Ministry of Higher Education and Scientific Research

Hassiba Benbouali University of Chlef

Faculty of Exact Sciences and Informatics

Department of Physics

THESIS

Presented for obtaining the diploma of

DOCTORATE

Field: Physics

Speciality: Theoretical physics

By

Halima BELKROUKRA

Theme:

Binary mixtures with a time-dependent variational approach: new families of breathers in two-component condensates

---

Examining Committee:

Hafid KHALFOUN	Professor	University of Chlef	President
Kamal BENCHEIKH	Professor	University of Setif	Examinator
Ahmed BOUHEKKA	Professor	University of Tissemsilt	Examinator
Ahmed HOCINE	MCA	University of Chlef	Examinator
Houria CHAACHOUA SAMEUT	MCA	University of Chlef	Supervisor
Mohamed BENAROUS	Professor	University of Chlef	Co-Supervisor

July 2022

Contents

List of Figures

Nomenclature

PDE: partial differential equation (PDE)
BEC: Bose Einstein Condensation.
DT : Darboux transformation
GPE : Gross-Pitaevskii Equation.
CGPE : Coupled Gross-Pitaevskii Equation.
RW: Rogue Waves.
KM: Kuznetsov-Ma breather .
AB: Akhmediev breather.
NLS: nonlinear Schrödinger equation.
DS: dark soliton.
BS: bright soliton.

Acknowledgments

This work is carried out within the laboratory of Theoretical Physics and Physics of Materials (LPTPM) of the Faculty of Exact Sciences and Informatics of the Hassiba Benbouali University of Chlef in Algeria.
I am very grateful to my supervisor, Dr. Chaachoua Sameut Houria, for her patience, guidance, support, and for her constant advice.
I warmly thank my co-supervisor, Pr. Benarous Mohamed, for his guidance and support. I am also thankful for his valuable suggestions, as I benefited greatly from his wealth of knowledge.
My heartfelt thanks also go to Pr. Khalfoun Hafid, who honored us by agreeing to chair the jury for this thesis.
I would like to thank Dr. Ahmed Hocine for agreeing to be part of the jury, and I am also grateful to him for his support.
My thanks also go to Pr. Bouhekka Ahmed for his interest in this work by agreeing to participate in the jury of this thesis.
I warmly thank Pr. Bencheikh Kamal, for his interest in this work by agreeing to take part in the jury of this thesis.
I would also like to thank all members of LPTPM. I am grateful for my parents, whose constant love and support keep me motivated and confident. My accomplishments and success are because they believed in me. My Deepest thanks to my siblings.

Overview

Our work is organized as follows.

In chapter 1, we give a short historic and background about the nonlinear Schrödinger equation (NLSE) and its solitonic solutions.

In chapter 2, we present the Balian-Vénéroni time dependent variational principle, which is the main tool to derive the Gross-Pitaevskii equation (GPE) and its generalizations in a mean field framework. Since these equations are highly nonlinear, they require special analytic tools. We present the general formalism of the Darboux transformation method and the Lax pair method. For readers who are not familiar with these methods, we present a simple example.

In chapter 3, we focus on two component condensates where we find solitonic solutions of the coupled Gross-Pitaevskii equations (CGPE). By transforming our model to a Manakov system via similarity transformation and employing Darboux transformation with zero seed, we observe that the introduction of an external trap leads to sudden shoots up in the atomic density indicating onset of dynamical instability.

We pursue our analysis in chapter 4 by finding another type of solitons, namely the Peregrine solitons. The Darboux transformation is used in two cases. The symmetric case with the same seed solutions and the nonsymmetric case. One also observes the onset of dynamical instability as the frequency of the harmonic trap is varied. By a specific choice of the spectral parameters, we show that these solitons may be stabilized. In chapter 5, we generalize our approach by letting free the spectral parameters. We find general families of solitonic solutions parametrized by the spectral parameters. We derive not only the Peregrine solitons found previously, but also the standard families of Ma and Akhmediev breathers as well as new general breathers and rogue waves. In all these cases, we show that, by modulating the trap frequency, we are be able to stabilize the solitons against dynamical instability.

In the last part of our work, we gather some conclusions and perspectives.

We have added an appendix at the end of the thesis where we present detailed derivations of the variational equations in the two component dilute Bose gas case.

Chapter 1

General Introduction

General Introduction

Nature is nonlinear, and that is why we find a large class of nonlinear problems in different disciplines of science, engineering, and technology. Most of the nonlinear phenomena are modeled in the form of nonlinear ordinary and partial differential equations PDEs. The discretization of nonlinear ordinary and partial differential equations provide the system of nonlinear equations. PDEs are often referred as Equations of Mathematical Physics (or Mathematical Physics but it is incorrect as Mathematical Physics is now a separate field of mathematics) because many of PDEs are coming from different domains of physics (acoustics, optics, elasticity, hydro and aerodynamics, electromagnetism, quantum mechanics, seismology etc). However PDEs appear in other fields of science as well (like quantum chemistry, chemical kinetics), some PDEs are coming from economics and financial mathematics, or computer science.

A nonlinear partial differential equation(PDE), such as the Korteweg de Vries (KdV) equation, nonlinear Schrödinger equation(NLSE), and sine-Gordon equation, are associated with an important mathematical property called integrability, where the system possesses infinitely many integrals of motion.

Soliton equations, in the mathematical sense of the term, provide remarkable examples of integrable systems with an infinite number of degrees of freedom. This is the reason why they have interested mathematicians , so much so that many works on solitons are strongly oriented towards the mathematical aspects of the theory. The discovery of solitons was one of the most important developments in the field of nonlinear dynamics. Solitons arise in a nonlinear partial differential equation. The discovery has also led to the development of analytical methods, devoted to solve these nonlinear PDEs . Getting the exact solutions of these integrable systems has become one of the important research topics in theory and practical applications. Quite a few approaches for finding exact solutions of such nonlinear systems are well established, such as the inverse scattering method [1], the Darboux transformation(DT) method [2, 3], the similarity transformation method [13, 14, 15, 16]. Among these approaches, the DT is well known to be a powerful method for finding exact solutions of integrable systems [2].

The Darboux transformation, or analogously Bäcklund or dressing transformation, applies only to systems of linear differential equations and cannot be applied directly to nonlinear differential equations. To be able to apply the DT to certain nonlinear differential equations,one finds a linear system of equations that is equivalent to a nonlinear differential equation. The relation between the linear system and the nonlinear differential equation is established through a consistency condition satisfied by the linear system. The Darboux transformation is then applied to the linear system resulting in transforming the equivalent nonlinear equation as well. The linear system is usually represented in terms of a pair of matrices called the Lax pair which must satisfy a consistency condition that is equivalent to the differential equation at hand. The difficulty is usually in finding this Lax pair. In addition to the Lax pair, one also needs to know an exact solution of the nonlinear differential equation. This exact solution is then used as a seed for the Darboux transformation to generate other exact solutions

$$\overbrace{\textit{Seed Solution} \rightarrow \textit{System} \rightarrow \textit{Lax pair}}^{\textbf{Integrable System}} \Rightarrow \textit{Darboux Transformation}$$

Analytic solutions of the nonlinear evolution equations, such as solitons, breathers and rogue waves, have received a large of research activities in many realms. When the effect of dispersion and nonlinearity is balanced in nonlinear waves during propagating, solitons will be formed.These waves keep their features (amplitudes, speeds and so on) unchanged, during their propagation and after interacting with each other. In many cases, they are considered as the ideal solution models in physics [11]. Solitons are localized wave packets that balance the wave dispersion with a focusing nonlinearity. They maintain their shape and amplitude while propagating with a constant velocity. When two or more solitons collide with each other, the solitons emerge from the collisions unchanged in shape, amplitude, and velocity, a particle-like behavior that earned them their name. Solitons are ubiquitous in physical systems, such as water waves, optical waves, plasma waves, matter waves, and biological systems, such as DNA.

As particular solutions of nonlinear systems, breathers propagate steadily and localize in either time or space, especially Akhmediev breather (AB)

[17, 18]and Kuznetsov-Ma breather (KM) [19]. Another special type of analytic solutions are the rogue waves which are localized in both space and time, and they have peak amplitudes usually are more than twice the background wave height [20, 21]. Besides, rogue waves appear from nowhere and disappear without a trace. But now, the study of the rogue waves is still only in its infancy. The mechanisms and probability of its occurrence are not clear. Because observing of rogue waves on the ocean is very difficult and dangerous, unreliable and few records and observations are available. Recently efforts have been made to explain the RW excitation through a nonlinear process. It has been found that the NLSE can describe many dynamical features of the RW. Certain kinds of exact solutions of NLSE have been considered to describe possible mechanism for the formation of RWs such as Peregrine soliton, time periodic breather or Ma soliton (MS) and space periodic breather or Akhmediev breather (AB) . As a consequence attempts have been made to construct RW solution through different methods for the NLSE and its higher derivative generalizations. One way of obtaining RW solution or Peregrine soliton for a given system is to first construct a breather solution, either AB or MS. From the latter, the RW solution can be deduced in an appropriate limit.

1.1 Bose-Einstein Condensation

1.1.1 A brief summary

Bose-Einstein Condensate is a new state of matter. This new phenomenon is reflected by the fact that a fraction of the total number of particles tend to occupy the lowest energy. When a gas of bosonic atoms is cooled below a critical temperature T_c, a large fraction of the atoms condenses into the lowest quantum state. This phenomenon was first predicted by Albert Einstein in 1925[23] and is a consequence of quantum statistics. It has been realized experimentally in 1995 in alkali gases. The award of the 2001 Nobel Prize in Physics to E. Cornell, C. Wieman, and W. Ketterle acknowledged the importance of the achievement. In this new state of matter, which is very dilute and at very low temperature,a macroscopic fraction of the atoms behave as a coherent matter wave similar to the coherent light wave produced by a laser. In the dilute limit, the condensate is well described by a mean-field theory and a macroscopic wave function. The properties of these gaseous quantum fluids constitute new a wide domain of research in low and high energy physics.
At almost zero temperature, when the lowest energy level is macroscopically occupied and when the gas is dilute enough such that particle interactions are weak, a common model to describe BEC is GPE. In order to describe two component condensates this model can be easily generalized to two coupled Gross Pitaevskii equations (CGPEs). The GPE model has proven to be a good description for many static and dynamical properties both for single condensate systems[30] and for condensate mixtures[31], even though thermal contributions and quantum fluctuations are not taken into account.

1.2 Soliton

1.2.1 An overview of the concept of solitons

The concept of solitons (solitary waves) plays a profoundly important role in modern physics and applied mathematics, extending beyond the bounds of these disciplines.The birth of the soliton was first recorded by John Scott Russell (1834) when he was investigating how improve the efficiency of designs for barges in canals[68]. After about 50 years being ignored this interesting discovery, Diederik Korteweg and his PhD student, Hendrik de Vries derived a nonlinear partial differential equation confirming mathematically the existence of Scotts solitary waves. They show that the change of the waves height in time is determined by nonlinear and dispersive effects. However, they did not find a general solution. In 1955, by means of the Los Alamos MANIAC computing machine, Enrico Fermi, John Pasta and Stanislaw Ulam (FPU) were exploring the energy equipartition in a slightly nonlinear mechanical system. It was expected that if all the energy was initially introduced in a single mode, the small nonlinearity would cause energy redistribution among all the modes, but this did not happen and the energy was periodically returning to the initially excited mode. Motivated to find an explanation for this phenomenon, Norman Zabusky and Martin Kruskal (1965) approximated the FPU system in the continuum limit using the KdV equation. They solved the equation numerically and reported that the solitary waves can pass through each other without change in their shape or speed, the only change found was a small phase shift after a collision. Zabusky and Kruskal then introduced for the first time the term soliton for this solitary waves, thus highlighting its particle-like behavior. In 1967 was discovered a method by Clifford Gardner, John Greene, Martin Kruskal and Robert Miura, to finding exact solutions (including soliton) of the KdV equation. At present this method is known as the inverse scattering method (ISM). It was later found that the ISM is more general and allows sought exact solitons in many others integrable nonlinear systems. In 1972 Vladimir Zakharov and Alexei Borisovich Shabat generalized the inverse scattering method and solved the nonlinear Schrödinger equation (NLSE). They demonstrated both integrability and existence of soliton solutions. The NLSE was found as a fundamental model in many important applications, in hydrodynamics, nonlinear optics, nonlinear acoustics, plasma waves, Bose-Eistein condensate, inter alia. Next, in 1973, Mark Ablowitz, David Kaup, Alan Newell and Harvey Segur also applied ISM finding the solitons in the sine-Gordon (SG) equation which admits kink and anti-kink solitons. The SG equation also appears in many physical applications, including the propagation of crystal defects and the propagation of fluxons (quantum units of magnetic flux) on long Josephson transmission lines. In 1973 Akira Hasegawa was the first to suggest that solitons could exist in optical fibers, due to a balance between self-phase modulation and anomalous dispersion

Figure 1.1: Recreating Russells soliton .
(Photography Cliris Eilbeck and Heriot-Watt University. 1995).

