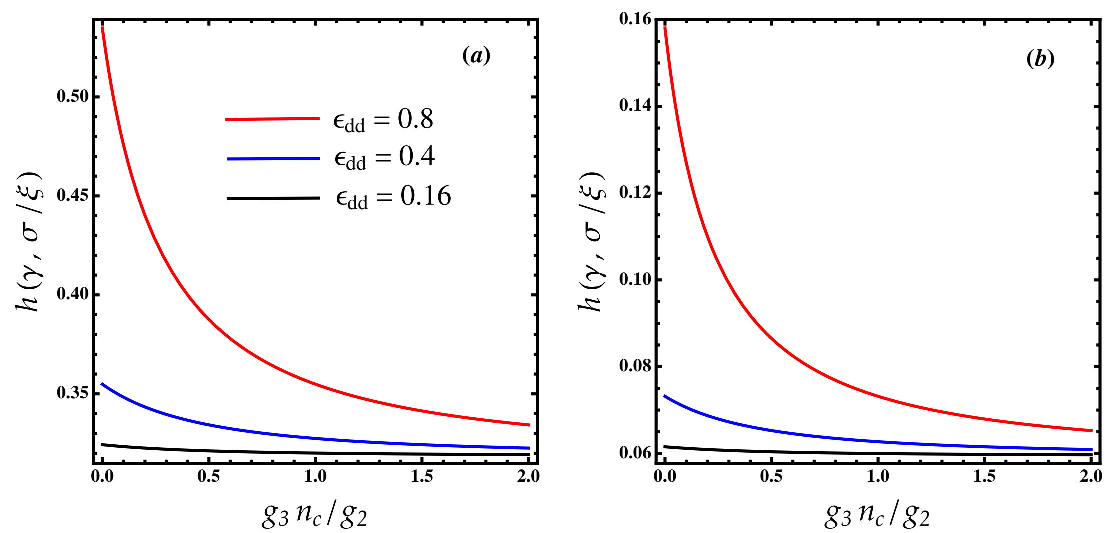


FIGURE 3.1: Disorder function $h(\gamma, \sigma/\xi)$, as a function of $g_3 n_c/g_2$ for several values of $\epsilon_{dd}$ for $\sigma/\xi = 0.4$ (a) and $\sigma/\xi = 1.2$ (b) [125].

For delta-correlated disorder where $\sigma/\xi \to 0$, the function $h(\gamma, 0) = \mathcal{Q}_{-1}(\gamma)$ and $n_R = n_{HM}\mathcal{Q}_{-1}(\gamma)$, where the contribution of the DDI is expressed by the functions

$\mathcal{Q}_j(x) = \int_0^1 dy(1 - x + 3xy^2)^{j/2}$ [**18**, **20**],[**70-73**]. Note that the functions $\mathcal{Q}_j(x)$ tend to unity for $\gamma = 0$ ($\mathcal{Q}_j(0) = 1$), and become imaginary for $\gamma > 0$.

Now, we focus ourselves to calculate quantum and thermal depletion in a disordered BEC. Integrals (2.18b) and (2.18c) yield, respectively

$$\frac{\tilde{n}_0}{n_c} = \frac{8}{3}\sqrt{\frac{n_c a^3}{\pi}}(1 + g_3 n_c/g_2)^{3/2}\mathcal{Q}_3(\gamma), \tag{3.8}$$

and

$$\frac{\tilde{n}_{th}}{n_c} = \frac{2}{3}\left(\frac{\pi T}{n_c g_2}\right)^2 \sqrt{\frac{n_c a^3}{\pi}}(1 + g_3 n_c/g_2)^{-1/2}\mathcal{Q}_{-1}(\gamma). \tag{3.9}$$

The anomalous density can be obtained via (2.19) with the help of the renormalization developped in [**59**]. After some algebra, we get

$$\frac{\tilde{m}_0}{n_c} = 8\sqrt{\frac{n_c a^3}{\pi}}\left(\frac{\epsilon_{dd}}{\gamma}\right)^{3/2}\mathcal{Q}_3(\gamma) \tag{3.10}$$

and

$$\frac{\tilde{m}_{th}}{n_c} = -\frac{2}{3}\left(\frac{\pi T}{n_c g_2}\right)^2 \sqrt{\frac{n_c a^3}{\pi}}\sqrt{\frac{\gamma}{\epsilon_{dd}}}\mathcal{Q}_{-1}(\gamma). \tag{3.11}$$

For $\epsilon_{dd} = 0$ and $g_3 = 0$, we recover the standard expressions for $\tilde{n}_0$, $\tilde{n}_{th}$, $\tilde{m}_0$ and $\tilde{m}_{th}$. When $g_3 = 0$, the expressions (3.8)-(3.11) reduce to that obtained in a dipolar BEC without TBI [**57**].

## 3.3 One-body density matrix

The one-body density matrix (first-order correlation function) is defined as $g^{(1)}(\mathbf{r}, \mathbf{r}', t, t') = \langle\hat{\psi}^\dagger(\mathbf{r}, t)\hat{\psi}(\mathbf{r}', t')\rangle$. In uniform case it depends only on the difference $|\mathbf{r} - \mathbf{r}'| = r$. Using the decomposition (3.2), expressing the noncondensed field operator as $\hat{\bar{\psi}} = (1/V)\sum_k [u_k \hat{b}_k e^{i\mathbf{k}.\mathbf{r}} - v_k \hat{b}_k^{\dagger} e^{-i\mathbf{k}.\mathbf{r}}]$, and then taking into account that $|\Phi(\mathbf{r},t)| = \sqrt{n_c}$. We thus, get

$$g^{(1)}(\mathbf{r}) = n_c + g_R^{(1)}(\mathbf{r}) + \int_0^{\infty} \frac{d\mathbf{k}}{(2\pi)^d} \left[v_k^2 + (u_k^2 + v_k^2) N_k\right] e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{3.12}$$

The second term $g_R^{(1)}(\mathbf{r}) = \int (d\mathbf{k}/(2\pi)^3) \langle |\beta_{\mathbf{k}}|^2 \rangle \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}}$ represents the disorder effects on the first order correlation function. The behavior of $g_R^{(1)}(\mathbf{r})$ is displayed in Fig.3.2.

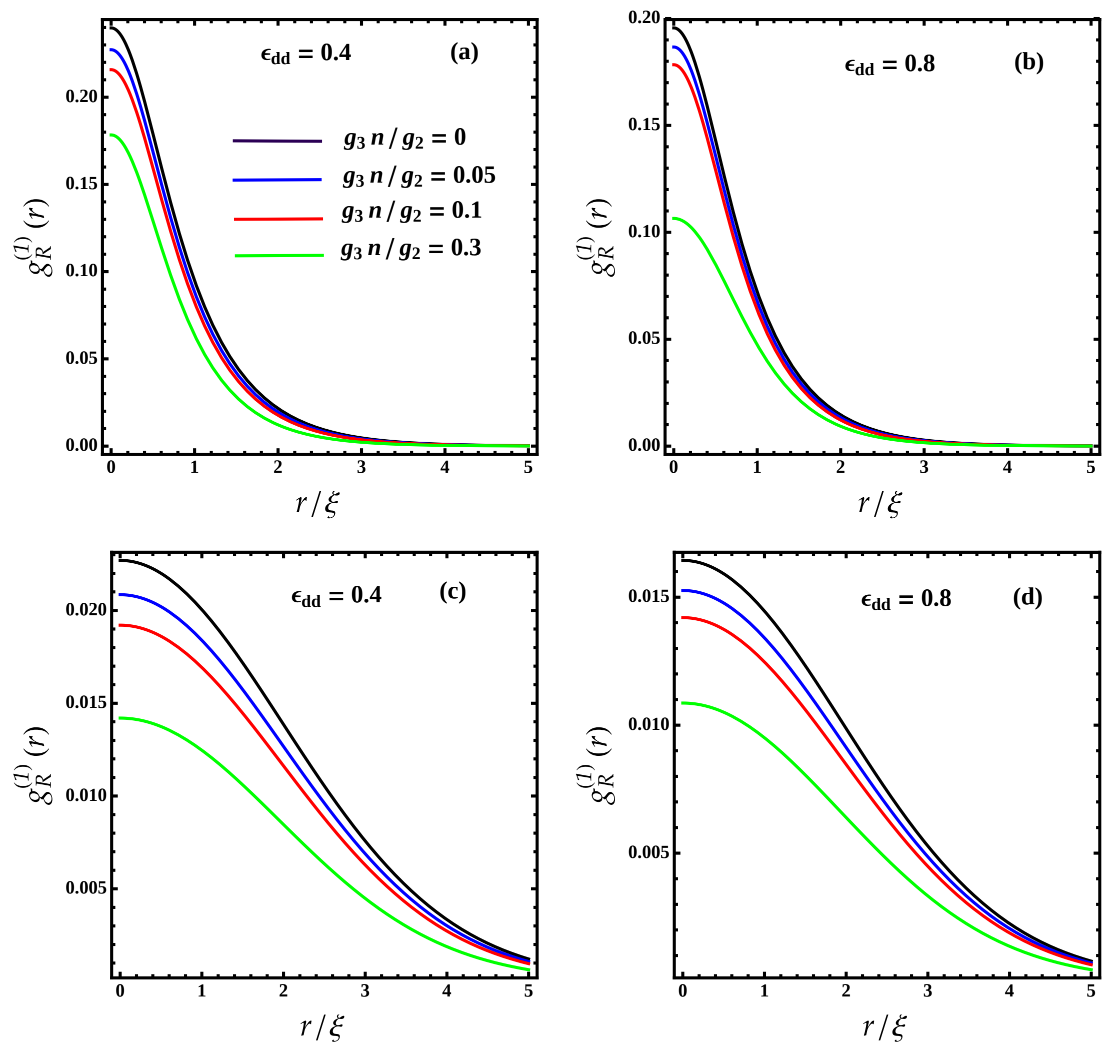


FIGURE 3.2: One-body density matrix due to the disorder corrections, $\mathrm{g}_R^{(1)}(r)$, for $\sigma/\xi = 0.2$ (a)-(b) and $\sigma/\xi = 1.2$ (c)-(d).[**125**].

We observe that for small disorder correlation length ($\sigma/\xi = 0.2$), $g_R^{(1)}(r)$ is decreasing with increasing the TBI and the DDI (see Fig.3.2. a-b ). The same behavior holds for large $\sigma$. Importantly, $g_R^{(1)}(r)$ vanishes at large distance $r$ in both cases signaling the non-existence of mini condensates formed by the localized particles in the respective minima of the external random potential. This does not mean that the long-range order of the whole system is destroyed.

The last term in Eq.(3.12) accounts for the quantum and thermal contributions to the one-body correlation function. One can easily show that this term decays at $r \to \infty$ and thus, $g^{(1)}(\mathbf{r})$ tends to $n_c$, revealing the existence of the long-range order (true condensate). Note that the DDI, the TBI and the temperature can also shift the one-body correlation function.