Solitons are the solutions of certain nonlinear partial differential equations, with special properties. Because of a balance between nonlinear and linear effects, the shape of soliton wave pulses does not change during propagation in a medium.
A soliton is a wave packet (a pulse) that maintains its shape while traveling at a constant speed, It arises because of the balance between the effects of the nonlinearity and the dispersion. Dispersion is the phenomenon in which the phase velocity of a wave depends on its frequency. Indeed,every wave packet can be thought of as consisting of plane waves of several different frequencies. Because of dispersion, waves with different frequencies will travel at different velocities and the shape of the pulse will therefore change over time.

Besides for a conservative system the soliton behaves like a particle i.e.

- It must maintain its shape when it travels at constant speed, reflecting a characteristic of the so-called solitary wave.

- When a soliton interacts with another soliton, it emerges from the collision unchanged except for a possibly phase shift, in other words, the amplitude, shape and velocity are conserved after the collision.

Mathematically, there is a difference between solitons and solitary waves. Solitons are localized solutions of integrable equations, while solitary waves are localized solutions of non-integrable equations. Another characteristic feature of solitons is that they are solitary waves that are not deformed after collision with other solitons. Thus the variety of solitary waves is much wider than the variety of the true solitons. Some solitary waves, for example, vortices and tornados are hard to consider as waves. For this reason, they are sometimes called soliton-like excitations. To avoid this bulky expression we shall often use the term soliton in all cases.

1.3 Classification of solitons

There are a few ways to classify solitons[73]. For example, as we known, there are topological and non-topological solitons. Independently of the topological nature of solitons, all solitons can be divided into two groups by taking into account their profiles: permanent and time dependent. For example, kink solitons have a permanent profile (in ideal systems), while all breathers have an internal dynamics, even, if they are static. So, their shape oscillates in time. The third way to classify the solitons is in accordance with nonlinear equations which describe their evolution. Here we discuss common properties of solitons on the basis of the four classification.

1.3.1 Classical and quantum solitons

A rough description of a classical soliton is that of a solitary wave which shows great stability in collision with other solitary waves. A solitary wave, as we have seen, does not change its shape, it is a disturbance $u(x-ct)$ which translating along the x-axis with speed c.[72] A remarkable example for this type is soliton solution for linear dispersion less equation or KdV equation. Quantum solitons for physical systems governed by quantum attractive nonlinear Schrödinger model and quantum Sine-Gordon model.

1.3.2 topological and non-topological solitons

In non-topologically soliton, for example the water canal solitary solution to the KdV equation means that the boundary conditions at infinity are topologically the same for the vacuum as for the soliton. The vacuum can be

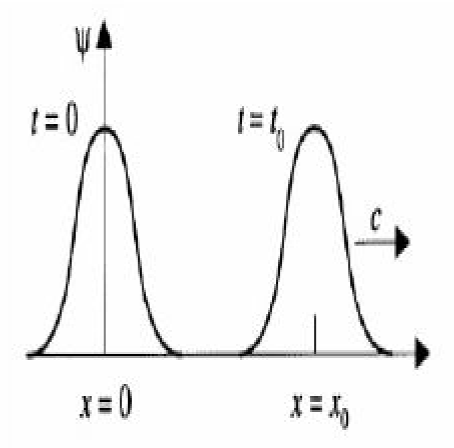

Figure 1.2: Classical soliton

non-degenerate but an additive conservation law is required. But topologically soliton need a degenerate vacuum. The boundary conditions at infinity are topologically different for the solitary wave than for a physical vacuum state. The solitary of topological soliton is due to the distinct classes of vacuum at the boundaries where these boundary conditions are characterized by a particular correspondence (mapping) between the group space and coordinate space, and because these mappings are not continuously deformable into one another they are topologically distinct.

In mathematics and physics, a topological soliton or a topological defect is a solution of a system of partial differential equations or of a quantum field theory homotopically distinct from the vacuum solution; it can be proven to exist because the boundary conditions entail the existence of homotopically distinct solutions. Typically, this occurs because the boundary on which the boundary conditions are specified has a non-trivial homotopy group which is preserved in differential equations; the solutions to the differential equations are then topologically distinct, and are classified by their homotopy class. Topological defects are not only stable against small perturbations, but cannot decay or be undone or be de-tangled, precisely because there is no continuous transformation that will map them (homotopically) to a uniform or "trivial" solution.[77]

In condensed matter physics, the theory of homotopy groups provides a natural setting for description and classification of defects in ordered systems. Topological methods have been used in several problems of condensed matter theory. Ponaru and Toulouse used topological methods to obtain a condition for line (string) defects in liquid crystals can cross each other without entanglement. It was a non-trivial application of topology that first led to the discovery of peculiar hydrodynamic behavior in the A-phase of superfluid Helium-3.

1.3.3 Different types of solitons

About thirty years ago a remarkable discovery was made in Los Alamos. Enrico Fermi, John Pasta, and Stan Ulam were calculating the flow of energy in a one dimensional lattice consisting of equal masses connected by nonlinear springs. They conjectured that energy initially put into a long-wavelength mode of the system would eventually be thermalized, that is, be shared among all modes of the system. This conjecture was based on the expectation that the nonlinearities in the system would transfer energy into higher harmonic modes. Much to their surprise the system did not thermalize but rather exhibited energy sharing among the few lowest modes and long time near recurrences of the initial state. This discovery remained largely a mystery until Norman Zabusky and Martin Kruskal started to investigate the system again in the early sixties. The fact that only the lowest order (long-wavelength) modes of the discrete Fermi-Pasta-Ulam lattice were active led them in a continuum approximation to the study of the nonlinear partial differential equation [78]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

This equation (the KdV equation) had been derived in 1885 by Korteweg and de Vries to describe long-wave propagation on shallow water. But until recently its properties were not well understood. From a detailed numerical study Zabusky and Kruskal found that stable pulse-like waves could exist in a system described by the KdV equation. A remarkable quality of these solitary waves was that they could collide with each other and yet preserve their shapes and speeds after the collision. This particle-like nature led Zabusky and Kruskal to name such waves solitons. The first success of the soliton concept was explaining the recurrence in the Fermi-Pasta-Ulam system. From numerical solution of the KdV equation with periodic boundary conditions (representing essentially a ring of coupled nonlinear springs), Zabusky and Kruskal made the following observations. An initial profile representing a long-wavelength excitation would break up into a number of solitons, which would propagate around the system with different speeds. The solitons would collide but preserve their individual shapes and speeds. At some instant all of the solitons would collide at the same point, and a near recurrence of the initial profile would occur. This success was exciting, of course, but the soliton concept proved to have even greater impact. In fact, it stimulated very important progress in the analytic treatment of initial-value problems for nonlinear partial differential equations describing wave propagation. During the past fifteen years a rather complete mathematical description of solitons has been developed. The amount of information on nonlinear wave phenomena obtained through the fruitful collaboration of mathematicians and physicists using this description makes the soliton concept one of the most significant developments in modern mathematical physics. The non dispersive nature of the soliton solutions to the KdV equation arises not because the effects of dispersion are absent but because they are balanced by nonlinearities in the system. The presence of both phenomena can be appreciated by considering simplified versions of the KdV equation. Eliminating the nonlinear term $u(\frac{\partial u}{\partial x})$ yields the linearized version

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.2}$$

The most elementary wave solution of this equation is the harmonic wave

$$u(x, t) = Ae^{i(kx+\omega t)} \tag{1.3}$$

Where k is the wave number and ω is the angular frequency.In order for the displacement $u(x, t)$ given by equation Eq. (1.1) to be solution of equation Eq. (1.2), ω and k must satisfy the relation

$$\omega = k^3 \tag{1.4}$$

Such a dispersion relation is a very handy algebraic description of a linear system since it contains all the characteristics of the original differential equation, Two important concepts connected with the phase velocity $v_p = \frac{\omega}{k}$ and the group velocity $v_g = \frac{\partial \omega}{\partial k}$. The phase velocity measures how fast a point of constant phase is moving, while the group velocity measures how fast the energy of the wave moves. The waves described by Eq. (1.2) are said to be dispersive because a wave with large k will have larger phase and group velocities than a wave with small k. Now, we eliminate the dispersive term $\frac{\partial^3 u}{\partial x^3}$ and consider the equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{1.5}$$

This simple nonlinear equation also admits wave solutions, but they are now of the form $u(x,t) = f(x - ut)$, where the function f is arbitrary.(that $f(x - ut)$) is a solution of Eq. (1.5) For waves of this form, the important thing to note is that the velocity of a point of constant displacement u is equal to that displacement. As a result, the wave breaks; that is, portions of the wave undergoing greater displacements move faster than, and therefore overtake, those undergoing smaller displacements, This multivaluedness is a result of the nonlinearity and, like dispersion, leads to a change in form as the wave propagates. A remarkable property of the KdV equation is that dispersion and nonlinearity balance each other and allow wave solutions that propagate without changing form , An example of such a solution is

$$u(x,t) = 3c\, sech^2[c^{\frac{1}{2}}(x - ct)/2] \tag{1.6}$$

where the velocity c can take any positive value and $sech^2(x) = \frac{1}{\cosh(x)}$. This is the one soliton solution of the KdV equation. not all nonlinear partial differential equations have soliton solutions. Those that do are generic and belong to a class for which the general initial-value problem can be solved by a technique called the inverse scattering transform, a brilliant scheme developed by Kruskal and his coworkers in 1967. With this method, which can be viewed as a generalization of the Fourier transform to nonlinear equations, general solutions can be produced through a series of linear calculations. During the solution process it is possible to identify new nonlinear modes generalized Fourier modes that are the soliton components of the solution and, in addition, modes that are purely dispersive and therefore often called radiation. Equations that can be solved by the inverse scattering transform are said to be completely integrable. The manifestation of balance between dispersion and nonlinearity can be quite different from system to system. Other equations thus have soliton solutions that are distinct from the bell-shaped solitons of the KdV equation. An example is the so-called nonlinear Schrödinger (NLS) equation. This equation is generic to all conservative systems that are weakly nonlinear but strongly dispersive. It describes the slow temporal and spatial evolution of the envelope of an almost monochromatic wave train. We present here a heuristic derivation of the NLS equation that shows how it is the natural equation for the evolution of a carrier-wave envelope. Consider a dispersion relation for a harmonic wave that is amplitude dependent:

$$\omega = \omega(k, |E|^2) \tag{1.7}$$

Here $E = E(x,t)$ is the slowly varying envelope function of a situation described by Eq. (1.7) occurs, for example. in nonlinear optical phenomena, where the dielectric constant of the medium depends on the intensity of the electric signal. Other examples include surface waves on deep water, electrostatic plasma waves, and bond-energy transport in proteins. By expanding Eq. (1.7)in a Taylor's series around ω_0 and k_0, we obtain

$$\omega(k) - \omega(k_0) = \frac{\partial\omega}{\partial k}|_0(k-k_0) + \frac{1}{2}\frac{\partial^2\omega}{\partial\omega^2}|_0(k-k_0)^2 + \frac{\partial\omega}{\partial(|E|^2)}|_0|E|^2 \quad (1.8)$$

We have expanded only to the first order in the nonlinearity but to the second order in the dispersion term, as we shall see only represents undistorted propagation of the wave with the group velocity $v_g = [\frac{\partial\omega}{\partial k}]_0$. If we now substitute the operators $i(\frac{\partial}{\partial t})$ for $\omega - \omega_0$ and $i(\frac{\partial}{\partial x})$ for $k - k_0$ in Eq(1.8) and let resulting expression operate on E ,we get

$$i[\frac{\partial E}{\partial t} + \frac{\partial\omega}{\partial k}|_0\frac{\partial E}{\partial x}] + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}|_0\frac{\partial^2 E}{\partial x^2} - \frac{\partial\omega}{\partial(|E|^2)}|_0|E|^2E = 0 \quad (1.9)$$

This is the nonlinear Schrödinger equation, so called because of its resemblance to the Schrödinger equation even though its derivation often has nothing to do with quantum mechanics. The first term of Eq. (1.9)represents undistorted propagation of the wave at the group velocity. and the second and third terms represent its linear and nonlinear distortion. respectively. This crude derivation of the NLS equation shows how it arises in systems with amplitude-dependent dispersion relations.It is often preferable to express Eq. (1.9) in a neater form. For this purpose we transform the variables x and t into z and τ , where $z = x - \frac{\partial\omega}{\partial k}|_0 t$ is a coordinate moving with group velocity and $\tau = \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}|t$ is the normalized time Eq. (1.9) then reduced

$$i[\frac{\partial E}{\partial\tau} + \frac{\partial^2 E}{\partial\tau^2} + 2k|E|^2E = 0 \quad (1.10)$$

where

$$k = \frac{-[\frac{\partial\omega}{\partial(|E|^2)}]_0}{[\frac{\partial^2\omega}{\partial k^2}]_0} \quad (1.11)$$

The NLS equation like the KdV equations completely integrable and has soliton solutions. The analytic form for a single-soliton solution is given by

$$E(z,\tau) = 2\eta\, sech^2[2\eta(\theta_0 - \eta z - 4\zeta\eta\tau)]exp^{-2i[\phi_0+2(\zeta^2-\eta^2)t]+\zeta z} \quad (1.12)$$

where ζ , η, θ_0, ϕ_0 are free parameters determining the speed,amplitude, initial position, and initial phase, respectively, of the soliton.Any initial excitation for the NLS equation will decompose into solitons and/or dispersive radiation. A monochromatic wave train solution $E(z,\tau) = E(\tau)$ is thus unstable to any z dependent perturbation and breaks up into separate and localized solitons. This phenomenon is called the Benjamin-Feir instability and is well known to any surfer on the beach who has noticed that every, say, seventh wave is the largest. The NLS equation is in a way more universal than the KdV equation since an almost monochromatic, small-amplitude solution of the KdV equation will evolve according to the NLS equation.The last type of soliton we mention, which is distinctly different from the kdv or NLS solitons, is one that represents topologically invariant quantities in a system. Such an invariant can be a domain wall or a dislocation in a magnetic crystal or a shift in the bond-alternation pattern in a polymer.The prototype of equations for such solitons is the Sine-Gordon equation,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin u \tag{1.13}$$

notice that this equation allows for an infinite number of trivial solution, namely $u = 0 \pm 2\pi \pm 4\pi$ systems with a multitude of such degenerate ground states also allow solutions that connect two neighboring ground states. Solutions of this type are often called kinks, and for the sine-Gordon equation they are exact solitons; that is, they collide elastically without generation of dispersive radiation. The analytic form is given by

$$u_+(x,t) = 4\tan^{-1} exp^{[\pm(x-x_0-ct)/(1-c^2)^{\frac{1}{2}}]} \tag{1.14}$$

where the solution u is often called an antikink. The parameter $c(-1 < c < 1)$ determines the velocity and x_0 the initial position, Other equations with degenerate ground states also have kink and antikink solutions, but they are not exact solitons like those of the sine-Gordon equation. It is interesting to note that small-amplitude solutions of the sine-Gordon equation also can be shown to evolve according to the NLS equation.[79]

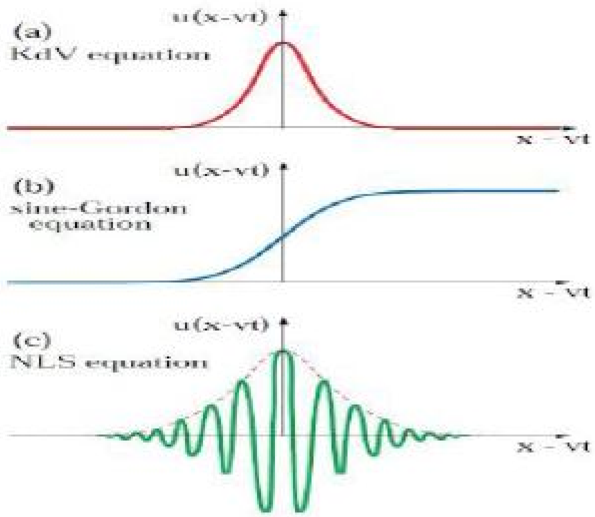

Figure 1.3: Schematic of the soliton solutions of: (a) the Korteweg-de Vries equation; (b) the Sine-Gordon equation, and (c) the nonlinear Schrödinger equation.

1.4 Solitons in Bose- Einstein condensates

The evolution of solitons is approximately described by the inhomogeneous nonlinear Schrödinger equation known as the Gross-Pitaevskii equation Introduced independently by Gross [25, 26, 27], and Pitaevskii [28, 29], the approximation stems from the fact that the Gross-Pitaevskii equation is a mean-field approximation of the exact N-particle Schrödinger equation. While in some cases the soliton dynamics obtained by these two equations disagree, the Gross-Pitaevskii equation often gives accurate results. Theoretical studies performed to account for the observed behavior of solitons were conducted by solving the Gross-Pitaevskii equation with numerical, perturbative, or variational methods. Much less effort was devoted to finding exact solutions of this equation.

In a BEC the fundamental excitations at zero-temperature are divided into collective and topological ones[69, 70]. The collective excitations are related to the density perturbations depending on the excitation wavelengthλ_e, relative to the size of the condensate R, thus when $\lambda_e < R$, the excitations are phonons (sound waves) and for $\lambda_e > R$, the excitations represent large-scale oscillations (breathing and quadruple modes). The topological excitations are solitons (bright and dark) and vortices (single vortices, vortex rings and vortex lattices) [69, 70, 71].

1.5 Dark Soliton

A dark soliton in BEC is a macroscopic excitation of the condensate with a corresponding positive scattering length, which is characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum[81]. The shape of the dip does not change due to the balance between kinetic energy and repulsive atom-atom collisions.

First created in 1987 in nonlinear optics [82], dark solitary waves have been created in atomic BECs in a controlled manner or through dynamical processes and are a topic of intense research . In another experiment, a pair of matter wave dark solitons was generated by merging two condensates initially prepared in a double well potential [83]. Finally, dark solitons have been generated through twocomponent BECs where the soliton exists in one component and is initially filled with the second component. The second component is then selectively removed.Various techniques have been used to create dark solitons through dynamical processes. Firstly, via a slow light technique, a disk shaped region of atoms was suddenly removed from the condensate generating counter propagating dark solitons [84]. In Reference [85] dark solitons emerged when a barrier, formed by a beam, swept through the condensate at intermediate speeds. For slow speeds, the fluid flow was steady while at fast speeds soliton formation ceased and the absence of excitations was surprisingly once again observed.[86]

1.6 Bright Soliton

In the case of negative scattering length (focusing nonlinearity in optics) the GPE has other kind of solutions called bright solitons. These are harder to observe due to the collapse of the system for sufficiently high number of atoms. Nevertheless, in 2002 two groups[87, 88], showed the generation and propagation of solitons in ^7Li condensates. The main difference in both experiments was the number of atoms allowing to create just one soliton[87], whereas in[88] were produced trains of several solitons and these trains oscillated in a weak attractive trapping potential. Later, were created bright solitons in ^{85}Rb[89].

1.7 Rogue waves

Rogue waves (RWs), similar to solitary waves, have no universal definition, and display the basic feature that RWs appear from nowhere and disappear without a trace. In general, the common criteria is available for RWs in the ocean, i.e., the height of a RW (vertical distance from trough to crest) is two or more times greater than the significant wave height (the average wave height among one third of the highest waves in a time series, e.g., usually of length 10 - 30 min) [90, 91, 92, 93, 94, 95]. The term RW (or freak wave) was first introduced in the scientific community due to Draper in 1964 [96]. RWs are also known as freak waves, monster waves, killer waves, abnormal waves, steep waves, giant waves, or extreme waves. More recently, the RW (or freak wave) was also coined rogon (or freakon) if they reappear virtually unaffected in size or shape shortly after their interactions [97].

The science of rogue waves in optics is now over five years old, and it has emerged as an area of broad interest to researchers across the physical sciences[98]. This area of study was initiated by the pioneering measurement of Solli et al [99] when analysing supercontinuum generation in optical fibres. Their measurements, using a novel dispersive Fourier transform technique to capture high-speed events in the time domain, observed extraordinarily high amplitude peaks at certain wavelengths in the chaotic spectrum from the supercontinuum. By analogy with the extreme waves in the ocean [100], of wide interest after 1995, such high amplitude pulses were described as optical rogue waves. This analogy between localized structures in optics and extreme waves on the ocean has opened up many possibilities for exploring extreme value dynamics in convenient table-top optical experiments. In addition to the proposed links with solitons suggested in [99], other recent studies, motivated from an optical context, have explored possible links with nonlinear breather propagation. There is now an international effort, worldwide, to study these extreme events in optics, both for their own intrinsic interest within their own domain of research, and also because of their links with the large amplitude ocean wave events [101] that have inspired their study.

The notion of rogue waves has lately expanded to many fields in science[98] . Careful studies on small scales may help to better understand rogue waves in the ocean. The analogy is mainly based on similar equations used to model rogue waves in various fields, including waves in the open ocean. However, specific features of waves in a laboratory also allow them to be considered as individual new directions in science. It would be hard to cover all these directions in a single volume. D.H. Peregrine, who was studying applied mathematics, primarily hydrodynamics, at the University of Bristol, described this particular soliton in 1983[134].

On the contrary, the fundamental soliton which has the property of keeping its characteristic shape unchanged during its propagation, the Peregrine soliton presents a double localization, both in the time domain and in the spatial domain. Thus, from a small oscillation on a continuous background, Peregrine's soliton develops, seeing its temporal duration decrease and its amplitude increase. At the point of maximum compression, its amplitude reaches three times the amplitude of the surrounding continuous background (if we reason in intensity as is the case in optics, it is a factor of 9 which separates the peak of the soliton from the surrounding background.). After this point of maximum compression, the wave sees its amplitude decrease and widen to finally disappear.

This behavior of the Peregrine soliton corresponds to the criteria usually used to qualify a rogue wave. Peregrine's soliton thus represents an attractive potential explanation for the formation of these waves of abnormally high amplitude which appear and disappear without leaving a trace.

- The Peregrine soliton can be seen as the limiting case of the space-periodic Akhmediev breather when the period tends to infinity. can also be seen as the limiting case of the time-periodic Kuznetsov-Ma breather when the period tends to infinity

Figure 1.4: A Rogue Wave is a short-lived large-amplitude[103, 104]

For BEC with attractive interactions, it admits bright soliton on zero background and rogue wave (RW) on plane wave background[105, 106]. For BEC with repulsive interactions, it admits dark soliton on a plane wave background [102]. Since there is no modulational instability for repulsive case, therefore rogue wave do not exist for repulsive scalar BEC. These characters just hold for scalar BEC systems.