## 3.4 Thermodynamic quantities

In this section, we calculate disorder corrections to some thermodynamic quantities such as the chemical potential and the ground state energy.
Within the realm of the HFB theory, the chemical potential can be written as

$$\mu = \mu_0 + \delta\mu + 2\mu_R, \tag{3.13}$$

where

$$\mu_0 = \bar{V}(0) n_c, \tag{3.14}$$

is the first-order chemical potential [**18**].
Corrections to the chemical potential due to the disorder effects are given as [**125**]

$$\mu_R = g_2 n_{\rm HM} (1 + g_3 n_c / g_2)^{1/2} H(\gamma, \sigma/\xi), \tag{3.15}$$

where

$$H(\gamma,\sigma/\xi)=\frac{1}{2}\int_0^{\pi}\mathrm{d}\theta\sin\theta\sqrt{1+\gamma(3\cos^2\theta-1)}S(\alpha), \tag{3.16}$$

Corrections to the chemical potential due to the quantum and thermal fluctuations are defined as : $\delta\mu=\sum_{\mathbf{k}}\bar{V}(\mathbf{k})\left[v_k(v_k-u_k)+(v_k-u_k)^2N_k\right]$ [**30**, **51**]. Nevertheless, this chemical potential cannot be evaluated straightforwardly since the zero-temperature term is ultraviolet divergent. Such a problem can be worked out either by using the dimensional regularization which is valid for very dilute gases [**18**, **56**, **103**] or by renormalizing the contact interaction through the $T-$ matrix approach [**104**]. After some algebra, the resulting corrections to the chemical potential read [**125**]

$$\begin{aligned}\frac{\delta\mu}{g_2n_c}&=\frac{32}{3}\sqrt{\frac{n_ca^3}{\pi}}(1+g_3n_c/g_2)^{5/2}\mathcal{Q}_5(\gamma)\\&+\frac{2}{3}\left(\frac{\pi T}{n_cg_2}\right)^2\sqrt{\frac{n_ca^3}{\pi}}(1+g_3n_c/g_2)^{1/2}\mathcal{Q}_1(\gamma).\end{aligned} \tag{3.17}$$

Importantly, for $g_3=0$, the total chemical potential (3.13) reduces to that obtained in[**18**]. For a cleaned ($R_0=0$) condensate with two-body contact interactions ($g_3=\epsilon_{dd}=0$), the obtained corrections to the chemical potential coincide with the seminal Lee-Huang-Yang quantum corrected equation of state [**55**].

The energy shift (2.17) due to the disorder effects is finite and it can be evaluated as

$$\frac{E_R}{N}=\frac{2mR_0}{\hbar^2}(1+g_3n_c/g_2)^{1/2}\sqrt{\frac{n_ca}{\pi}}h_1(\gamma,\sigma/\xi), \tag{3.18}$$

where the function

$$h_1(\gamma,\sigma/\xi)=\frac{1}{2}\int_0^{\pi}\mathrm{d}\theta\sin\theta\sqrt{1+\gamma(3\cos^2\theta-1)}S_1(\alpha), \tag{3.19}$$

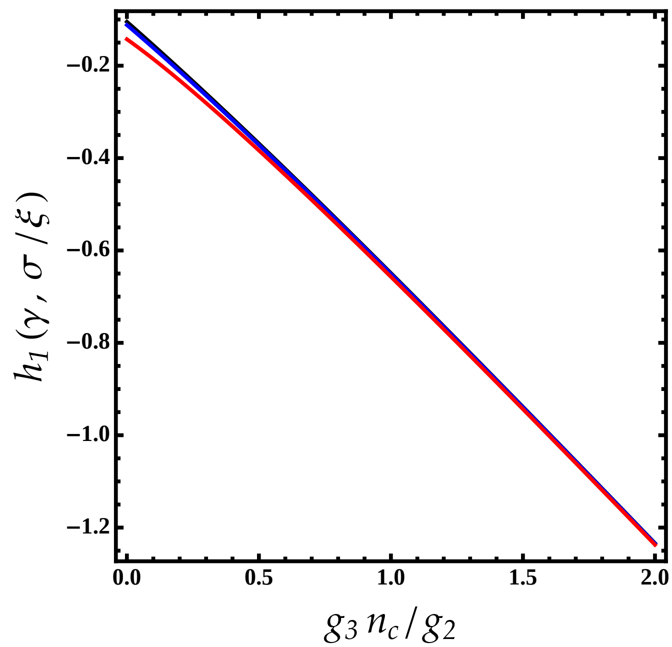


FIGURE 3.3: Disorder energy function $h_1(\gamma, \sigma/\xi)$ versus $g_3 n_c/g_2$ for $\sigma/\xi = 0.5$. Black line : $\epsilon_{dd} = 0.16$, blue line : $\epsilon_{dd} = 0.4$, and red line : $\epsilon_{dd} = 0.8$ [**125**].

and the function $S_1(\alpha) = e^{2\alpha}\text{erfc}(\sqrt{2\alpha}) - \sqrt{1/2\alpha}$. The disorder energy function $h_1(\gamma, \sigma/\xi)$ is decreasing with $g_3$ as is seen in Fig.3.3 indicating that the TBI lead to lower the energy due to the disorder fluctuations which is in agreement with the above results. We observe also that for $g_3 n_c/g_2 \leq 0.7$, the DDI effects on the energy are more pronounced.

In conclusion, we investigated in this chapter the properties of dipolar Bose gas with TBI subjected to a correlated Gaussian disorder. We showed that the DDI may lead to arrest transport of atoms under disorder augmenting the glassy fraction inside the condensate, while the presence of the TBI may lead to a diffusive motion of particles. The combined effects of the DDI, TBI, and temperature found to crucially affect the chemical potential and the ground state energy of the system.

## CHAPITRE 4

## TWO-DIMENTIONAL DIPOLAR BOSE GAS IN WEAK DISORDER POTENTIAL

In past two decade, the experimental progress of the ultracold gases in 2D [**183-190**] has attracted much attention. The properties of these fluids are radically different from those in 3D. The famous Mermin-Wagner-Hohenberg theorem [**191**, **192**] states that long-wavelength thermal fluctuations destroy long-range order in a homogeneous 1D Bose gas at all temperatures and in a homogeneous 2D Bose gas at any nonzero temperature, preventing formation of condensate. Since the earlier works of Schick [**193**] and Popov [**194**], several theoretical studies of fluctuations, scattering properties, and the appropriate thermodynamics have been performed in [**156**], [**195-201**] .

In the last years, 2D or quasi-2D dipolar gases have attracted a great deal of interest due to their intriguing scattering properties. Interesting structural properties emerge in such systems is the presence of the low-lying roton minimum in the excitation spectrum [**38**, **50**] and the possibility of the crystallization of solid bubble into a lattice superstructure, resulting in a global supersolid phase [**50**, **115**, **202**]. The appearance of the roton-maxon character in the excitation spectrum of pancake dipolar condensates has been predicted first by Santos et al.[**111**] where it has been shown that upon further decreasing the confining trap frequency the roton energy drops to zero triggering a dynamical instability. Since that time, there has been a number of recent theoretical studies proposing schemes to detect rotons and characterize their effects on the properties of dipolar Bose-Einstein condensate (BEC) (e.g., see [**43**, **52**, **154**],[**203-208**]). The experimental observation of the roton modes has been reported very recently in Ref.[**209**] using momentum-distribution measurements in dipolar quantum gases of highly-magnetic Er atoms.

The roton-maxon character of the Bogoliubov spectrum was originally observed in $^4$He superfluid and arises due to the strong isotropic repulsion between the atoms in the liquid [**210-214**]. However, the nature of the roton in the context of quasi-2D dipolar BECs is radically different. In these dilute systems, the roton is originated from the anisotropy of the DDI interaction. It was found that the presence of the roton minimum in 2D dipolar bosons leads to reduce the condensed fraction even at zero temperature [**215-217**]. Finite temperature Monte Carlo simulations [**58**] have revealed that the rotonization of the spectrum can decrease the Kosterlitz-Thouless superfluid transition temperature. As for the pancaked dipolar BEC, it has been pointed out that the roton modes serve to change the sign of the anomalous density near the trap center for large values of DDI [**51**].

The study on dirty dipolar boson systems is still inadequate. In this chapter, we investigate the properties of a quasi-2D dipolar Bose gas subjected to a weak random potential with Gaussian correlation by using the Huang-Meng-Bogoliubov theory which marked an important step towards a quantitative description of dirty dipolar Bose systems as we have seen in previous Chapters. Before doing so, we first review the main features of a cleaned homogeneous dipolar Bose gases in a quasi-2D geometry. We derive analytical expressions for the condensate fluctuations and thermodynamic quantities such as the chemical potential, the ground state energy and the sound velocity for Gaussian correlated disorder potential in the roton regime. We analyze the behavior of noncondensed and anomalous densities in terms of the temperature and the interaction strength. Moreover, we calculate the corrections to the sound velocity due to the correlated disorder and the superfluid fraction with respect of the system parameters. It is found that the presence of a disordered potential in the regime where the roton develops in the excitations spectrum strongly enhances the fluctuations, the thermodynamics and the superfluidity.

## 4.1 Cleaned two-dimensional homogeneous dipolar Bose gases

We consider a dilute Bose-condensed gas of dipolar bosons tightly confined in the axial direction $z$ by an external potential $V(\mathbf{r}) = m^2 z^2/2$ and assume that in the $x, y$ plane the translational motion of atoms is free. The dipole moments $d$ are oriented perpendicularly to the $x, y$ plane. In the ultracold limit $kr_* \ll 1$, where $r_* = md^2/\hbar^2$ is a characteristic range of the DDI, the momentum representation of the two-body interaction potential $V(\mathbf{r} - \mathbf{r}')$ is given as [**50**]

$$V(\mathbf{k}) = g_{2D}(1 - C|\mathbf{k}|), \tag{4.1}$$

where $C = 2\pi d^2/g_{2D}$, $g_{2D} = g_{3\text{D}}/\sqrt{2}l_0$ is the 2D contact interaction coupling constant which strongly depends on the strength of the transverse confinement $l_0 = \sqrt{\hbar/m}$, and $g_{3\text{D}} = 4\pi\hbar^2 a/m$ with $a$ being the $s$-wave scattering length ($a > 0$ throughout this section). Another model for the effective quasi-2D potential was proposed in Ref.[**218**] $V(k) = g_{2D}[1-Ckl_0 \exp(k^2 l_0^2/2)\text{Erfc}(kl_0/\sqrt{2})]$, where Erfc is the complementary error function. Expanding this potential which can be obtained by integration of the full 3D dipolar interaction over the transverse harmonic oscillator at small momenta leads to $V(k) = g_{2D}(1 - Cl_0 k)$, with $l_0$ adjusts the scale for the strength of the linear term and can be set to unity since the ratio between the dipolar length and the trap length is the most important. Therefore, both potentials require a high momentum cut-off when calculating the beyond-mean field corrections. The large momentum behavior of both potentials is different, the potential of Ref.[**218**] is constant $(-\sqrt{2/\pi})$ for large $k$, while the potential (4.1) is linear in $k$.