1.8 Breather

Breathers are solutions of nonlinear equations with a profile which is usually localized in space and periodic in time, even for equations with constant coefficients. Such solutions usually come from a compound of two or more solitons located at the same spatial position. After the first experimental manipulation of Bose-Einstein condensates (BECs) [107, 108, 109], studies on bright and dark solitons [110, 111, 112] have triggered a lot of new investigations, with a diversity of scenarios being proposed and tested [113, 114, 115, 116, 117]. In particular, the presence of experimental techniques for manipulating the strength of the effective interaction between trapped atoms [118] leads us to believe that in BECs we have an excellent opportunity to investigate breathers of atomic matter waves taking advantage of Feshbach-resonance management [119, 120, 121, 122, 123, 124, 125, 126, 127]. Breathers or breathing solutions are nonlinear excitations which concentrate energy in a localized and oscillatory manner. the breather excitations play an important role, directly affecting the electronic, magnetic, optical, vibrational and transport properties of the systems. In the above mentioned studies, one usually considers genuine breathers, i.e., solutions which oscillate in time when the nonlinear equation presents constant coefficients (i.e., without modulation). However, in a more realistic scenario the several parameters that characterize the physical systems may depend on both space and time, leading to breather solutions that can be modulated in space and time. The presence of nonuniform and time-dependent parameters opens interesting perspectives not only from the theoretical point of view, for investigation of nonautonomous nonlinear equations, but also from the experimental point of view, for the study of the physical properties of the systems.

- It is possible to find analytical solutions of the nonlinear Schrödinger equation which have periodicity properties in the spatial (x) and temporal (t) directions. These solutions are linked together, as shown in the next figure

Akhmediev breather

Another family of structures undergoing compression-decompression phases was discovered by Akhmediev et al. [132, 133]. Like Ma, the authors considered a plane wave having a periodic disturbance. But they were particularly

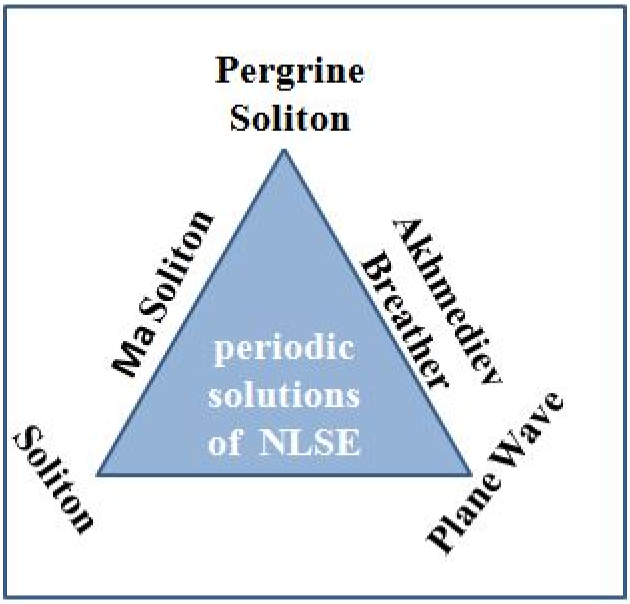

Figure 1.5: Diagram of periodic solutions of the NLSE

interested in the solutions describing the modulation instability and, consequently, the limiting condition is a return to the initial state when the distance tends towards infinity. Note that the term breather is not totally appropriate because there is, in this case only one breath, that is to say the amplitude of the solutions increases to reach its maximum at the distance tends towards 0 then decreases symmetrically to disappear forever. These structures are, now named Akhmediev Breathers (AB).

Kuznetsov-Ma Soliton

Among these solutions, the first family of breathers, or breathing solitons, was discovered by Kuznetsov in 1977 [128] then by Kawata and Inoue [129]. Finally, Ma [130] made a complete description of it in 1979. The latter solved the ESNL by considering, for initial state, a slightly disturbed plane wave and, for the boundary conditions, a return to the initial state when time tends towards infinity. He thus showed the existence of a family of periodic solitary waves in space, surrounded by residual dispersive waves of small amplitudes. These solitons on a continuous background "breathe" (succession of compression and decompression) and are therefore called breathers or in this precise case, the solitons of Ma (or even solitons of Kuznetsov-Ma). As can be seen in the diagram , one of the borderline cases of the Ma solitons is the standard soliton. Note that structures which "breathe" have also been shown by Zakharov and Shabat when the non-linearity is defocusing [131]. The main difference is that these solitons are black instead of shiny.

1.8.1 Breather solution of NLS equation

breather type solutions to the dimensionless NLS equation[135, 136, 137]:

$$i\frac{\partial\psi}{\partial\xi} + \frac{1}{2}\frac{\partial^2\psi}{\partial\tau^2} + |\psi|^2\psi = 0 \tag{1.15}$$

The envelope$\psi(\xi,\tau)$ is a function ofξ (propagation distance) and τ
The solution:

$$\psi(\xi,\tau) = e^{i\xi}[1 + \frac{2(1-2a)\cosh(b\xi) + ib\sinh(b\xi)}{\sqrt{2}\cos(\omega\tau) - \cosh(b\xi)}] \tag{1.16}$$

The solution's properties are determined by one positive parameter $a(a \neq 1/2)$ through arguments $b = [8a(1-2a)]1/2$ and $\omega = 2(1-2a)1/2$. Over the range $0 < a < 1/2$. the solution is the Akhmediev breather (AB), which is shown in Figure(1.6.**a**). the limit $a \to 1/2$. which gives the Peregrine Soliton. Figure(1.6.**b**).
The PS, given by:

$$\psi(\xi,\tau) = [1 - \frac{4(1+2i\xi)}{1+4\tau^2+4\xi^2}] \tag{1.17}$$

corresponds to a single pulse with localization in time (τ) as well as along the propagation direction (ξ) as shown) .

When $a > 1/2$, the parameters ω and b become imaginary, and the solution exhibits localisation in the temporal dimension ξ but periodicity along the propagation direction τ. This is the Kuznetsov-Ma soliton which is shown in Figure(1.6.**c**).

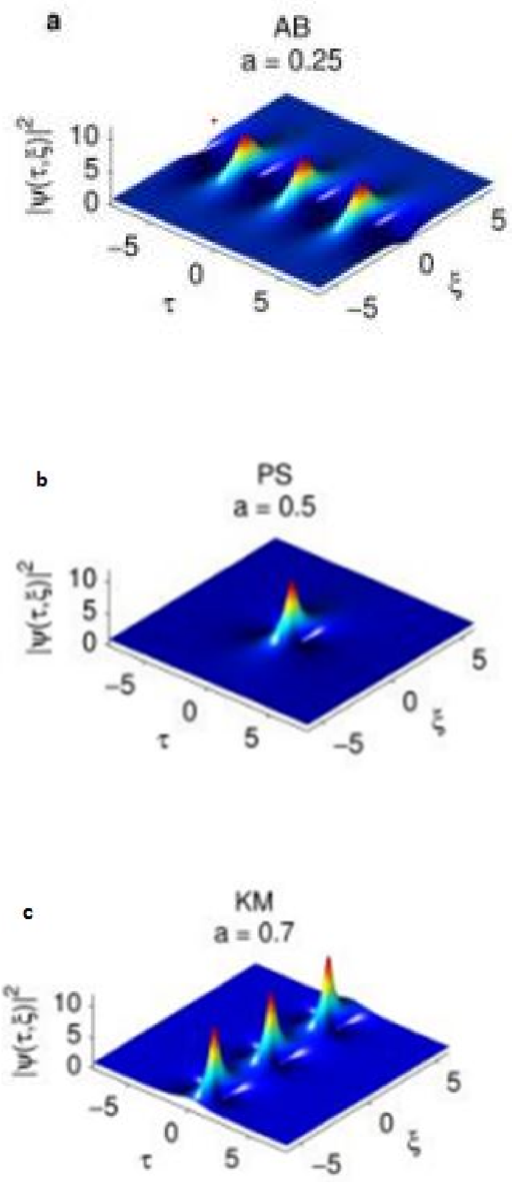

Figure 1.6: The Solutions of NLSE **(a:AB)**Breathers-Akhmediev, **(b:PS)** peregrine Soliton(Rogue wave),and **(c:KM)** Kuznetsov-Ma Breather.

Recently there has been a growing interest in the Gross-Pitaevskii (GP) equations [138, 139] describing two-component Bose-Einstein condensates (BEC) in external trap potentials[140, 141, 142, 143, 144, 145, 146, 147, 148, 149] . In the absence of the confining potential, the GP equations reduce to the coupled non-linear Schrödinger (NLS) equations which play an important role in optics [153]. Coupled GP equations are also used to describe Josephson-type oscillations between two coupled BEC [146, 147, 148], spin-mixing dynamics of spinor BEC [149, 150, 151, 152], or to explore such interesting field of matter waves as possible atomic soliton lasers [141, 154, 155].

Chapter 2

Variational Derivation of the Coupled Gross- Pitaevskii Equations

2.1 Time Dependent Variational

Variational methods in physics and applied mathematics were formulated long ago [168, 169, 170, 171, 172, 173] It was Maupertuis [171], who wrote in 1774 the celebrated statement: *Nature, in the production of its effects, does so always by simplest means.* Since that time variational methods have become an increasingly popular tool in mechanics, hydrodynamics,theory of elasticity, etc. Moreover, the variational methods are useful and workable tools for many areas of the quantum theory of atoms and molecules [170, 174, 175, 176], statistical many-particle physics and condensed matter physics. The variational methods have been applied widely in quantum-mechanical calculations [170, 174, 175, 176], in theory of many-particle interacting systems [157, 158, 159, 160, 161, 162, 163] and theory of transport processes [177, 178]. As a result of these efforts many important and effective methods were elaborated by various researchers. From the other hand the study of the quasi-particle excitations in many-particle systems has been one of the most fascinating subjects for many years [164, 156, 166, 167]. The quantum field-theoretical techniques have been widely applied to the statistical treatment of a large number of interacting particles. Many-body calculations are often done for model systems of statistical mechanics using the perturbation expansion. The basic procedure in many-body theory is to find the relevant unperturbed Hamiltonian and then take into account the small perturbation operator. This procedure, which works well for the weakly interacting systems, needs the suitable reformulation for the many-body systems with complicated spectra and strong interaction. In general case, a many-particle system with interactions is very difficult to solve exactly, except for special simple cases. Theory of molecular (or mean) field permits one to obtain an approximate solution to the problem. The effect of all the other particles on any given particle is approximated by a single averaged effect, thus reducing a many-body problem to a single-body problem.

The time-dependent variational principle of Balian and Vénéroni (BV)[179].It retain: is one of the most powerful tools since, not only does it retain the essential features of the physics behind the previous approximations, but it also allows one to bypass some (if not all) of the ad-hoc assumptions. This well-known advantage of this (and any) variational principle faces however a major difficulty related to the choice of the trial spaces. A (difficult) compromise must be found between a correct description of the physics on one hand, and the tractability of the calculations on the other.

2.1.1 The Balian-Vénéroni Variational Method

Balian and Vénéroni propose a variational principle appropriate for deriving approximations of the expectation value of a given observable A at a time t_1, when the density operator D of the system is known at an earlier time t_0. The action I , which involves as a variational set, two time-dependent operators $A(t)$ and $D(t)$, is written as

$$I = Tr(AD)_{t_1} - \int_{t_0}^{t_1} dt Tr(A(t))(\frac{dD(t)}{dt} + \frac{i}{\hbar}[H, D(t)]) \tag{2.1}$$

The symbol Tr stands for a trace taken over a complete basis of the Fock space. And H is the Hamiltonian of the system assumed time independent.