Applying the standard Bogoluibov theory presented in chapter 1, we obtain for the excitations spectrum [**50**]

$$\varepsilon_k = \sqrt{E_k^2 + 2ng_{2D}E_k(1 - Ck)} \tag{4.2}$$

To zero order the chemical potential is $\mu = ng_{2D}$. For small momenta the excitations are sound waves, $\varepsilon_k = \sqrt{ng_{2D}/m}k$. The dependence of $\varepsilon_k$ on $k$ remains monotonic with increasing $k$ if $C \leq \sqrt{8}\xi/3$ (see Fig. 4.1). For the constant $C$ in the interval

$$\frac{\sqrt{8}}{3}\xi \leq C \leq \xi, \tag{4.3}$$

the excitation spectrum has a roton-maxon structure. It is then convenient to represent $\varepsilon_k$ in the form :

$$\varepsilon_k = \frac{\hbar^2 k}{2m}\sqrt{(k - k_r)^2 + k_\Delta^2}, \tag{4.4}$$

where $k_r = 2C/\xi^2$ and $k_\Delta = \sqrt{4/\xi^2 - k_r^2}$. If the roton is close to zero, then $k_r$ is the position of the roton, and

$$\Delta = \hbar^2 k_r k_\Delta/2m = 2ng_{2D}C\sqrt{mng_{2D}/\hbar^2 - C^2(mng_{2D}/\hbar^2)^2} \tag{4.5}$$

is the height of the roton minimum (see Fig 4.1). For $C = \xi$ the roton minimum touches zero, and at larger $C$ the uniform Bose condensate becomes dynamically unstable.

It should be noted that the coupling constant $g_{2D}$ can be tuned by using Feshbach resonances or by modifying the frequency of the tight confinement. Therefore, although the range of $C$ given by Eq.(4.3) is rather narrow, it can be reached without serious difficulties. The condition $C = 2\pi d^2/g_{2D} = \xi$ is reduced to $(mg_{2D}/2\pi\hbar^2) = a/\sqrt{2}l_0 \simeq 2\pi n r_*^2$. For dysprosium atoms we have the dipole-dipole distance $r_* \simeq 200$ Å, and at 2D densities $\sim 10^9$ cm$^{-2}$ the roton-maxon spectrum is realized for the 3D scattering length $a$ of several tens of angstroms at the frequency of the tight confinement of 10 kHz leading to the confinement length $l_0$ about 1000 Å.

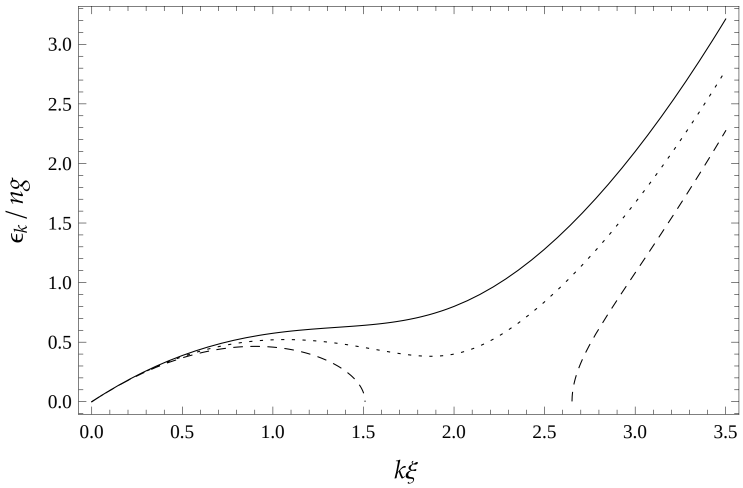


FIGURE 4.1: Excitation energy $\varepsilon_k$ of the quasi-2D dipolar BEC as a function of momentum $k$ for several values of $k_r$. The solid curve ($k_r\xi = 1.84$) shows a monotonic dependence $\varepsilon_k$, the dotted curve ($k_r\xi = 1.96$) is $\varepsilon_k$ with the roton-maxon structure, and the dashed curve ($k_r\xi = 2.08$) corresponds to dynamically unstable BEC [**50**].

### 4.1.1 Condensed depletion

At zero temperature, the evaluation of the integral (1.27) in 2D geometry requires special care due to the crucial contribution to the beyond mean field terms of the transverse trap modes of the contact interactions. The large-momentum divergence originating from the dipolar term $-g_{2D}Ck$ (valid only for $k \ll 1/r_*$) is another issue of the integral

$$\delta\mu_{\text{LHY}} = \frac{1}{2}\int \frac{d\mathbf{k}}{2\pi)^2} V_{\mathbf{k}} \left[\frac{E_k}{\varepsilon_k} - 1\right]. \tag{4.6}$$

One possibility to solve this problem is to work with an arbitrary $\Lambda$-cutoff. In the case of contact interactions, the potential (4.1) takes the form $V(k) = g_{2D}$ for $k < \Lambda$, and 0 otherwise. Then, if $\Lambda$ is larger than typical momenta in the gas, the obtained LHY corrections are cutoff-independent and in good agreement with the existing literature (see e.g. [**156**],[**219-221**]). Now if one applies this method to the dipolar interaction case, it turns out that the resulting corrections to the equations of state are cutoff-dependent (the cutoff is not larger than the roton momentum) due to the special character of the DDI (see e.g [**222**]). Another possible route to compute the LHY corrections (4.6) is to take into account the full transverse structure. Obtaining reasonable stable corrections within this technique is also a tedious and time-consuming task (diagonalizing the Bogoliubov-De Gennes equations is extremely difficult both analytically and numerically) [**50**].

To circumvent this problem, a high-momentum cutoff is considered here which is valid in the ultracold regime $k \ll 1/r_*$ [**50**]. Despite it gives qualitative correct results, it renders much simpler the calculations and captures the main features of the system at hand [**50**]. The choice of this momentum cutoff is not only motivated by computational convenience, but also the obtained corrections will be insensitive to the cutoff in contrast to the $\Lambda$-cutoff method. This leads [**50**]

$$\frac{\tilde{n}}{n}=\frac{mg_{2D}}{4\pi\hbar^2}\left[1-k_r\xi-\frac{3(k_r\xi)^2}{4}+\frac{(k_r\xi)^2}{2}\ln\!\left(\frac{\xi}{r_*(2-k_r\xi)}\right)\right]. \tag{4.7}$$

In the absence of the dipole-dipole interaction ($r_* = 0$ and $k_r = 0$) we recover the usual result for the 2D BEC with short-range interparticle repulsion, $\tilde{n} = n(mg_{2D}/4\pi\hbar^2)$. For $\Delta \ll ng_{2D}$ we have $(2 - k_r\xi) \simeq (k_\Delta\xi)^2/4$ and Eq.(4.7) transforms to

$$\frac{\tilde{n}}{n} \simeq \frac{mg_{2D}}{\pi\hbar^2}\ln\left(\frac{2ng_{2D}}{\Delta}\zeta\right); \quad \Delta \ll ng_{2D}, \tag{4.8}$$

where $\zeta = \sqrt{2\pi\hbar^2/e^2mg_{2D}}$.

As we see from Eq. (4.8), for the roton minimum close to zero a small condensate depletion and small fluctuations of the density require the inequality

$$\frac{mg_{2D}}{\pi\hbar^2}\ln\left(\frac{2ng_{2D}}{\Delta}\right) \ll 1. \tag{4.9}$$

It differs only by a logarithmic factor $\ln(2ng_{2D}/\Delta)$ from the small parameter of the theory, $(mg_{2D}/2\pi\hbar^2) \ll 1$, in the absence of the roton.

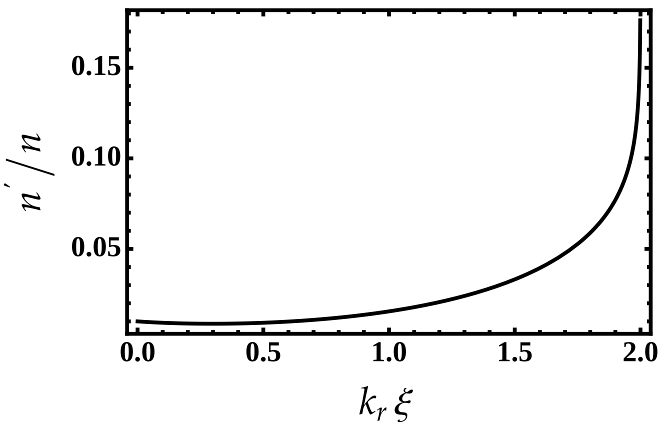


FIGURE 4.2: Non-condensed fraction as a function of $k_r\xi$ for $mg_{2D}/4\pi\hbar^2 = 0.01$ ($\xi/r_* = 100/k_r\xi$) [**50**].

We thus conclude that at $T = 0$ the validity of the Bogoliubov approach is guaranteed by the presence of the small parameter (4.9). For the dysprosium example given after Eq.(4.5) we have $ng_2D$ about 5 nK, and the criterion (4.9) is satisfied for the roton energy above 2 nK.

### 4.1.2 Equation of state

Using the same high-momentum cutoff scheme, the LHY corrections is obtained in[**50**]

$$\frac{\delta\mu_{\rm LHY}}{E_0} = (4\pi^{3/2}/b^2)^2 nr_*^2 \left\{1 - 2b(nr_*^2)^{1/2} - 3b^2nr_*^2 + 2b^2nr_*^2 \ln\left[1/2(1-b\sqrt{nr_*^2})\right]\right\}, \tag{4.10}$$

where $E_0 = \hbar^2/mr_*^2$ and $b = \sqrt{2\pi^{3/2}l_0/a}$. In the absence of the DDI, Eq.(4.10) excellently agrees with the usual short-range 2D Bose gas (see e.g. [**156**, **221**]). When the roton minimum is close to zero i.e. $C = \xi$, one has $\delta\mu_{\rm LHY}/E_0 \simeq$

$(8\pi^{3/2}/b^2)^2 nr_*^2 \ln\left[1/\sqrt{b^2nr_*^2(1-b^2nr_*^2)}\right]$. The quantum corrections (4.10) are important to halt the collapse of the system when the roton touches zero (roton instability) conducting to a stable droplet state [**31**].

### 4.1.3 Fluctuations at finite temperatures

As we have stated in the introduction, in 2D at finite temperatures, long-wave fluctuations of the phase destroy the condensate [**191**, **192**, **194**]. There is the so-called quasicondensate, or condensate with fluctuating phase. In this state fluctuations of the density are suppressed but the phase still fluctuates. The transition from a non-condensed state to quasi-BEC is of the Kosterlitz-Thouless type and it occurs through the formation of bound vortex-antivortex pairs [**20**]. Somewhat below the Kosterlitz-Thouless transition temperature the vortices are no longer important, and in the weakly interacting regime that we consider the phase coherence length $l_\phi$ is exponentially large. Thermodynamic properties, excitations, and correlation properties on a distance scale smaller than $l_\phi$ are the same as in the case of a true BEC. Moreover, for realistic parameters of quantum gases, $l_\phi$ exceeds the size of the system [**221**], so that one can employ the ordinary BEC theory.