The equations for $A(t)$ and $D(t)$ are obtained by writing the stationarity of Eqs. (2.1) with respect to $D(t)$ and $A(t)$, respectively. They read

$$Tr\delta A(t)(\frac{dD(t)}{dt} + \frac{i}{\hbar}[H, D(t)]) = 0 \tag{2.2}$$

$$Tr\delta D(t)(\frac{dA(t)}{dt} - \frac{i}{\hbar}[A(t), H]) = 0 \tag{2.3}$$

and are subjected to the boundary conditions

$$D(t_0) = D_0, A(t_1) = A \tag{2.4}$$

When the variations $\delta A(t)$ and $\delta D(t)$ are unrestricted, we deduce from Eqs. (2.2) and Eqs. (2.3) the evolution equations for $D(t)$ and $A(t)$:

$$i\hbar \frac{dD(t)}{dt} = [H, D(t)] \tag{2.5}$$

$$i\hbar \frac{dA(t)}{dt} = -[H, A(t)] \tag{2.6}$$

We recognize the exact Liouville-von Neumann equation for the operator density $D(t)$. and the exact Heisenberg equation for the observable $A(t)$,

2.1.2 Variational Derivation of the CGPE

According to the preceding discussion, we have now to choose trial spaces for $D(t)$ and $A(t)$ in order to get approximate dynamics.
Consider a system composed of two species of bosons (labeled 1 and 2) of which the creation and annihilation field operators are $\psi_i(r)$, $\psi_i^\dagger(r)$, (i=1,2), satisfying

$$[\psi_i(r), \psi_i^\dagger(r')] = \delta_{ij}\delta(r - r')$$

$$[\psi_i(r), \psi_i(r')] = [\psi_i^\dagger(r), \psi_i^\dagger(r')] = 0$$

We consider a gaussian density operator [180]:

$$D(t) = \exp Q(t) \tag{2.7}$$

where $Q(t)$ is a one- body operator

$$\begin{aligned}
Q(t) = \nu_1(t) &+ \int_r [\lambda_1(r,t)\psi_1^\dagger(r) + \lambda_2(r,t)\psi_2^\dagger(r) + \lambda_1^*(r,t)\psi_1(r) + \lambda_2^*(r,t)\psi_2(r)] \\
&+ \int_{r,r'} [\bar{\psi_1}(r)s_1(r,r',t)\bar{\psi_1}(r') + \bar{\psi_1^\dagger}(r)s_1^*(r,r',t)\bar{\psi_1^\dagger}(r') + \bar{\psi_1^\dagger}(r)s_2(r,r',t)\bar{\psi_1}(r')] \\
&+ \int_{r,r'} [\bar{\psi_2}(r)s_3(r,r',t)\bar{\psi_2}(r') + \bar{\psi_2^\dagger}(r)s_3^*(r,r',t)\bar{\psi_2^\dagger}(r') + \bar{\psi_2^\dagger}(r)s_4(r,r',t)\bar{\psi_2}(r')] \\
&+ \int_{r,r'} [\bar{\psi_1}(r)s_5(r,r',t)\bar{\psi_2}(r') + \bar{\psi_1^\dagger}(r)s_5^*(r,r',t)\bar{\psi_2^\dagger}(r')] \\
&+ \int_{r,r'} [\bar{\psi_1^\dagger}(r)s_6(r,r',t)\bar{\psi_2}(r') + \bar{\psi_2^\dagger}(r)s_6^*(r,r',t)\bar{\psi_1}(r')]
\end{aligned}$$

$$ \tag{2.8} $$

and a trial observable $A(t)$ also as a one- body operator

$$
\begin{aligned}
A(t) = \nu_2(t) &+ \int_r [u_1(r,t)\psi_1^\dagger(r) + u_2(r,t)\psi_2^\dagger(r) + u_1^*(r,t)\psi_1(r) + u_2^*(r,t)\psi_2(r)] \\
&+ \int_{r,r^{'}} [u_3(r,r^{'},t)\bar{\psi}_1(r)\bar{\psi}_1(r^{'}) + u_3^*(r,r^{'},t)\bar{\psi}_1^\dagger(r)\bar{\psi}_1^\dagger(r^{'}) + u_4(r,r^{'},t)\bar{\psi}_1^\dagger(r)\bar{\psi}_1(r^{'})] \\
&+ \int_{r,r^{'}} [u_5(r,r^{'},t)\bar{\psi}_2(r)\bar{\psi}_2(r^{'}) + u_5^*(r,r^{'},t)\bar{\psi}_2^\dagger(r)\bar{\psi}_2^\dagger(r^{'}) + u_6(r,r^{'},t)\bar{\psi}_2^\dagger(r)\bar{\psi}_2(r^{'})] \\
&+ \int_{r,r^{'}} [u_7(r,r^{'},t)\bar{\psi}_1(r)\bar{\psi}_2(r^{'}) + u_7^*(r,r^{'},t)\bar{\psi}_1^\dagger(r)\bar{\psi}_2^\dagger(r^{'})] \\
&+ \int_{r,r^{'}} [u_8(r,r^{'},t)\overline{\psi_1^\dagger}(r)\bar{\psi}_2(r^{'}) + u_8^*(r,r^{'},t)\bar{\psi}_2^\dagger(r)\bar{\psi}_1(r^{'})]
\end{aligned}
\tag{2.9}
$$

we have now to compute the functional of Eqs. (2.1) and then write the stationarity conditions with respect to the parameters of $A(t)$. We compute separately $TrA(t)D(t)$ and $TrA(t)[H,D(t)]$.

$$
\begin{aligned}
Tr[AD] &= Z\langle A\rangle \\
&= Z[\nu_2(t) + \int_r [u_1(r,t)\langle\psi_1^\dagger(r)\rangle + u_2(r,t)\langle\psi_2^\dagger(r)\rangle + u_1^*(r,t)\langle\psi_1(r)\rangle + u_2^*(r,t)\langle\psi_2^\dagger(r)\rangle] \\
&+ \int_{r,r^{'}} [u_3(r,r^{'},t)\langle\bar{\psi}_1(r)\bar{\psi}_1(r^{'})\rangle + u_3^*(r,r^{'},t)\langle\bar{\psi}_1^\dagger(r)\bar{\psi}_1^\dagger(r^{'})\rangle + u_4(r,r^{'},t)\langle\bar{\psi}_1^\dagger(r)\bar{\psi}_1(r^{'})\rangle] \\
&+ \int_{r,r^{'}} [u_5(r,r^{'},t)\langle\bar{\psi}_2(r)\bar{\psi}_2(r^{'})\rangle + u_5^*(r,r^{'},t)\langle\bar{\psi}_2^\dagger(r)\bar{\psi}_2^\dagger(r^{'})\rangle + u_6(r,r^{'},t)\langle\bar{\psi}_2^\dagger(r)\bar{\psi}_2(r^{'})\rangle] \\
&+ \int_{r,r^{'}} [u_7(r,r^{'},t)\langle\bar{\psi}_1(r)\bar{\psi}_2(r^{'})\rangle + u_7^*(r,r^{'},t)\langle\bar{\psi}_1^\dagger(r)\bar{\psi}_2^\dagger(r^{'})\rangle] \\
&+ \int_{r,r^{'}} [u_8(r,r^{'},t)\langle\bar{\psi}_1^\dagger(r)\bar{\psi}_2(r^{'})\rangle + u_8^*(r,r^{'},t)\langle\bar{\psi}_2^\dagger(r)\bar{\psi}_1(r^{'})\rangle]]
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
Tr[A\frac{dD}{dt}] &= \frac{d}{dt}(TrA(t)D(t))_A \\
&= \frac{\dot{Z}}{Z}TrAD + Z[\int_r [u_1(r,t)\dot{\phi}_1^*(r) + u_2(r,t)\dot{\phi}_2^*(r) + u_1^*(r,t)\dot{\phi}_1(r) + u_2^*(r,t)\dot{\phi}_2] \\
&+ \int_{r,r^{'}} [u_3(r,r^{'},t)\dot{\tilde{m}}_{11}(r,r^{'}) + u_3^*(r,r^{'},t)\dot{\tilde{m}}_{11}^*(r,r^{'}) + u_4(r,r^{'},t)\dot{\tilde{n}}_{11}(r,r^{'})] \\
&+ \int_{r,r^{'}} [u_5(r,r^{'},t)\dot{\tilde{m}}_{22}(r,r^{'}) + u_5^*(r,r^{'},t)\dot{\tilde{m}}_{22}^*(r,r^{'}) + u_6(r,r^{'},t)\dot{\tilde{n}}_{22}(r,r^{'})]]
\end{aligned}
\tag{2.11}
$$

In Eqs. (2.11), we have introduced the quantities $\phi_i(r,t)$,$\tilde{n}_{ii}(r,r^{'},t)$ and $\tilde{m}_{ii}(r,r^{'},t)$ defined as:

$$
\begin{aligned}
&\phi_i(r,t) = \langle \psi_i(r) \rangle \\
&\tilde{n}_{ii}(r,r^{'},t) = \langle \bar{\psi}_i^\dagger(r)\bar{\psi}_i(r^{'}) \rangle \\
&\tilde{m}_{ii}(r,r^{'},t) = \langle \bar{\psi}_i(r)\bar{\psi}_i(r^{'}) \rangle
\end{aligned}
\tag{2.12}
$$

and $\bar{\psi}_i = \psi_i - \langle \psi_i \rangle$.
The second term containing the Hamiltonian writes

$$
\begin{aligned}
&TrA_[H,D] = Z\langle [A,H] \rangle \\
&= Z[\int_r dr[u_1(r,t)\langle [\psi_1^\dagger,H] \rangle + u_2(r,t)\langle [\psi_2^\dagger,H] \rangle + u_1^*(r,t)\langle [\psi_1,H] \rangle + u_2^*(r,t)\langle [\psi_2,H] \rangle] \\
&+ \int_{r,r^{'}} drdr^{'}[u_3(r,r^{'},t)\langle [\bar{\psi}_1\bar{\psi}_1,H] \rangle + u_3^*(r,r^{'},t)\langle [\bar{\psi}_1^\dagger\bar{\psi}_1^\dagger,H] \rangle + u_4(r,r^{'},t)\langle [\bar{\psi}_1^\dagger\bar{\psi}_1,H] \rangle] \\
&+ \int_{r,r^{'}} drdr^{'}u_5(r,r^{'},t)\langle [\bar{\psi}_2\bar{\psi}_2,H] \rangle + u_5^*(r,r^{'},t)\langle [\bar{\psi}_2^\dagger\bar{\psi}_2^\dagger,H] \rangle + u_6(r,r^{'},t)\langle [\bar{\psi}_2^\dagger\bar{\psi}_2,H] \rangle]
\end{aligned}
\tag{2.13}
$$

To compute (2.13)explicitely, one should specify the Hamiltonian. Let us consider the general two- body hamiltonian

$$
\begin{aligned}
H = &\sum_{i=1}^{2}\int_r [\psi_i^\dagger(r)h_i\psi_i(r)] + \frac{1}{2}\sum_{i=1}^{2}\int_{r,r^{'}} g_{ii}[\psi_i^\dagger(r)\psi_i^\dagger(r^{'})\psi_i(r^{'})\psi_i(r)] \\
&+ \int_{r,r^{'}} g_{12}[\psi_1^\dagger(r)\psi_2^\dagger(r^{'})\psi_2(r^{'})\psi_1(r)]
\end{aligned}
\tag{2.14}
$$

g_{ii} and g_{12} are respectively the intra and interspecies coupling strengths. Moreover h_i is the single particle Hamiltonian $h_i = \frac{\hbar^2}{2m_i}\nabla^2 + V_{ext}(r)$, where $V_{ext}(r)$ is the trapping field. The dynamics of $D(t)$ is obtained by writing the stationarity of Eqs. (2.11),(2.13) with respect to the parameters of $A(t)$. This leads to the following equations for the expectation values (2.13) (For more details, see Appendix.)

$$\begin{aligned} i\hbar\dot{\phi}_1 &= (h_1 + g_{11}(n_1 + \tilde{n}_{11}) + g_{12}n_2)\phi_1 + g_{11}\tilde{m}_{11}\phi_1^* \\ i\hbar\dot{\phi}_2 &= (h_2 + g_{22}(n_2 + \tilde{n}_{22}) + g_{12}n_1)\phi_2 + g_{22}\tilde{m}_{22}\phi_2^* \end{aligned} \tag{2.15}$$

$$\begin{aligned} i\hbar\dot{\tilde{n}}_{11} &= g_{11}(\phi_1^2\tilde{m}_{11}^* - \phi_1^{*2}\tilde{m}_{11}) \\ i\hbar\dot{\tilde{n}}_{22} &= g_{22}(\phi_2^2\tilde{m}_{22}^* - \phi_2^{*2}\tilde{m}_{22}) \end{aligned} \tag{2.16}$$

$$\begin{aligned} i\hbar\dot{\tilde{m}}_{11} &= 2(h_1 + 2g_{11}n_1 + g_{12}n_2)\tilde{m}_{11} + 2g_{11}(\phi_1^2 + \tilde{m}_{11})\tilde{n}_{11} \\ i\hbar\dot{\tilde{m}}_{22} &= 2(h_1 + 2g_{22}n_2 + g_{12}n_1)\tilde{m}_{22} + 2g_{22}(\phi_2^2 + \tilde{m}_{22})\tilde{n}_{22} \end{aligned} \tag{2.17}$$

This set of equations constitute the time- dependent Hartree-Fock-Bogoliubov equations for a bose-bose mixture. They describe the dynamics of the mixture beyond the mean-field approximations as they couple the condensates with the thermal and quantum fluctuations depicted by the $\tilde{n}_{ii}$ and $\tilde{m}_{ii}$. At ultra low temperatures, the quantum depletion is extremely small. Hence, one may set $\tilde{n}_{ii} = \tilde{m}_{ii} \simeq 0$ and get the coupled Gross- Pitaevskii equations.

$$\begin{aligned} i\hbar\dot{\phi}_1 &= (h_1 + g_{11}n_1 + g_{12}n_2)\phi_1 \\ i\hbar\dot{\phi}_2 &= (h_2 + g_{22}n_2 + g_{12}n_1)\phi_2 \end{aligned} \tag{2.18}$$

where $n_1 = |\phi_1|^2 + \tilde{n}_{11}$ is the density of the condensate 1. Since, we set the system at $T = 0$, we have $n_1 = |\phi_1|^2$ and the system Eqs. (2.18)is a closed set of PDE.

Although appearing quite simple, the equations (2.18) possess a considerable set of different solutions. which belong to very distinct regim. Indeed, due the non- linearities, the solutions are very sensitive to the initial and boundary solutions. Hence, although several numerical algorithms have been developed and used to solve Eqs. (2.18), many questions remain unanswered. In particular , how does a set of solutions move continuously toward another set in parameter space. This kind of questions require an analytic answer.
In what follows, we will use a well-know method, the Darboux transformation method (DT) together with the Lax pair method to generate analytically a large class of solutions.

2.2 The Darboux transformation and Lax pairs methods

2.2.1 Introduction

The Darboux transformation, or analogously Bäcklund or dressing transformation, applies only to systems of linear differential equations and cannot be applied directly to nonlinear differential equations. To be able to apply the Darboux transformation to a certain nonlinear differential equation, one finds a linear system of equations that is equivalent to a nonlinear differential equation. The relation between the linear system and the nonlinear differential equation is established through a consistency condition satisfied by the linear system. The Darboux transformation is then applied to the linear system resulting in transforming the equivalent nonlinear equation as well. The linear system is usually represented in terms of a pair of matrices called the Lax pair which must satisfy a consistency condition that is equivalent to the differential equation at hand. The difficulty is usually in finding this Lax pair. It is known for some nonlinear differential equations such as the Kortwegde Vries (KdV) equation, the sine-Gordon equation and the nonlinear Schrodinger equation [183]. In addition to the Lax pair, one also needs to know an exact solution of the nonlinear differential equation. This exact solution is then used as a seed for the Darboux transformation to generate other exact solutions [184].

2.2.2 Integrable System

The concept of completely integrable system arose in the 19th century in the context of finite-dimensional .

one can, in a heuristic way, define the subject that we are interested in, saying: *An integrable model consists of non-linear differential equations which can be solved analytically, at least "in principle"* . The resulting quantum system is universally called integrable. For infinite-dimensional systems and finite lattice systems, however, there is less agreement on the notion of integrability. Often the term integrable is used as a synonym for solved or soluble. The connection with the concept of completely integrable system was first made by Zakharov and Faddeev (1971). Kruskal and coworkers had shown that the KdV equation has an infinite number of conservation laws, and that there exists a linearizing transformation, which maps the initial value $u(0, x)$ for the KdV Cauchy problem to spectral and scattering data of the Schrödinger operator $-d^2/dx^2 - u(0, x)$. The nonlinear evolution yielding $u(t, x)$ then transforms into an essentially linear time evolution of these data, so that $u(t, x)$ can be constructed via the inverse map, the so-called Inverse Scattering Transform (IST). Inspired by these findings, Zakharov and Faddeev showed that the KdV equation may be viewed as an infinite-dimensional classical integrable system, the spectral and scattering data being the action-angle variables, the IST the (inverse of the) action-angle map, and the infinity of conserved quantities the Poisson commuting Hamiltonians. Important integrable equations of a different kind include higher-dimensional PDEs with soliton solutions (such as the Kadomtsev-Petiashvili and Davey-Stewartson equations) and soliton lattices (the infinite Toda lattice being a prime example). A further large class of equations consists of integrable discretizations of soliton equations

Since mathematicians as well as physicists are quite interested in these models, today there are several works, looking through their theoretical and experimental aspects. The formal characteristics of the integrable models, concepts such as infinite dimensional Lie algebras and their representations and new subjects that were born in the core of differential geometry as well as in the Sturm-Liouville problem ,like B*ä*cklund, Moutard, Darboux transformations and so on , are the main interests shared by mathematicians. Physicists, naturally, are interested in the possibility of applying these models in physical phenomena, besides, the solitonic solutions of these models emerge as a good opportunity to test new ideas in the areas of non-linear optics, hydrodynamics, condensed matter, continuous mechanics, plasma physics and high energy physics [185, 186, 187]. In fact, solitons are the strongest tools for non-perturbative approach in various theories, from the hydrodynamics to string theory.

2.2.3 Darboux Transformation for Nonlinear Partial Differential Equations

For nonlinear partial differential equations, Darboux transformation is applied in an indirect manner. One starts by finding a linear system of equations for an auxiliary field ψ in the form $\Psi_x = U.\Psi$ and $\Psi_x = V.\Psi$,where $\Psi_x = \frac{\partial\Psi}{\partial x}$, the order of the matrices Ψ, U and V depends on the equation to be solved as will be seen next. The pair of matrices U and V known as the Lax pair,are functionals of the solution of the differential equation. The consistency condition of the linear system $\Psi_{xt} = \Psi_{tx}$ should be equivalent to the differential equation. The linear system and hence its consistency condition are covariant under the Darboux transformation. Therefore, applying the Darboux transformation on the linear system results in a new consistency condition which is equivalent to a new differential equation that is covariant with the old one. The new differential equation is satisfied by a new solution. In the following we describe this procedure in a more detailed manner.

2.3 Darboux transformation

Consider the following version of the Darboux transformation

$$\Psi[1] = \Psi\,\Lambda - \sigma\,\Psi, \tag{2.19}$$

where $\Psi[1]$ is the transformed field,Λ is a constant diagonal matrix and $\sigma = \Psi_0 \Lambda\,\Psi_0^{-1}$. Here Ψ_0 is a known solution of the linear system, (2.29) and (2.30). To be able to find such a solution the coefficients of the linear system should be known explicitly. These coefficients are functionals of the solution of the differential equation Q. Thus, determining the coefficients of the linear system requires knowing an exact solution of the differential equation. This solution is known as the seed solution. which we denote here by Q_0. It is in the very nature of the Darboux transformation method that new exact solutions are only obtained from other exact solutions. The transformed field $\Psi[1]$ is required to be a solution of a linear system that is covariant with the system (2.29) and (2.30) ,namely

$$\boldsymbol{\Psi[1]_x = U_0[1] \cdot \Psi[1] + U_1[1] \cdot \Psi[1] \cdot \Lambda} \tag{2.20}$$

and

$$\boldsymbol{\Psi[1]_t = V_0[1] \cdot \Psi[1] + V_1[1] \cdot \Psi[1] \cdot \Lambda + V_2[1] \cdot \Psi[1] \cdot \Lambda^2} \tag{2.21}$$

Requiring the system (2.20) and (2.21) to be covariant with the system (2.29) and (2.30) leads to a consistency condition

$$\boldsymbol{U_0[1]_t - V_0[1]_x + [U_0[1], V_0[1]] = 0}, \tag{2.22}$$

that is covariant with (2.31)similar to(2.35), this new consistency condition will be equivalent to a differential equation that is covariant with (2.36).

$$\boldsymbol{U_0[1]_t - V_0[1]_x + [U_0[1], V_0[1]]} = \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix} = \boldsymbol{0}, \tag{2.23}$$

This means that $Q[1]$ is a new solution of the same differential equation that Q_0 is a solution for. To find $U_0[1]$ and $V_0[1]$ and hence $Q[1]$ we substitute for $\Psi[1]$ from (2.19)in (2.20) and (2.21) using (2.29) and (2.30) and then equating the coefficients of Λ^0 and Λ^1 to zero we get

$$\mathbf{U_0}[\mathbf{1}] = \sigma\,\mathbf{U_0}\,\sigma^{-\mathbf{1}} + \sigma_{\mathbf{x}}\,\sigma^{-\mathbf{1}}, \tag{2.24}$$

$$\mathbf{U_0}[\mathbf{1}] = \mathbf{U_0} + [\mathbf{U_1}, \sigma], \tag{2.25}$$

Where σ^{-1} is the inverse of σ. The new solution $Q[1]$ can be calculated using either of these two equations which can be shown to be equivalent. Notice that the quantities on the right-hand side are calculated using the seed solution Q_0.

To summarize, a nonlinear differential equation can be solved with the Darboux transformation method by first finding an exact (seed) solution,Q_0, to the differential equation and finding a linear system for an auxiliary field Ψ that is associated to the differential equation through its consistency condition. Using the seed solution, a solution of the linear system,Ψ_0, is found. The linear system is then transformed into a new one via the Darboux transformation. Thus, the coefficients of the new linear system which are functionals of the new solution of the differential equation,$Q[1]$ will be related to the coefficients of the original linear system which are functionals of Q_0. This relation gives the new solution $Q[1]$ in terms of the seed solution Q_0.

2.4 Lax pair

Consider the general form of the nonlinear partial differential equation

$$F[Q(x,t), Q^{\star}(x,t), Q_t(x,t), Q_{xx}(x,t)] = 0 \tag{2.26}$$

The auxiliary field is represented by a 2×2 matrix

$$\mathbf{\Psi} = \begin{pmatrix} \psi_1(x,t) & \psi_2(x,t) \\ \phi_1(x,t) & \phi_2(x,t) \end{pmatrix}, \tag{2.27}$$

The linear system of equations of the auxiliary field is formally written as an expansion in powers of the matrix

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.28}$$

as follows

$$\mathbf{\Psi_x} = \mathbf{U_0} \cdot \mathbf{\Psi} + \mathbf{U_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda} \tag{2.29}$$

and

$$\mathbf{\Psi_t} = \mathbf{V_0} \cdot \mathbf{\Psi} + \mathbf{V_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda} + \mathbf{V_2} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda^2} \tag{2.30}$$

Here $U_{0,1}$ and $V_{0,1,2}$ are in principle functionals of Q and $Q^\star$ and their partial derivatives. The expansions were terminated at the linear and quadratic powers of Λ Eqs. (2.29) and Eqs. (2.30) respectively. To satisfy both Eqs. (2.29) and Eqs. (2.30), ψ must obey the consistency condition $\Psi_{xt} = \Psi_{tx}$ which leads to

$$\mathbf{U_{0t}} - \mathbf{V_{0x}} + [\mathbf{U_0}, \mathbf{V_0}] = \mathbf{0}, \tag{2.31}$$

$$\mathbf{U_{1t}} - \mathbf{V_{1x}} + [\mathbf{U_0}, \mathbf{V_1}] + [\mathbf{U_1}, \mathbf{V_0}] = \mathbf{0}, \tag{2.32}$$

$$\mathbf{V_{2x}} + [\mathbf{V_2}, \mathbf{U_0}] + [\mathbf{V_1}, \mathbf{U_1}] = \mathbf{0}, \tag{2.33}$$

$$[\mathbf{U_1}, \mathbf{V_3}] = \mathbf{0}, \tag{2.34}$$

where $[\mathbf{X}, \mathbf{Y}]$ denotes the commutator of $\mathbf{X}$ and $\mathbf{Y}$. These equations are obtained by equating the coefficients Λ^0, Λ^1 ,Λ^2 and Λ^3 in Ψ_{xt} to the corresponding ones in Ψ_{tx}. The matrices U_0 and V_0 are the lax pair of (2.36). It is the consistency condition,(2.31), that should be equivalent to the differential equation

$$\mathbf{U_{0t}} - \mathbf{V_{0x}} + [\mathbf{U_0}, \mathbf{V_0}] = \begin{pmatrix} 0 & F \\ -F^* & 0 \end{pmatrix}, \tag{2.35}$$

2.4.1 Example of Solitonic Solutions of the nonlinear Schrödinger equation

In this section, we illustrate the preceding technique on a simple example.