At finite temperature, the LHY thermal fluctuations reads

$$\delta\mu_{\rm LHY}^{\rm th} = \int V(\mathbf{k})\frac{E_k}{\varepsilon_k}[\exp(\varepsilon_k/T)-1]^{-1}\frac{d\mathbf{k}}{(2\pi)^2}. \tag{4.11}$$

In contrast to the zero temperature case, integral (4.11) is finite (convergent). At low $T$, the main contribution to Eq.(4.11) comes from the phonon branch. This yields [**31**]

$$\frac{\delta\mu_{\rm LHY}^{\rm th}}{E_0} = \frac{b^2}{4\pi^{3/2}}\left[\frac{\zeta(3)}{\sqrt{2}}(nr_*^2)^{-2}\left(\frac{T}{E_0}\right)^3 - \frac{b}{120\pi^2}(nr_*^2)^{-5/2}\left(\frac{T}{E_0}\right)^4\right], \tag{4.12}$$

where $\zeta(3)$ is the Riemann Zeta function. Notice that at $T > \mu_0$, the leading term for the chemical potential coincides with that of an ideal gas.

Assuming now that the roton energy $\Delta$ is very small (at least $\Delta \ll T$), Eq.(4.12) turns out to be given as [**50**] :

$$\frac{\delta\mu}{\mu} \simeq \frac{2mg_{2D}}{\hbar^2}\frac{T}{\Delta}; \quad \Delta \ll T. \tag{4.13}$$

Equation (4.13) clearly shows that the rotonization of the spectrum can strongly increase thermal fluctuations of the density and destroy the Bose-condensed state even at very low $T$.

We now calculate the density of the normal component in the presence of the roton. If the roton minimum is close to zero and $\Delta \ll T$, then the momenta near the roton minimum are the most important, and the integration over $dk$ yields [**50**] :

$$\frac{n_T}{n} = \frac{2mg_{2D}}{\hbar^2}\frac{T}{\Delta}. \tag{4.14}$$

The employed approach requires the condition $n_T \ll n$ because we used the spectrum of excitations obtained by the Bogoliubov method. Again, at temperatures $T \lesssim \Delta$ we should have the inequality $(2mg_{2D}/\hbar^2)T/\Delta \ll 1$.

One can conclude that at finite temperatures, the Bogoliubov approach requires the inequality

$$\frac{mg}{\hbar^2}\left(\frac{2ng_2D}{\Delta}\right)^2\frac{T}{\Delta} \ll 1, \tag{4.15}$$

whereas for $T \lesssim \Delta$ it is sufficient to have criterion (4.9).

## 4.2 Two-dimentional dipolar Bose gas with weak disorder

In this section we study the effects of a weak disorder potential on a dipolar Bose gas quasi-2D in the regime where rotons develop. We report on the behavior of a number of key quantities that characterize the system, including the condensate depletion, the ground-state energy, the sound velocity and superfluid fraction.

### 4.2.1 Fluctuations and Thermodynamics

In what follows, we consider the case of a weak external random potential with Gaussian correlation which can be written in the momentum space as Eq. 2.25. Indeed, this type of disorder potential makes our study substantially more detailed, general and rigorous since uncorrelated random potentials are usually crude approximations of realistic disorder, for which $\sigma$ can be significantly large.

Assuming now that the roton is close to zero and the roton energy is $\Delta \ll \mu$, we have the coefficient $C$ close to $\xi$, and $k_r \simeq 2/\xi$. Then, using Eq.(2.25) for the contribution of momenta near the roton minimum at $T = 0$, we obtain :

$$\frac{n_R}{n} = \frac{mg_{2D}}{4\hbar^2}\left(\frac{2\mu}{\Delta}\right)^3 R\, e^{-2\sigma^2/\xi^2}; \quad \Delta \ll \mu, \tag{4.16}$$

where $R = R_0/ng_{2D}^2$ is a dimensionless disorder strength.
For $\sigma/\xi \to 0$, the disorder fluctuation (4.16) reduces to that of dipolar BEC with $\delta$-correlated disorder [**92**].

Integrals (2.18) and (2.19) are logarithmically divergent at large momenta because of the dipolar contribution to the interaction strength $-g_{2D}Ck$ [**50**]. To overcome this problem, we can resort to a high momentum cut-off $1/r_*$. Inserting the resulting expressions in (2.18) and (2.19), we obtain for the condensate depletion and the anomalous fraction :

$$\frac{\tilde{n}}{n} \approx \frac{\tilde{m}}{n} \simeq \frac{mg_{2D}}{\pi\hbar^2}\left[\ln\left(\frac{2\mu}{\Delta}\zeta\right) + \frac{\pi}{4}\left(\frac{2\mu}{\Delta}\right)^3 R\, e^{-2\sigma^2/\xi^2}\right], \tag{4.17}$$

The leading term in Eq.(4.17) was first obtained in the recent work [**50**], while the second term represents the disorder correction to the noncondensate and anomalous fractions [**93**]. For $\sigma/\xi \gg 1$, the disorder effects become negligible and hence, the condensed fraction takes the form $n_c/n \simeq 1 - (mg_{2D}/\pi\hbar^2)\ln(2\mu\zeta/\Delta)$. Furthermore, equation (4.17) clearly shows that the anomalous density and the condensate depletion are comparable in the roton branch.

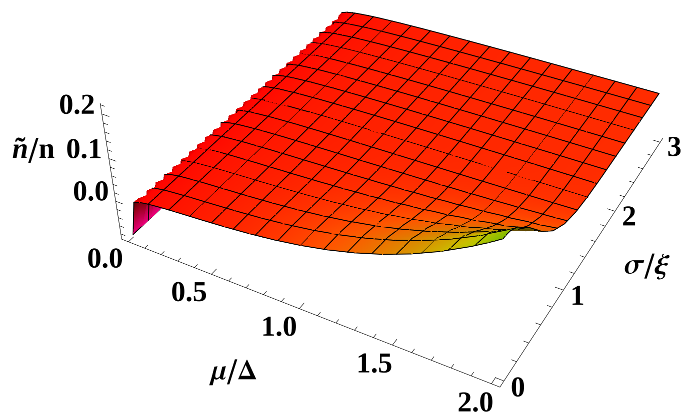


FIGURE 4.3: Quantum depletion of dirty dipolar condensate, as a function of $\mu/\Delta$ and $\sigma/\xi$ for $mg_{2D}/4\pi\hbar^2 = 0.01$ and $R = 0.1$.[**93**]

Figure.4.3 shows that in the absence of the random external potential i.e. $R = 0$, the noncondensed fraction grows logarithmically (see Eq. (2.18)) when the roton energy $\Delta$ goes to zero yielding the transition to a supersolid state [**50**, **111**, **202**]. In the presence of the disorder potential the ratio of the correlation length and the healing length $\sigma/\xi$ decreases the condensate depletion according to the function $e^{-2\sigma^2/\xi^2}$.

The Bogoliubov approach assumes that the condensate depletion should be small. We thus conclude from Eq. (2.18) that at $T = 0$ and for the roton minimum close to zero, the validity of the Bogoliubov approach is guaranteed by the inequalities $\big(mg_{2D}/\pi\hbar^2\big)\ln(2\mu/\Delta\zeta) \ll 1$, and $\big(mg_{2D}/\hbar^2\big)\,(2\mu/\Delta)^3 R\, e^{-2\sigma^2/\xi^2} \ll 1$. For $R = 0$ and in the absence of the roton, this condition differs only by the logarithmic factor $\ln(2ng_{2D}/\Delta)$ from the universal small parameter of the theory, $\big(mg_{2D}/\pi\hbar^2\big) \ll 1$.

However, the situation changes in the calculation of the correction to the ground-state energy due to the external random potential. When the roton minimum is approaching to zero, we get from (2.17)

$$\frac{E_R}{E_0} = -\frac{2mg_{2D}}{\hbar^2}\left(\frac{2\mu}{\Delta}\right) R\, e^{-2\sigma^2/\xi^2}; \quad \Delta \ll \mu. \tag{4.18}$$

Equation (4.18) shows that $E_R$ linearly depends on $ng_{2D}/\Delta$, and decreases with increasing $\sigma/\xi$. Furthermore, the correction (4.18) is negative which means that the random potential leads to lower the total energy of the system.

The correction to the chemical potential due to disorder effects is then obtained easily through $\partial E_R/\partial N$

$$\frac{\mu_R}{\mu} = -\frac{mg_{2D}}{\hbar^2}\left(\frac{2\mu}{\Delta}\right)^3 R\, e^{-2\sigma^2/\xi^2}; \;\; \Delta \ll \mu. \tag{4.19}$$

The shift of the ground-state energy due to quantum fluctuations can be given as

$$\frac{E'}{E_0} \simeq 1 + \frac{2mg_{2D}}{\pi\hbar^2} + \frac{2mg_{2D}}{\pi\hbar^2} \ln\left(\frac{2\mu}{\Delta}\right); \quad \Delta \ll \mu. \tag{4.20}$$

Note that quantum fluctuations correction to the chemical potential can be calculated straightforwardly using $\partial E'/\partial N$ (see e.g. [**50**]).

The correction to the sound velocity can be simply calculated via $mc_s^2 = n\partial\mu/\partial n$ [**50**, **56**, **166**] as

$$\frac{c_s^2}{c_{s0}^2} = 1 + \frac{mg_{2D}}{\pi\hbar^2}\left[2\ln\left(\frac{2\mu}{\Delta}\right) + \left(\frac{2\mu}{\Delta}\right)^2\right] + \frac{mg_{2D}}{\hbar^2}\left[\left(\frac{2\sigma^2}{\xi^2} + \frac{3}{2}\right)\left(\frac{2\mu}{\Delta}\right)^3 - \frac{3}{2}\left(\frac{2\mu}{\Delta}\right)^5\right] R\, e^{-2\sigma^2/\xi^2}, \tag{4.21}$$

where $c_{s0} = \sqrt{\mu/m}$ is the zeroth order sound velocity. The second and the third terms originate from quantum fluctuations while the last term comes from the disorder contribution. For $\sigma/\xi \to 0$, the sound velocity (4.21) becomes identical to that obtained in quasi-2D dipolar BEC with delta-correlated disorder [**92**].

Equation (4.21) shows that the main correction to the sound velocity due to the disorder potential is negative $\sim -(2\mu/\Delta)^5 \, R \, e^{-2\sigma^2/\xi^2}$.

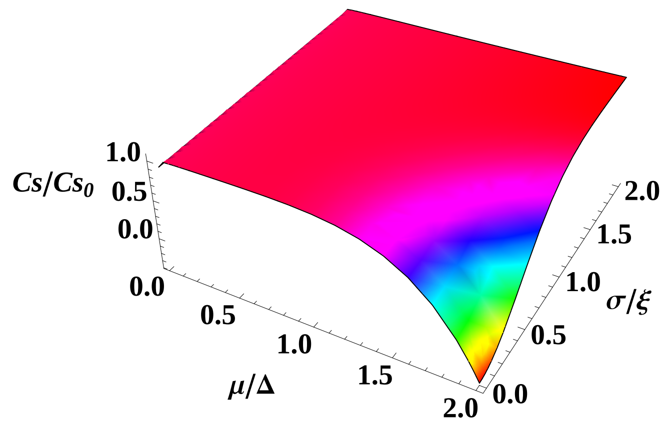


FIGURE 4.4: Sound velocity as a function of $\mu/\Delta$ and $\sigma/\xi$. Parameters are the same as in Fig.4.3 [**93**].

We see from Fig.(4.4) that for $\sigma \ll \xi$, $c_s$ is practically constant in the range $0 < \Delta \leq \mu$, while it reduces and vanishes at $\Delta \simeq \mu/2$. This value can be changed with increasing or decreasing the disorder strength $R$. For $\sigma > \xi$, $c_s$ rises with rising $\mu/\Delta$.

It is worth stressing that, in 3D disordered BECs with a pure contact interaction, the sound velocity has been calculated with different approaches leading to different predictions. For instance, standard perturbation theory predicts an increased $c_s$ in Bose gas with an uncorrelated disorder [**76**, **223**]. On the other hand, the extended Bogoliubov approach developed in [**224**, **225**] and the mean field theory of Ref [**159**] provide a reduced sound velocity.

### 4.2.2 Superfluid fraction of quasi-2D bose gas

In the context of the liquid helium, it has been shown that the position of the roton minimum influences the phenomenon of superfluidity [**226**]. Here we look how the interplay of the rotonization and external disorder potential can affect the superfluid fraction of a quasi-2D Bose gas with DDI.

The superfluid fraction $n_s/n$ can be found from the normal fraction $n_n/n$ which is determined by the transverse current-current correlator $n_s/n = 1 - n_n/n$. We apply a Galilean boost with the total momentum of the moving system $\hat{\mathbf{P}}_{\mathbf{v}} = \hat{\mathbf{P}} + mvN$, where $\hat{\mathbf{P}} = \sum_k \hbar\mathbf{k}\, \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ and $v$ is the liquid velocity. In the $d$-dimensional case, the superfluid fraction reads

$$\frac{n_s}{n} = 1 - \frac{2}{dTn}\int \frac{d^dk}{(2\pi)^d}\left[\frac{E_k}{4\sinh^2(\varepsilon_k/2T)} + \frac{nR_kE_k^2}{\varepsilon_k^3}\coth\left(\frac{\varepsilon_k}{2T}\right)\right]. \tag{4.22}$$

At very low temperature we can put $\coth(\varepsilon_k/2T) = 2T/\varepsilon_k$. Thus, Eq. (4.22) reduces to

$$\frac{n_s}{n} = 1 - \frac{4}{d}\frac{n_R}{n} - \frac{2}{dTn}\int \frac{d^dk}{(2\pi)^d}\left[\frac{E_k}{4\sinh^2(\varepsilon_k/2T)}\right]. \tag{4.23}$$

Equation.(4.23) clearly shows that the ratio between the normal fluid density and the corresponding condensate depletion increases to 2 in 2D and to 4 in 1D, in contrast to the familiar 4/3 in 3D geometry obtained earlier in [**75**, **76**, **78**]. Remarkably, the superfluid density (4.23) is a scalar quantity contrary to the 3D case where it has been found that $n_s$ is a tensorial quantity[**82**, **178**] due to the anisotropy of the DDI.

Assuming now that the roton minimum is close to zero, then the momenta near the roton minimum are the most important, this yields at $T = 0$ :

$$\frac{n_s}{n} = 1 - \frac{mg_{2D}}{2\hbar^2}\left(\frac{2\mu}{\Delta}\right)^3 R\, e^{-2\sigma^2/\xi^2}; \quad \Delta \ll \mu. \tag{4.24}$$

This equation shows that for $\sigma/\xi \gg 1$, $n_s \sim n$ in contrast to $n_c$ where this latter remains small even for $\sigma/\xi \gg 1$ owing to the quantum fluctuation described by the logarithmic term.

Now we turn to analyze numerically the normal fraction of the superfluid as a function of the ratio $\sigma/\xi$ and the strength of disorder for different positions of the roton minimum using the standard Monte Carlo method. The results are depicted in Fig.4.5.

We observe that for $\sigma \geq \xi$, the normal fraction vanishes and thus, the system becomes completely superfluid for any value of the disorder strength and the roton position. The reason is that when the healing length of the BEC is smaller than the correlation length of the disorder potential, the kinetic energy term is small and the BEC density simply follows the spatial modulations of the potential and hence, the condensed particles will not localize [**227**]. This result excellently coincides with our analytical predictions (4.24). Whereas, for $\sigma < \xi$, $n_n/n$ is increasing with $R$ and $C/\xi$. One can observe from the same figure that when the roton minimum is very close to zero ($C \sim \xi$) and for a large value of $R$, the normal fraction is significant which makes it possible to destroy superfluidity even at very low temperature (see Fig.4.5.c). This is attributed to the fact that the particles are localized in the respective minima of the external random potential and thus form distributed randomly obstacles for the motion of the superfluid. However this localization is different from Anderson localization of Bogoliubov quasiparticles observed by Lugan et al. [**228**, **229**]. The Bogoliubov quasiparticles experience a randomness mediated by the inhomogeneous condensate background, which responds nonlinearly and nonlocally to an effective potential that is different from the usual bare disorder [**225**, **228**, **229**, **230**]. Therefore, the localization properties are changed compared to bare particles although the general symmetry class is the same [**229**, **230**].

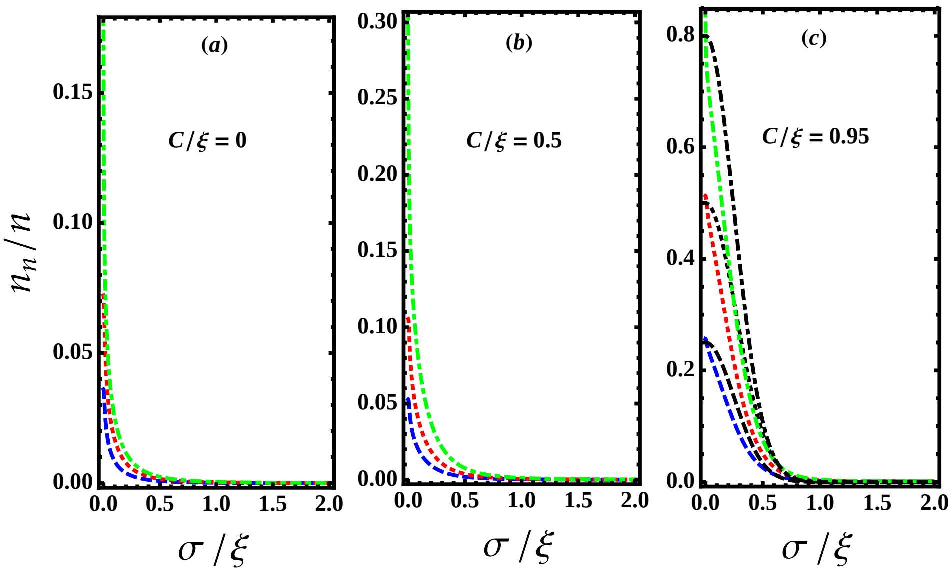


FIGURE 4.5: Normal fraction from Eq.(4.22), as a function of $\sigma/\xi$ for $R = 0.05$ (Blue dashed), $R = 0.1$ (Red dotted) and $R = 0.15$ (Green dotdashed). Parameters are : $T/\mu = 0.2$ and $mg/4\pi\hbar^2 = 0.01$. Black lines represent analytical solutions [**93**].

In conclusion, we have investigated a dilute 2D dipolar Bose gas with dipoles oriented perpendicularly to the plane subjected to a weak Gaussian correlated disorder potential in the roton regime. Using the Bogoliubov-Huang-Meng approach, we have derived analytical expressions for the condensate depletion, the ground state energy, the equation of state, the sound velocity and the superfluid fraction. Our analysis signifies that in the limit $\sigma/\xi \to 0$, the disorder potential strongly enhances the fluctuations and the thermodynamic quantities. This may lead to the transition of a non-trivial quantum phase (disordered supersolid state). We have pointed out also that the peculiar interplay of rotonization induced by DDI and disorder may lead to strongly depress the superfluid density in the roton's region due to the localization of the particles in the respective minima of the external random potential.

The findings of this chapter open up new prospects for investigating effects of a disorder potential on 2D dipolar BEC in a bilayer system, the subject of the next chapter.

# CHAPITRE 5

# EFFECTS OF WEAK DISORDER ON TWO-DIMENSIONAL BILAYERED DIPOLAR BOSE-EINSTEIN CONDENSATES

Ultracold dipolar gases in layered structures have attracted considerable attention [**15-18**, **84**, **112-123**]. Unlike single layers, bilayered configurations in quasi-2D geometry, the dipolar interaction between particles in different layers shows attractive regions that makes possible the formation of dimers. These setups exhibit many interesting phenomena namely : the formation of conventional and unconventional superfluids of polar molecules [**15-18**, **112**, **114**, **116**, **117**, **119**], soliton molecules [**120**] and the enhacement of the roton instability [**113**, **122**] due to the interlayer effects. The Fermi-polaron problem has been also discussed in such a bilayer system [**123**].

However, the contemporary problem of disordered ultracold dipolar bosons in bilayer systems has never been analyzed in the literature. Due to the availability of creating this bilayered configuration experimentally by means of a 1D subwavelength lattice, it is then instructive to study disordered BEC with DDI in bilayer arrangements. Such systems enable us to unveil the intriguing role of disorder, the interlayer effects and the dipolar interactions.

In this chapter, we investigate the problem of a disordered quasi-2D bilayered dipolar BEC with dipoles are oriented perpendicularly to the layers and in same (i.e parallel, denoted $\uparrow\uparrow$) /opposite (i.e antiparallel, denoted $\downarrow\uparrow$), directions in different layers (see Figs.5.1). To this end, we use the Bogoliubov-Huang-Meng theory [**77**]. Many studies have confirmed recently the effectiveness of this method in treating dirty dipolar Bose gases [**74**, **56**, **16**, **96**, **57**, **131**]. We quantitatively examine the effects of varying polarization direction and interlayer DDI on the excitations spectrum, glassy fraction, one-body density matrix and the superfluid fraction. Importantly, we find

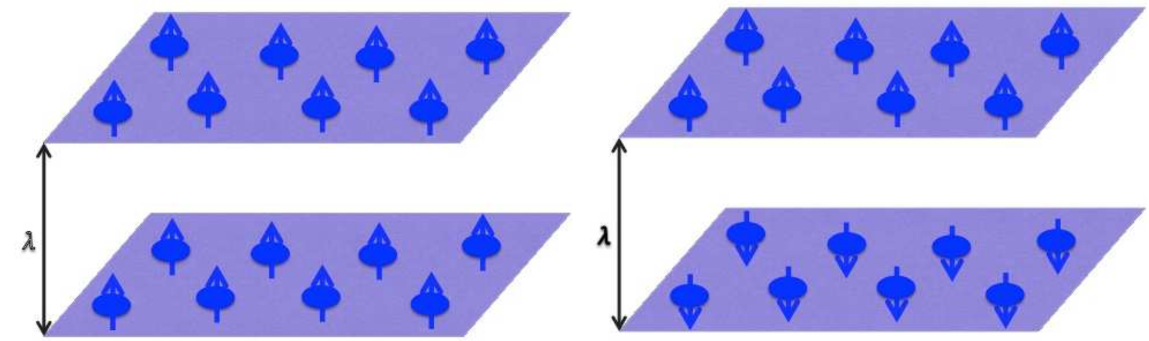


FIGURE 5.1: Schematic representaion of the dipolar bilayer systems under consideration : (left) dipoles oriented in same directions in different layers (parallel configuration, ↑↑). (right) dipoles oriented in opposite directions in different layers (antiparallel configuration, ↓↑)[**126**].

that in the parallel configuration, the interlayer DDI causes delocalization of particles enabling the transition to the superfluid phase. Surprisingly, in the antiparallel arrangement, the bosons strongly fill the potential wells formed by disorder fluctuations depressing both the condensate and the superfluidity due to the intriguing interplay of the disorder and the interlayer DDI. Our results reveal also that beyond a certain temperature depending on the polarization direction, the superfluid fraction vanishes.