The NLSE equation can be written as:

$$i\,\psi_t + 1/2\,\psi_{xx} + \psi^*\,|\psi|^2 = 0 \tag{2.36}$$

where $\psi(x;t)$ is the wave function.

Calculating the Lax Pair

The method is described in the previous section here we summarize the basic equations explained before In order to derive the Lax pair we start by writing the following two equations:

$$\mathbf{\Psi_x} = \mathbf{U_0} \cdot \mathbf{\Psi} + \mathbf{U_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda} \tag{2.37}$$

$$\mathbf{\Psi_t} = \mathbf{V_0} \cdot \mathbf{\Psi} + \mathbf{V_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda} + \mathbf{V_2} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda^2} \tag{2.38}$$

For an auxiliary field$\mathbf{\Psi}$. Here subscripts t and x represent the derivative with respect to time and position, respectively.The matrices $\mathbf{U_0}$, $\mathbf{U_1}$, $\mathbf{V_0}$, $\mathbf{V_1}$, and $\mathbf{V_2}$ are functions of the solution of the given NLSE and its space derivatives. For obtaining the integrability condition for the NLS equation let us apply the following form for the $\mathbf{U}$ and $\mathbf{V}$ matrices:

$$\mathbf{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{2.39}$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \tag{2.40}$$

Using the compatibility condition$\Psi_{xt} = \Psi_{tx}$ and:

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.41}$$

one finds:

$$\mathbf{U_{0\,t}} - \mathbf{V_{0\,x}} + [\mathbf{U_0}, \mathbf{V_0}] = \mathbf{0}, \tag{2.42}$$

$$\mathbf{U_{1\,t}} - \mathbf{V_{1\,x}} + [\mathbf{U_0}, \mathbf{V_1}] + [\mathbf{U_1}, \mathbf{V_0}] = \mathbf{0}, \tag{2.43}$$

$$\mathbf{V_{2\,x}} + [\mathbf{V_2}, \mathbf{U_0}] + [\mathbf{V_1}, \mathbf{U_1}] = \mathbf{0}, \tag{2.44}$$

$$[\mathbf{U_1}, \mathbf{V_2}] = \mathbf{0}, \tag{2.45}$$

Existence of the Lax pair assures the integrability of the given NLSE, the given NLSE equation can be solved analytically. In Ref [184], the author derived a systematic approach to find the Lax pair. In this approach, the Lax pair $\mathbf{U_0}$ and $\mathbf{V_0}$ is constructed in terms of Ψ the solution of the NLSE and its time and position derivatives with unknown coefficients up to a certain order depending on the given NLSE. By equating the compatibility condition matrix, Equation (2.42), and the matrix constructed from the original NLSE, we get a set of equations for the unknown coefficients in $\mathbf{U_0}$ and $\mathbf{V_0}$ in terms of the coefficients of the original NLSE. Solving this set of equations gives $\mathbf{U_0}$ and $\mathbf{V_0}$ for the given NLSE.
In the second step, we construct arbitrary $\mathbf{U_1}$ and $\mathbf{V_1}$ as we do for $\mathbf{U_0}$ and $\mathbf{V_0}$. Using the second compatibility condition, Equation (2.43), we get a set of equations for the unknown coefficients. By solving this set of equations for unknown coefficients, we get $\mathbf{U_1}$ and $\mathbf{U_1}$. In the same manner we find the matrix $\mathbf{V_2}$ by solving the two compatibility conditions Equation (2.44) and Equation (2.45). The Lax pair $\mathbf{U}$ and $\mathbf{V}$ is expanded in powers of $\psi(x,t)$ and its derivatives, as follows (2.93) and (2.40), where

$u_{011} = f_1(x,t) + f_2(x,t)\psi(x,t)$
$u_{012} = f_3(x,t) + f_4(x,t)\psi(x,t)$
$u_{021} = f_5(x,t) + f_6(x,t)\psi^*(x,t)$
$u_{022} = f_7(x,t) + f_8(x,t)\psi^*(x,t)$
and
$v_{011} = g_1(x,t) + g_2(x,t)\psi(x,t) + g_3(x,t)\psi_x(x,t) + g_4(x,t)\psi(x,t)\psi^*(x,t)$
$v_{012} = g_5(x,t) + g_6(x,t)\psi(x,t) + g_7(x,t)\psi_x(x,t) + g_8(x,t)\psi(x,t)\psi^*(x,t)$
$v_{021} = g_9(x,t) + g_{10}(x,t)\psi^*(x,t) + g_{11}(x,t)\psi^*_x(x,t) + g_{12}(x,t)\psi(x,t)\psi^*(x,t)$
$v_{022} = g_{13}(x,t) + g_{14}(x,t)\psi^*(x,t) + g_{15}(x,t)\psi^*_x(x,t) + g_{16}(x,t)\psi(x,t)\psi^*(x,t)$

Employing the systematic search approach we find the complete set of matrices for equation (2.42), requires:
$g_3 = f_2 = g_{15} = f_8 = g_{14} = g_8 = g_{12} = g_9 = g_2 = f_3 = g_5 = g_{13} = g_1 = f_5 = f_1 = f_7 = g_{10} = g_6 = 0$
This results in many of the coefficients to be equal to constant,namely:
$f_6 = -ip_2, f_4 = ip_1, g_{11} = \frac{-p_2}{2}, g_7 = \frac{-p_1}{2}, g_4 = -\frac{ip_1p_2}{2}, g_{16} = \frac{ip_1p_2}{2}, p_1 = \frac{1}{p_2}, p_2 = -i;$
we obtain the following lax pair for nonlinear Schrödinger equation:

$$\mathbf{U_0} = \begin{pmatrix} 0 & \psi(x,t) \\ \psi^*(x,t) & 0 \end{pmatrix} \tag{2.46}$$

and

$$\mathbf{V_0} = \frac{\mathbf{i}}{\mathbf{2}} \begin{pmatrix} \psi(x,t)\psi^*(x,t) & \psi_x(x,t) \\ \psi^*_x(x,t) & -\psi(x,t)\psi^*(x,t) \end{pmatrix} \tag{2.47}$$

In the same manner we calculate $\mathbf{U_1}$,$\mathbf{V_1}$:
We suppose that:

- $u_{111} = f_{11}(x,t) + f_{12}(x,t)\psi(x,t) + f_{13}(x,t)\psi^*(x,t)$
- $u_{112} = f_{14}(x,t) + f_{15}(x,t)\psi(x,t) + f_{16}(x,t)\psi^*(x,t)$
- $u_{121} = f_{17}(x,t) + f_{18}(x,t)\psi(x,t) + f_{19}(x,t)\psi^*(x,t)$
- $u_{122} = f_{110}(x,t) + f_{111}(x,t)\psi(x,t) + f_{112}(x,t)\psi^*(x,t)$

and

- $v_{111} = g_{11}(x,t) + g_{12}(x,t)\psi(x,t) + g_{13}(x,t)\psi^*(x,t) + g_{14}(x,t)\psi_x(x,t) + g_{15}(x,t)\psi^*_x(x,t) + g_{16}(x,t)\psi(x,t)\psi^*(x,t)+)$
- $v_{112} = g_{17}(x,t) + g_{18}(x,t)\psi(x,t) + g_{19}(x,t)\psi^*(x,t) + g_{100}(x,t)\psi_x(x,t) + g_{111}(x,t)\psi^*_x(x,t) + g_{112}(x,t)\psi(x,t)\psi^*(x,t))$

- $v_{121} = g_{113}(x,t)+g_{114}(x,t)\psi(x,t)+g_{115}(x,t)\psi^*(x,t)+g_{116}(x,t)\psi_x(x,t)+g_{117}(x,t)\psi_x^*(x,t)+g_{118}(x,t)\psi(x,t)\psi^*(x,t)$

- $v_{122} = g_{119}(x,t)+g_{120}(x,t)\psi(x,t)+g_{121}(x,t)\psi^*(x,t)+g_{122}(x,t)\psi_x(x,t)+g_{123}(x,t)\psi_x^*(x,t)+g_{124}(x,t)\psi(x,t)\psi^*(x,t)$

Employing the systematic search approach we find the complete set of matrices for equation (2.42), requires:
$g_{12} == g_{18} = f_8 = g_{114} = g_{13} = g_{120} = g_{19} = g_{115} == g_{14} = g_{121} = g_{11} = f_{17} = f_{14} = g_{119} = 0$
This results in many of the coefficients to be equal to constant,namely:
$g_{133} = -g_{17} i \frac{\psi^*(x,t)}{\psi(x,t)}, f_{11} = -f110, f_{110} = -1, g_{17} = -i\psi(x,t);$
we obtain the following lax pair :

$$\mathbf{U_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.48}$$

and

$$\mathbf{V_1} = -\mathbf{i}\begin{pmatrix} 0 & \psi(x,t) \\ -\psi^*(x,t) & 0 \end{pmatrix} \tag{2.49}$$

We do the same things with $\mathbf{V_2}$

- $v_{211} = g_{21}(x,t)+g_{22}(x,t)\psi(x,t)+g_{23}(x,t)\psi^*(x,t)+g_{14}(x,t)\psi_x(x,t)+g_{15}(x,t)\psi_x^*(x,t)+g_{16}(x,t)\psi(x,t)\psi^*(x,t)$

- $v_{212} = g_{27}(x,t)+g_{28}(x,t)\psi(x,t)+g_{29}(x,t)\psi^*(x,t)+g_{110}(x,t)\psi_x(x,t)+g_{111}(x,t)\psi_x^*(x,t)+g_{112}(x,t)\psi(x,t)\psi^*(x,t)$

- $v_{221} = g_{213}(x,t)+g_{214}(x,t)\psi(x,t)+g_{215}(x,t)\psi^*(x,t)+g_{116}(x,t)\psi_x(x,t)+g_{117}(x,t)\psi_x^*(x,t)+g_{118}(x,t)\psi(x,t)\psi^*(x,t)$

- $v_{222} = g_{219}(x,t)+g_{220}(x,t)\psi(x,t)+g_{221}(x,t)\psi^*(x,t)+g_{122}(x,t)\psi_x(x,t)+g_{123}(x,t)\psi_x^*(x,t)+g_{124}(x,t)\psi(x,t)\psi^*(x,t)$

Employing the systematic search approach we find the complete set of matrices for equation (2.42), requires:
$g_{112} == g_{28} = f_8 = g_{115} = g_{116} = g_{110} = g_{118} = g_{215} == g_{25} = g_{214} = g_{111} = f_{213} = f_{27} = g_{117} = 0$
This results in many of the coefficients to be equal to constant,namely:
$g_{21} = i, g_{219} = -i;$

$$\mathbf{V_2} = \mathbf{i}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.50}$$

Darboux transformation:

After finding Lax pair we use the DT

$$\mathbf{U_0} = \begin{pmatrix} 0 & \psi(x,t) \\ -\psi^*(x,t) & 0 \end{pmatrix}, \tag{2.51}$$

And

$$\mathbf{V_0} = \mathbf{i/2} \begin{pmatrix} 0 & \psi(x,t)\,\psi(x,t)^* \\ \psi(x,t)\,\psi(x,t)^* & 0 \end{pmatrix}, \tag{2.52}$$

And

$$\mathbf{U_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.53}$$

$$\mathbf{V_1} = \mathbf{i} \begin{pmatrix} 0 & \psi(x,t) \\ -\psi(x,t)^* & 0 \end{pmatrix}, \tag{2.54}$$

$$\mathbf{V_2} = \mathbf{i} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.55}$$

Also

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{2.56}$$

Where

$$\mathbf{U_{0\,t}} - \mathbf{V_{0\,x}} + [\mathbf{U_0}, \mathbf{V_0}] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.57}$$

$$\mathbf{U_{1\,t}} - \mathbf{V_{1\,x}} + [\mathbf{U_0}, \mathbf{V_1}] + [\mathbf{U_1}, \mathbf{V_0}] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.58}$$

$$\mathbf{V_{2\,x}} + [\mathbf{V_2}, \mathbf{U_0}] + [\mathbf{V_1}, \mathbf{U_1}] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.59}$$