## 5.1 Model

We consider a dilute Bose-condensed gas of dipolar bosons subjected to an external random potential loaded in a quasi-2D bilayer setup. Assuming vanishing hopping between layers and dipole moments $d$ are aligned perpendicularly to the plane of motion (cf. Figs 5.1). In a quasi-2D bilayer, the secondly quantized Hamiltonian

(2.9) truns out to be given as :

$$\hat{H}=\sum_j\Bigg[\sum_{\mathbf{k}}E_k\hat{a}^\dagger_{j\mathbf{k}}\hat{a}_{j\mathbf{k}}+\frac{1}{S}\sum_{\mathbf{k},\mathbf{p}}U_{j,\mathbf{k-p}}\hat{a}^\dagger_{j\mathbf{k}}\hat{a}_{j\mathbf{p}}+\frac{1}{2S}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}}V_{jj}(|\mathbf{q}-\mathbf{p}|)\hat{a}^\dagger_{j,\mathbf{k+q}}\hat{a}^\dagger_{j,\mathbf{k-q}}\hat{a}_{j,\mathbf{k+p}}\hat{a}_{j,\mathbf{k-p}},$$
$$+\frac{1}{2S}\sum_{j'}\sum_{\mathbf{k},\mathbf{q},\mathbf{p}}V_{jj'}(|\mathbf{q}-\mathbf{p}|)\hat{a}^\dagger_{j,\mathbf{k+q}}\hat{a}^\dagger_{j',\mathbf{k-q}}\hat{a}_{j',\mathbf{k+p}}\hat{a}_{j,\mathbf{k-p}}\Bigg], \tag{5.1}$$

where $j=\pm1$ is the layer index, $S$ is the surface area, $E_k=\hbar^2k^2/2m$ is the energy of free particle, $\hat{a}^\dagger_{\mathbf{k}}$, $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators of particles, and $U$ is the disorder potential which is described by vanishing ensemble averages $\langle U(\mathbf{r})\rangle=0$ and a finite correlation of the form $\langle U(\mathbf{r})U(\mathbf{r}')\rangle=R(\mathbf{r},\mathbf{r}')$. In quasi-2D geometry, at large interparticle separations $r$ the intralayer interaction reads $V_{jj}(r)=d^2/r^3=\hbar^2r_*/mr^3$ [**13**], where $r_*=md^2/\hbar^2$ is the characteristic dipole-dipole distance, $d$ is the dipole moments, and $m$ is the particle mass. In momentum space it can be written as [**122**]

$$V_{jj}(\mathbf{k})=g(1-C|\mathbf{k}|), \tag{5.2}$$

where $g=g_{3D}/\sqrt{2}l_0$ is the 2D short-range coupling constant, $l_0=\sqrt{\hbar/m}$, and is the confinement frequency, and $C=2\pi\hbar^2r_*/mg$.

The interlayer interaction potential ($j\neq j'$) is given by [**94**, **121**, **101**, **231**]

$$V_{jj'}(r)=V_{\uparrow\uparrow,\downarrow\uparrow}(r)=\pm\,d^2\frac{r^2-2\lambda^2}{(r^2+\lambda^2)^{5/2}}. \tag{5.3}$$

The potential $V_{jj'}(r)$ is attractive at large/short distances $r$ depending on the dipoles orientation leading to the formation of an interlayer bound state. The corresponding Fourier transform is given by $V_{\uparrow\uparrow,\downarrow\uparrow}(\mathbf{k})=\int d\mathbf{r}V_{\uparrow\uparrow,\downarrow\uparrow}(\mathbf{r})e^{-i\mathbf{kr}}$. After some algebra, we obtain for the two configurations [**126**] :

$$V_{\uparrow\uparrow,\downarrow\uparrow}(\mathbf{k})=\mp\frac{2\pi\hbar^2}{m}r_*|\mathbf{k}|e^{-|\mathbf{k}|\lambda}, \tag{5.4}$$

For $k\lambda \ll 1$, $V(k) = (2\pi\hbar^2/m)\, r_* k$. This linear dependence on $k$ originates from the so-called anomalous contribution to scattering [**231**]. Now, we address the regime of weak interactions i.e. $mg/2\pi\hbar^2 \ll 1$ and $r_* \ll \xi$, with $\xi = \hbar/\sqrt{mng}$ being the healing length, and sufficiently weak external disorder potential. The Hamiltonian (5.1) can be diagonalized using the Bogoliubov-Huang-Meng transformation [**84**] : $\hat{a}_k = u_k \hat{b}_k - v_k \hat{b}^\dagger_{-k} - \beta_k$, where $\hat{b}^\dagger_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ are operators of elementary excitations. The Bogoliubov functions $u_k, v_k$ are expressed in a standard way : $u_k, v_k = (\sqrt{\varepsilon_k/E_k} \pm \sqrt{E_k/\varepsilon_k})/2$, $\beta_{\mathbf{k}} = \sqrt{n/S} U_k E_k/\varepsilon_k^2$, where $S$ is the surface area. The Bogoliubov excitations energy reads [**126**]

$$\varepsilon_{k\,\uparrow\uparrow,\downarrow\uparrow} = \sqrt{E_k^2 + 2ngE_k(1 - Ck \mp Cke^{-k\lambda})}, \tag{5.5}$$

For $k\lambda \gg 1$, the interlayer DDI vanishes and thus, the spectrum (5.5) reproduces analytically the roton-maxon structure seen in the 2D ordinary dipolar BEC (i.e. single layer) [**50**].

For $k\lambda \ll 1$, one has $\varepsilon_{k\uparrow\uparrow} = \sqrt{E_k^2 + 2ngE_k(1 - 2Ck)}$ which is similar to the single layer spectrum, while $\varepsilon_{k\downarrow\uparrow} = \sqrt{E_k^2 + 2ngE_k}$ is equivalent to the spectrum of a nondipolar BEC. One can conclude that for a bilayer system of dipoles with the antiparallel polarization of dipolar moments in two layers, the interlayer effects is important only for large enough interlayer distance $\lambda$ in stark contrast with the parallel configuration. At low momenta $k \to 0$, the excitations are linear in $k$ (phonon regime) $\varepsilon_k = \hbar c_s k$, where $c_s = \sqrt{ng/m}$ is the sound velocity, it does not depend neither on the interlayer DDI nor on the intralayer DDI regardless the value of $\lambda$ and the polarization directions (see Fig.5.2). At higher momenta, it becomes quadratic as in the nondipolar case (see Fig.5.2) for any interlayer distance. The dispersion relation changes it behavior and exhibits roton-maxon structure at intermediate $k$ as is shown in Figs.5.2.(a) and (b). The position and the energy of the roton strongly depend on the effect of varying polarization orientation and interlayer DDI. For instance, the roton can be formed in the dispersion spectrum for very small $\lambda$ in the configuration ↑↑, while in the arrangement ↓↑, the roton can be observed only for large $\lambda$. The roton instability can be identified by $d\varepsilon_k/dk|_{k=k_r} = 0$. The roton minimum touches zero at $k_r = 0$ leading to roton instability. Another feature of the spectrum (5.5) is that it is independent of the random potential.

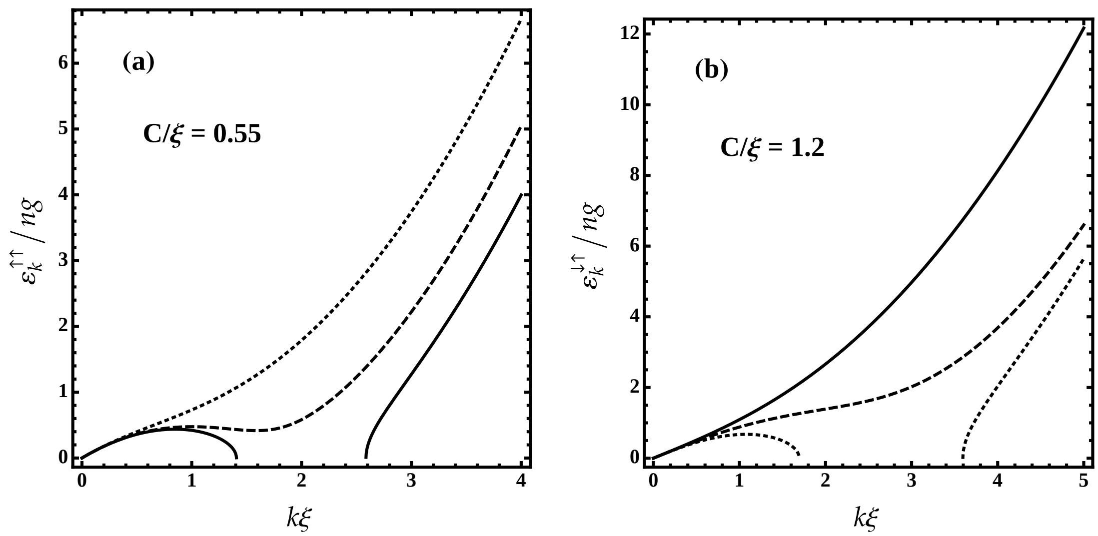


FIGURE 5.2: The Bogoliubov excitations spectra (a) $\varepsilon_{k\uparrow\uparrow}$ and (b) $\varepsilon_{k\downarrow\uparrow}$ from Eq.(5.5) for several values of $\lambda$. Solid line : $\lambda = 0.05$. Dashed line : $\lambda = 0.2$. Dotted line : $\lambda = 1.1$ [**126**].

## 5.2 Noncondensed density

At zero temperature, the noncondensed density is defined as $\tilde{n} = \tilde{n}_0 + n_R$, where

$$\tilde{n}_0 = \frac{1}{2}\int \frac{d\mathbf{k}}{(2\pi)^2}\left[\frac{E_k + gn(1 - Ck \mp Cke^{-k\lambda})}{\varepsilon_k} - 1\right], \tag{5.6}$$

accounts for the quantum fluctuations contribution to the noncondensed density. For vanishing $\lambda$, the $\tilde{n}_0$ reduces to that of a single layer system. The glassy fraction reads

$$n_R = n \int \frac{d\mathbf{k}}{(2\pi)^2} R_k \frac{E_k^2}{\varepsilon_k^4}, \tag{5.7}$$

Let us now complete this work by looking at the disorder relevant regime, focussing on correlated Gaussian environment. We try to understand the interplay of interlayer DDI, disorder effects, and polarization direction. As in the case of a single layer, integrals (5.6) and (5.7) over infinite momentum space are logarithmically divergent and require a special care. Therefore, to be quantitative, we solve them numerically using the standard Monte Carlo method [**96**] in the limit $k \ll 1/r_*$. Figure.5.3 shows that in the setup $\uparrow\uparrow$, the glassy fraction $n_R$ is decreasing with $\lambda$ indicating that the interlayer effects lead to tune the disorder fluctuations ensuring the existence of the condensate even for relatively large disorder strength. Conversely, in the arrangement $\downarrow\uparrow$, when the two layers are well separated ($\lambda/\xi \lesssim 0.7$), $n_R$ substantially increases results in the disappearance of the condensate. The disorder fraction becomes important when the roton minimum is close to zero (diverges at $k_r = 0$) yielding the transition to a novel quantum phase [**50**] (see right panels). For $\sigma > \xi$, the disorder effects is not important in both configurations regardless the polarization directions.

We observe also that in the absence of the random external potential i.e. $R = 0$, the total noncondensed density, $\tilde{n}^{\uparrow\uparrow}$, lowers for $\lambda/\xi \lesssim 0.2$ and then grows logarithmically for $\lambda/\xi > 0.2$, where the condensate becomes completely depleted due to the DDI (see bottom panel left). However, the situation is inverted in the configuration $\downarrow\uparrow$ (see bottom panel rigth). The presence of the disorder potential augments the condensate depletion notably for large $\lambda$ as is seen in the same figure.5.3(bottom panel).

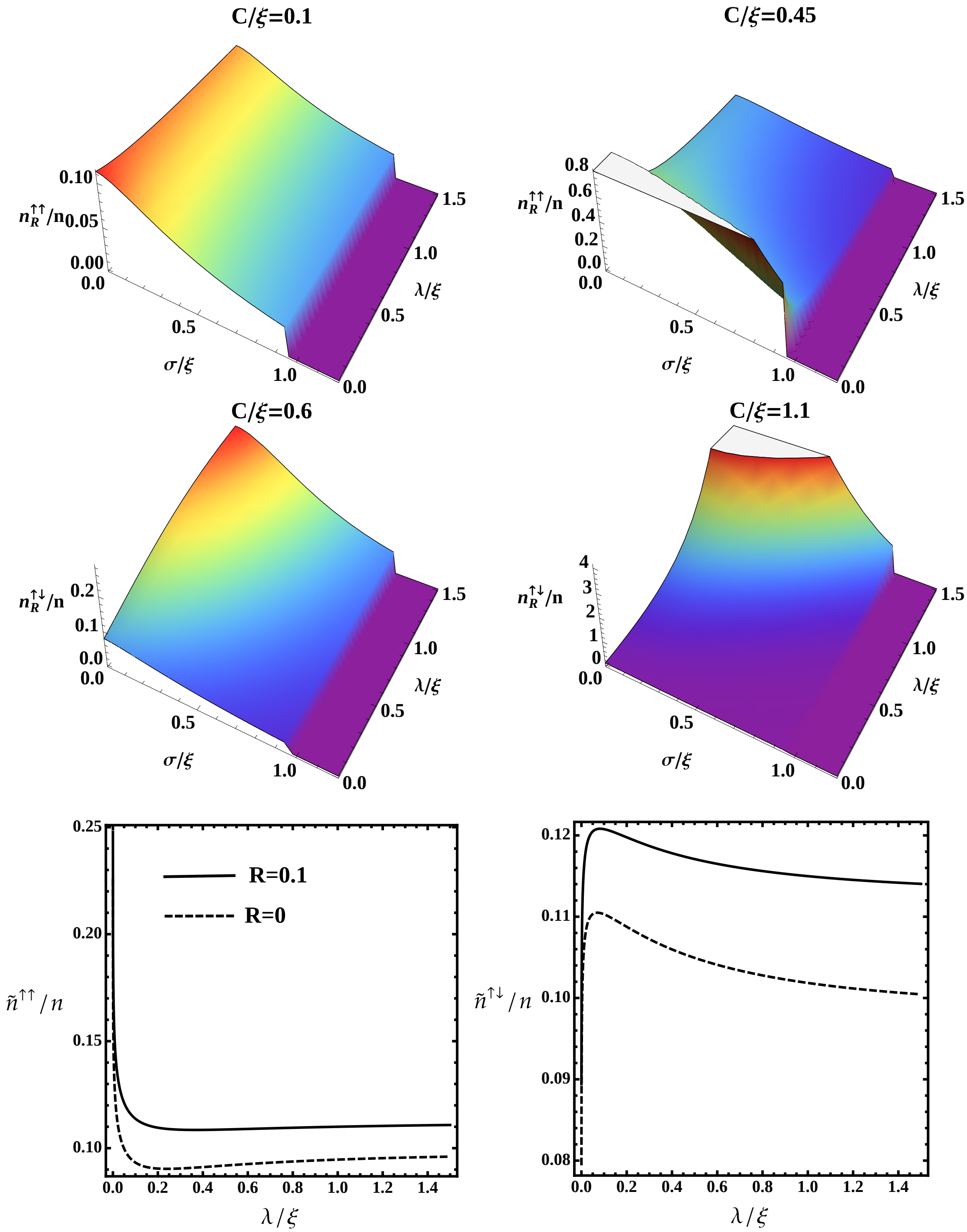


FIGURE 5.3: Glassy fractions $n_R^{\uparrow\uparrow}$ (top panel) and $n_R^{\downarrow\uparrow}$ (middle panel) from Eq.(5.7) as a function of $\lambda/\xi$ and $\sigma/\xi$. Parameters are : $mg/4\pi\hbar^2 = 0.01$ and R=0.1. Total condensate depletion as a function of $\lambda/\xi$ for $C/\xi = 0.3$ and $\sigma/\xi = 0.2$ (bottom panel). Here $R = R_0/ng^2$ [**126**].

## 5.3 One-body density matrix

At zero temperature, the one-body density matrix is defined as [**125**] $g_1(\mathbf{r}) = n_c + \int \tilde{n}\, e^{i\mathbf{k}.\mathbf{r}} d\mathbf{k}/(2\pi)^2$, where $n_c$ is the condensed density. The numerical simulation of this integral reveals that when $C/\xi$ is small, the first order correlation functions $g_1^{\uparrow\uparrow}(\mathbf{r})$ and $g_1^{\downarrow\uparrow}(\mathbf{r})$ decay at large distance and go to their constant value $n$ (see Fig.5.4 left panels). This is a genuine signature of the existence of a true BEC at zero temperature in quasi-2D geometry. In such a case the interlayer distance and the polarization direction play a minor role ; they only slightly shift $g_1^{\uparrow\uparrow}(\mathbf{r})$ and $g_1^{\downarrow\uparrow}(\mathbf{r})$ near the center. For large $C/\xi$ (i.e. when the roton minimum close to zero) and depending on the interlayer space, $g_1^{\uparrow\uparrow}(\mathbf{r})$ and $g_1^{\downarrow\uparrow}(\mathbf{r})$ display oscillations at small distances (see right panels figure 5.4). This signals the destruction of the off-diagonal long-range order (i.e., BEC). One can conclcude that below a certain critical intralayer coupling $\lambda_c$ which relies on the polarization direction, the BEC remains stable. For $\lambda > \lambda_c$, the system undergoes instability opening the door to a new phase transition.

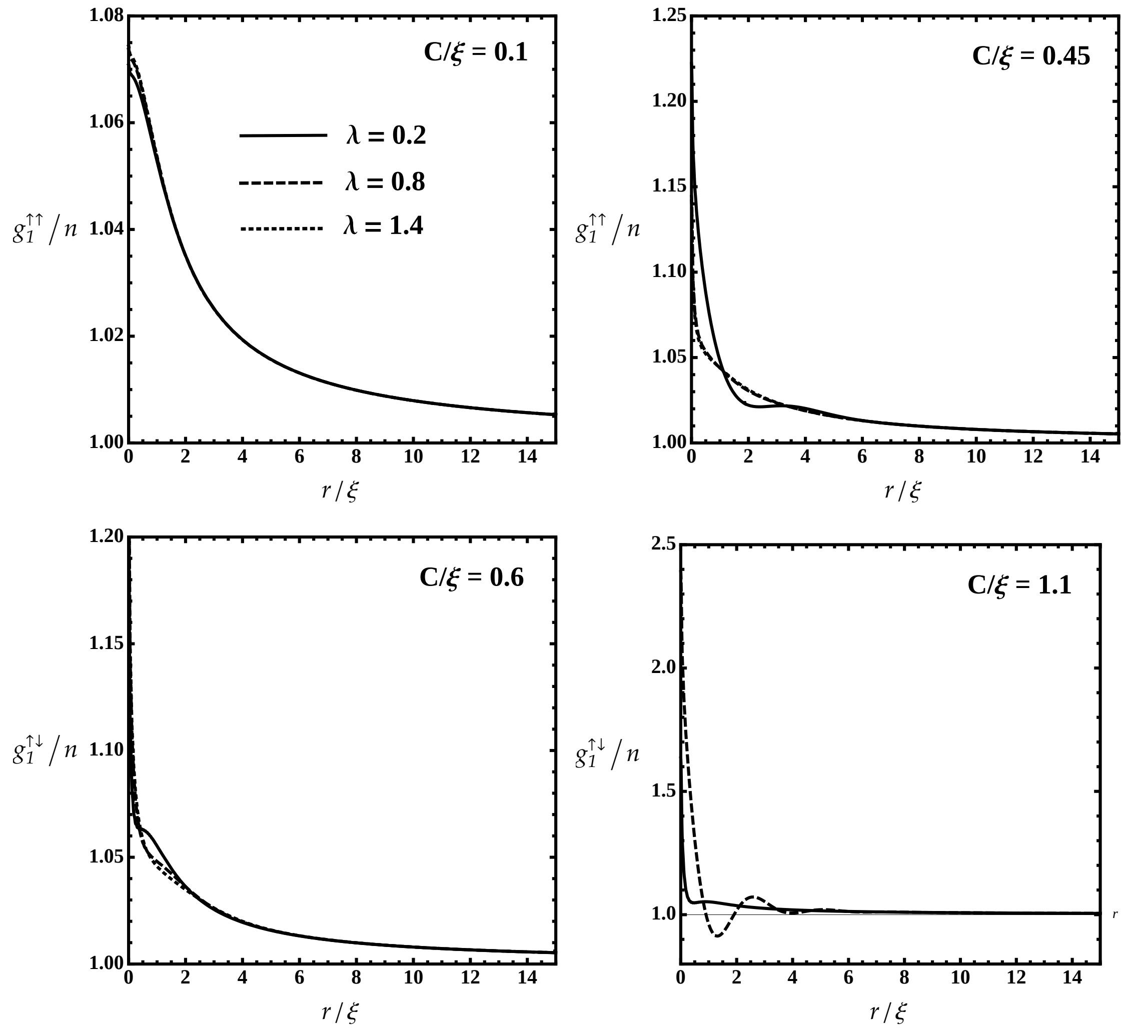


FIGURE 5.4: One-body density correlation function for several values of interlayer separation $\lambda$. Parameters are : $mg/4\pi\hbar^2 = 0.01$ and $\sigma/\xi = 0.2$[**126**].

## 5.4 Superfluidity in a quasi-2D bilayer

Quasi-2D superfluidity can be well understood in the framework of the Berezinskii-Kosterlitz-Thouless theory [**158**, **194**, **232**]. The relation between the disorder potential, DDI and the superfluidity in quasi-2D geometry has been explained in details in our recent papers [**158**, **194**, **232**]. The superfluid fraction $n_s/n$ is defined as

$n_s/n = 1 - n_n/n - n_R^{th}/n$ [**29**], where

$$\frac{n_n}{n} = \frac{2}{dTn} \int \frac{d^dk}{(2\pi)^d} \frac{E_k}{4\sinh^2(\varepsilon_k/2T)}, \tag{5.8}$$

is the normal fraction of the superfluid.

And

$$\frac{n_{Rth}}{n} = \frac{2}{dTn} \int \frac{d^dk}{(2\pi)^d} \frac{nR_k E_k^2}{\varepsilon_k^3} \coth\left(\varepsilon_k/2T\right), \tag{5.9}$$

represents the disorder thermal contribution to the superfluid fraction. At temperatures $T \to 0$, it reduces to $n_{Rth}/n = 4n_R/dn$. In quasi-2D one has $n_{Rth} = 2n_R$ which leads to considerably lower the superfluid fraction. Another important remark is that the superfluid fraction is no longer a tensorial quantity in opposite to the 3D dirty dipolar Bose gas case [**16**, **57**, **76**, **84**, **94**, **96**, **125**, **178**] since the dipoles are assumed to be perpendicular to the plane. However, in the case of dipolar BECs with tilted dipoles, the superfluid becomes anisotropic.

Figure.5.5 shows that $n_{Rth}$ is increasing with temperature in both configurations. We see also that $n_{Rth}^{\uparrow\uparrow}$ lowers with $\lambda$ at any temperatures. For instance, at temperatures $T/ng < 0.1$, $n_{Rth}^{\uparrow\uparrow} \approx n$ at $\lambda \geq \xi$ which means that the whole system becomes practically superfluid. Whereas, in the configuration $\downarrow\uparrow$, $n_{Rth}^{\downarrow\uparrow}$ augments with both temperature and interlayer spacing. For example, at $T/ng > 3$ and for $\lambda > 1.2\,\xi$, the superfluid fraction vanishes. This implies that the condensed particles are localized prohibiting the superfluid flow results in the formation of the so-called Bose glass phase.

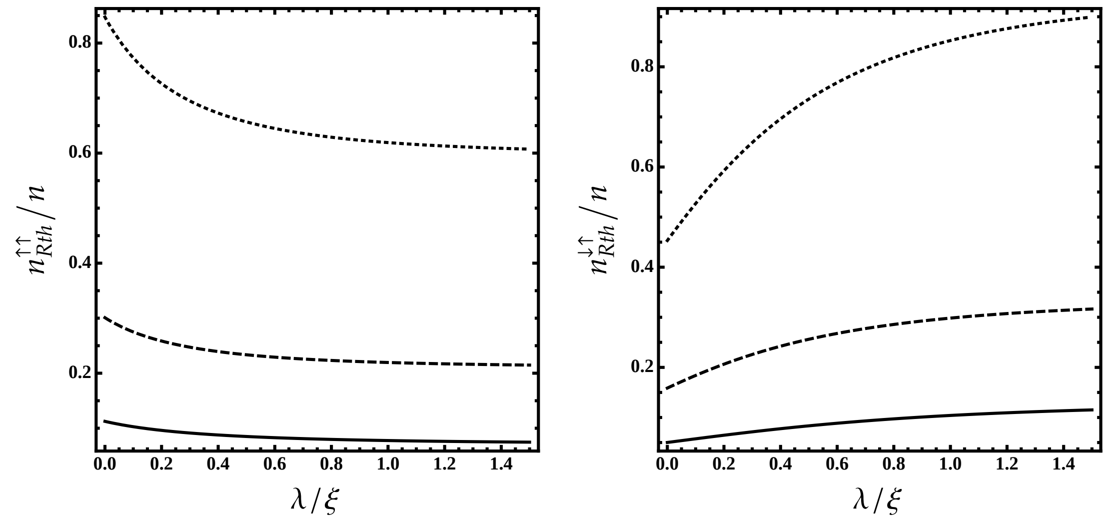


Figure 5.5: Disorder thermal corrections to the superfluid fraction from Eq.(5.9), as a function of $\lambda/\xi$. Parameters are : $mg/4\pi\hbar^2 = 0.01$, $\sigma/\xi = 0.2$, $C/\xi = 0.1$ (left) and $C/\xi = 0.6$ (rigth). Solid line : $T/ng = 0.1$. Dashed line : $T/ng = 1$. Dotted line : $T/ng = 3$. [**126**]

## 5.5 Conclusion

In this chapter we studied the implications of varying polarization orientation and interlayer DDI on the propertites of quasi-2D bilayered dipolar Bose gases in a random environment at zero temperature. We calculated analytically and numerically the dispersion relation, the condensate depletion, the first-order correlation function and the superfluidity using the Bogoliubov theory. Our analysis revealed that the competition between the disorder potential and the interlayer DDI may significantly enhance the rotonization and the glassy fraction inside the condensate. In the parallel configuration, interlayer coupling may lead to delocalize atoms rising both the condensed and the superfluid fractions. However, the situation is completely different in the antiparallel arrangement where the condensate and the superfluid fraction are decreased. This important result has never been addressed before in the literature. We showed in addition that in the roton regime, the long-range order is distroyed and hence, the condensate and the superfluidity disappear in both configurations. Whereas, for small values of DDI, the coherence of the BEC remains insensitive to interlayer distance. It was found also that a true BEC exists up to certain critical temperature which depends on the interlayer distance and the polarization direction. We believe that our results provide new insights to understand these exotic systems, and opening new prospects for realizing dirty dipolar Bose gases.

## CONCLUSIONS AND PERSPECTIVES

In this thesis, we systematically studied dipolar Bose gases in weak disorder potential in various contexts which led to an original and fascinating physics in each case. To this end, we used both analytical and numerical methods based on the Bogoliubov-Huang-Meng theory. This formalism provides a successful quantitative description of the homogeneous dipolar BEC in weak disorder.

In chapter 1, we briefly discussed the main ingredients of weakly interacting dipolar BEC and introduced the Bogoliubov theory that allows us to model such systems at both zero and finite temperatures. We assessed the quantum and thermal fluctuations and various thermodynamic quantities.

In chapter 2, we studied the properties of a dipolar Bose gas in the presence of a weak random potential with a Gaussian correlation at finite temperature. During this journey we took the opportunity to review some statistical properties of a disorder potential. Within the Bogoliubov-Huang-Meng approach, we derived useful analytical expressions for the condensate fluctuations due to disorder, as well as the corresponding corrections to the noncondensed and the anomalous densities, the ground state energy and the superfluid fraction in the homogeneous case. Furthermore, we pointed out that the interplay of the DDI and the external random potential makes both the BEC and the superfluidity anisotropic. Such an anisotropy is found to be weak in the parallel component of the superfluid density while it becomes strong in the perpendicular component. We also demonstrated that for a strong disorder strength the system introduces an unusual quantum regime where the superfluid fraction is smaller than the condensate fraction.

Until now, there has been very little work for ultracold disordered dipolar gases with TBI. With this in mind, in chapter 3 we extended the Bogoliubov-Huang-Meng theory applicable to the dipolar bosonic gas with TBI subjected to a correlated Gaussian disorder at both zero and finite temperatures. We showed that the intriguing interplay of the disorder, DDI and TBI plays a fundamental role in the physics of the system. We pointed out in particular that the DDI may lead to arrest transport of atoms under disorder augmenting the glassy fraction inside the condensate, while the presence of the TBI may lead to a diffusive motion of particles. Our results revealed that the one-body density matrix is a decreasing function with the TBI. We calculated in addition the chemical potential of a disordered dipolar BEC and ultraviolet divergences are removed by means of dimensional regularization. The combined effects of the DDI, TBI, and temperature found to crucially affect the chemical potential and the ground state energy of the system. In the absence of the TBI, we recovered the results obtained for disordered BEC with two-body interactions.

In chapter 4, we presented a detailed study of the physics of a quasi-2D dipolar BEC. By lowering the interparticle contact interaction, we bring the system into a new finite-momentum minimum which is called roton. We showed that such a rotonization may strongly enhance the fluctuations and suppress the superfluidity leading to uncover a novel quantum phase transition. On the other hand, we investigated a dilute 2D dipolar Bose gas with dipoles oriented perpendicularly to the plane subjected to a weak Gaussian correlated disorder potential in the roton regime. Our analysis signifies that when the strength and the correlation length of the disorder is very small, the disorder potential strongly enhances the fluctuations and the thermodynamic quantities. In contrast to the 3D case, the superfluid density is isotropic and found to be strongly depressed in the roton's region results in the localization of the particles in the respective minima of the external random potential due to the peculiar interplay of rotonization induced by DDI and disorder.

Finally in chapter 5, we studied the implications of varying polarization orientation and interlayer DDI on the propertites of quasi-2D bilayered dipolar Bose gases in a random environment at zero temperature. Here the dipoles are oriented perpendicularly to the layers and in parallel/antiparallel configurations. We calculated analytically and numerically the dispersion relation, the condensate depletion, the first-order correlation function and the superfluidity. Our analysis revealed that the competition between the disorder potential and the interlayer DDI may significantly enhance the rotonization and the glassy fraction inside the condensate. In the parallel configuration, interlayer coupling may lead to delocalize atoms rising both the condensed and the superfluid fractions. However, the situation is completely different in the antiparallel arrangement where the condensate and the superfluid fraction are decreased. This important result has never been addressed before in the literature. We showed in addition that in the roton regime, the long-range order is distroyed and hence, the condensate and the superfluidity disappear in both configurations as in the case of a disordered single component BEC. Whereas, for small values of DDI, the coherence of the BEC remains insensitive to interlayer distance. It was found also that a true BEC exists up to certain critical temperature which depends on the interlayer distance and the polarization direction.

Promising candidates for the experimental realization of such dirty dipolar BECs are atomic species with highly magnetic dipolar interaction such as Dy (magnetic moment $10\mu_B$) [**10**] or polar heteronuclear molecules such as KRb (magnetic moment 0.6 Debye)[**13**].

# Outlook

There is significant scope for future work in disordered dipolar Bose gases. One promising appplication of our work would be the study of the role of the TBI in the Anderson localization of Bogoliubov quasi-particles. Experimentally, the Anderson localization of Bogoliubov quasi-particles should be observable with Bragg-spectroscopy techniques. Our results open new prospects for understanding transport properties which could be studied by a direct numerical simulation of the time-dependent nonlocal GP equation in a disordered potential. Another promising avenue we are pursuing is the implementation of a disorder potential in the droplet state. Finally, an important extension of our work would be to analyze the possible formation of the superglass state in 2D disordered weakly-interacting dipolar Bose gas loaded in optical lattices by developping beyond-mean-field approach. The study of the effects of weak disorder in 2D dipolar gas with tilting angle where the interaction in the plane becomes anisotropic is also an interesting avenue that could be explored in the future.

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