Here is time-dependent constant and is constants of integration (independent of x and t), The application of the equation (2.60), (2.61)and (2.62)

$$\mathbf{\Psi_x} = \mathbf{U_0} \cdot \mathbf{\Psi} + \mathbf{U_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda}; \tag{2.60}$$

$$\mathbf{\Psi_t} = \mathbf{V_0} \cdot \mathbf{\Psi} + \mathbf{V_1} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda} + \mathbf{V_2} \cdot \mathbf{\Psi} \cdot \mathbf{\Lambda^2}; \tag{2.61}$$

and

$$\mathbf{\Psi} = \begin{pmatrix} \psi_1(x,t) & \psi_2(x,t) \\ \phi_1(x,t) & \phi_2(x,t) \end{pmatrix}, \tag{2.62}$$

Solution of Nonlinear Schrödinger equation

Solitonic Solution

In this cases we use the seed : $\psi_0(x,t) = 0$
We derive the compatibility equations, we found 8 equations: 4 equations for time and 4 equations for space:

$$\psi_{1x} = \psi_0\phi_1 + \psi_1\lambda_1 \tag{2.63}$$

$$\psi_{2x} = \psi_0\phi_2 + \psi_2\lambda_2 \tag{2.64}$$

$$\phi_{1x} = -\psi_0^*\psi_1 - \phi_1\lambda_1 \tag{2.65}$$

$$\phi_{2x} = -\psi_0^*\psi_2 - \phi_2\lambda_2 \tag{2.66}$$

$$\psi_{1t} = \frac{1}{2}i(2\lambda_1\psi_0\phi_1 + 2\lambda_1^2\psi_1 + \psi_0\psi_0^*\psi_1 + \phi_1\psi_{0x}) \tag{2.67}$$

$$\psi_{2t} = \frac{1}{2}i(2\lambda_2\psi_0\phi_2 + 2\lambda_2^2\psi_2 + \psi_0\psi_0^*\psi_2 + \phi_2\psi_{0x}) \tag{2.68}$$

$$\phi_{1t} = \frac{1}{2}i(-2\lambda_1^2\phi_1 - \psi_0\psi_0^*\phi_1 - 2\lambda_1\psi_0^*\psi_1 + \psi_1\psi_{0x}) \tag{2.69}$$

$$\phi_{2t} = \frac{1}{2}i(-2\lambda_2^2\phi_2 - \psi_0\psi_0^*\phi_2 - 2\lambda_2\psi_0^*\psi_2 + \psi_2\psi_{0x}) \tag{2.70}$$

Using the seed $\psi_0 = 0$,$\psi_0^* = 0$, our equations are simplified into:

$$-\lambda_1\psi_1(x,t) \ + \psi_{1x}(x,t) = 0 \tag{2.71}$$

$$-\lambda_2\psi_2(x,t) \ + \psi_{2x}(x,t) = 0 \tag{2.72}$$

$$\lambda_1\phi_1(x,t) + \phi_{1x}(x,t) = 0 \tag{2.73}$$

$$\lambda_2\phi_2(x,t) + \phi_{2x}(x,t) = 0 \tag{2.74}$$

$$-i\lambda_1^2\psi_1(x,t) + \psi_{1t}(x,t) = 0 \tag{2.75}$$

$$-i\lambda_2^2\psi_2(x,t) + \psi_{2t}(x,t) = 0 \tag{2.76}$$

$$i\lambda_1^2\phi_1(x,t) + \phi_{1t} = 0 \tag{2.77}$$

$$i\lambda_2^2\phi_2(x,t) + \phi_{2t} = 0 \tag{2.78}$$

After solving the system, the solution reads:

$$\psi_1(x,t) = \exp{(x\lambda_1)}c_1(t) \tag{2.79}$$

$$\psi_2(x,t) = \exp{(x\lambda_2)}c_2(t) \tag{2.80}$$

$$\phi_1(x,t) = \exp{(-x\lambda_1)}c_3(t) \tag{2.81}$$

$$\phi_2(x,t) = \exp{(-x\lambda_2)}c_4(t) \tag{2.82}$$

Where

$$c_1(t) = \exp{\left(it\lambda_1^2\right)}c_1 \tag{2.83}$$

$$c_2(t) = \exp{\left(it\lambda_2^2\right)}c_2 \tag{2.84}$$

$$c_3(t) = \exp{\left(-it\lambda_1^2\right)}c_3 \tag{2.85}$$

$$c_4(t) = \exp{\left(-it\lambda_2^2\right)}c_4 \tag{2.86}$$

So we simplify the solution into:

$$\psi_1(x,t) = c_1\exp{\left(x\lambda_1 + it\lambda_1^2\right)} \tag{2.87}$$

$$\psi_2(x,t) = c_2\exp{\left(x\lambda_2 + it\lambda_2^2\right)} \tag{2.88}$$

and get:

$$\phi_1(x,t) = c_3\exp{\left(-x\lambda_1 - it\lambda_1^2\right)} \tag{2.89}$$

$$\phi_2(x,t) = c_4\exp{\left(-x\lambda_2 - it\lambda_2^2\right)} \tag{2.90}$$

where: c_1 , c_2 , c_3 , c_4, are real arbitrary constants .

- Consider the following version of the Darboux transformation:

$$\sigma = \psi_{\mathbf{0}} \cdot \mathbf{\Lambda} \cdot \psi_{\mathbf{0}}^{\mathbf{-1}}, \tag{2.91}$$

$$\mathbf{U_{01}} = \mathbf{U_0} + \mathbf{U_1}\sigma - \sigma\mathbf{U_1} \tag{2.92}$$

Where

$$\mathbf{U_{01}} = \begin{pmatrix} 0 & \frac{2c_{10}c_{20}\exp(2x(\lambda_1+\lambda_2)+2it(\lambda_1^2+\lambda_2^2))(\lambda_1-\lambda_2))}{(-c_1c_4\exp(2\lambda_1(x+it\lambda_1))} \\ \frac{2c_{30}c_{40}(\lambda_1-\lambda_2))}{(-c_1c_4\exp(2\lambda_1(x+it\lambda_1))} & 0 \end{pmatrix} \tag{2.93}$$

And using the following substitutions: $\psi_2 = \phi_1^*$, and $\phi_2 = -\psi_1^*$.
After some simplifications the nonlinear Schrödinger equation will have the solution:

$$\psi(x,t) = \frac{16(c_2^* c_3^*)^2 e^{2\lambda_2(x+it\lambda_2)}(\lambda_2+\lambda_2^*)^2}{-1+4(c_2^* c_3^*)^2 e^{(x+it(\lambda_2-\lambda_2^*))((\lambda_2+\lambda_2^*))}} \tag{2.94}$$

where c_2^* and c_3^* are constants of integration. The solution is depicted on figure (2.1).

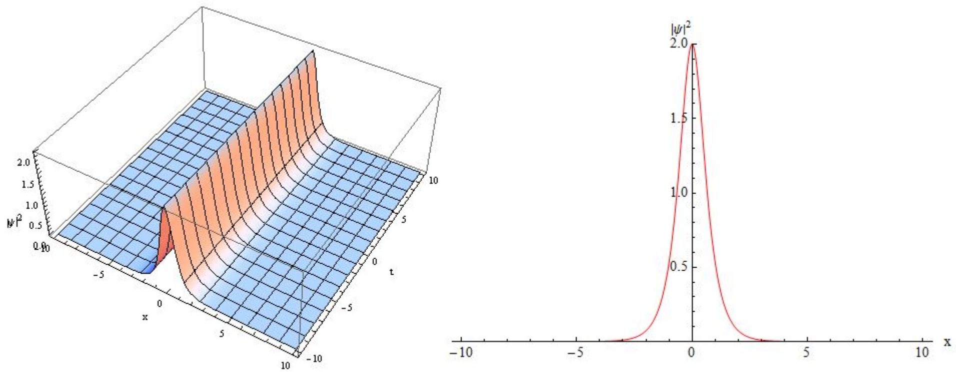

Figure 2.1: solitonic solution $|\psi|^2$ constant coeffcients, namely,$c_2^* = 1, c_3^* =$ 1:for $\lambda_2 = 1, \lambda_2^* = 1$.

Chapter 3

Solitonic solution of two Coupled Gross-Pitaevskii equations

3.1 Introduction

It is well known that nonlinear Schrödinger(NLS)-type equations play a prominent role in nonlinear physical systems, such as nonlinear optics [188] and Bose Einstein condensates [189]. In these physical systems, the nonlinear coefficient can be positive or negative, depending on the physical situations [190]. For example, the nonlinearity in optics corresponds to the focusing or defocusing case. It can be positive or negative when the interaction between the atoms is repulsive or attractive in Bose Einstein condensates [189].

3.2 The Model

For a two- component condensate, the dynamical Eqs. (2.18) may be written

$$i\,\frac{\partial \psi_1}{\partial t} + \frac{\partial_1^2 \psi_1}{\partial x^2} + V(x,t)\psi_1 + \left[g_{11}\,|\psi_1|^2 + g_{12}\,|\psi_2|^2\right]\psi_1 = 0 \tag{3.1}$$

$$i\,\frac{\partial \psi_2}{\partial t} + \frac{\partial_2^2 \psi_2}{\partial x^2} + V(x,t)\psi_2 + \left[g_{21}\,|\psi_1|^2 + g_{22}\,|\psi_2|^2\right]\psi_2 = 0 \tag{3.2}$$

when we set $\hbar = 1$, $m_1 = m_2 = 1$, and we use undimensioned space and time variables. Moreover, $V(x,t) = \frac{1}{2}\omega^2(t)x^2$ is the trapping potential.

3.3 Similarity transformation and analytical setup from GPE to Manakov system

According to the general method, let us set

$$\psi_1 = Q_1(X,T)e^{\int h(t)dt + ia(x,t)} \tag{3.3}$$

$$\psi_2 = Q_2(X,T)e^{\int h(t)dt + ia(x,t)} \tag{3.4}$$

where,

$$X = xe^{2\int h(t)\,dt} - 2\int h_1(t)e^{2\int h(t)\,dt}\,dt,$$

$$T = \int e^{4\int h(t)\,dt}\,dt,$$

$$a(x,t) = -\frac{1}{2}x^2h(t) + xh_1(t) + h_2(t)$$

$$g_{11} = g_{12} = g_{21} = g_{22} = \exp^{(}2\int h(t)dt),$$

$$\omega^2(t) = h^2(t) - \frac{h'(t)}{2}, h(t) = \frac{h_1'(t)}{2h_1(t)}$$

will reduce the coupled GP equation to the following coupled equations of the form[191, 192]

$$iQ_{1t} = [-\frac{1}{2}Q_{1xx} - (|Q_1|^2 + |Q_2|^2)Q_1] \tag{3.5a}$$

$$iQ_{2t} = [-\frac{1}{2}Q_{2xx} - (|Q_1|^2 + |Q_2|^2)Q_2] \tag{3.5b}$$

3.4 Finding the Lax Pair

Applying the Darboux transformation (DT) [193, 243] method on the generalized CNLS equation requires finding a linear system of equations for an auxiliary fields $\mathbf{\Phi}(x,t)$. The linear system is usually written in compact form in terms of the pair of matrices as follows

$$\mathbf{\Phi}_x = \mathbf{U\Phi}, \tag{3.6a}$$

$$\mathbf{\Phi}_t = \mathbf{V\Phi}, \tag{3.6b}$$

$\mathbf{U}$ and $\mathbf{V}$, known as the Lax pair, are functionals of the solutions of the model equations. The consistency condition of the linear system $\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}$ must be equivalent to the model equation under consideration. We find the following linear system which corresponds to the class of generalized CNLS .

$$\mathbf{\Phi}_x = \mathbf{U}_0\,\mathbf{\Phi} + \mathbf{U}_1\,\mathbf{\Phi}\,\mathbf{\Lambda} \tag{3.7a}$$
$$\mathbf{\Phi}_t = \mathbf{V}_0\,\mathbf{\Phi} + \mathbf{V}_1\,\mathbf{\Phi}\,\mathbf{\Lambda} + \mathbf{V}_2\,\mathbf{\Phi}\,\mathbf{\Lambda}^2 \tag{3.7b}$$

where,

$$\mathbf{\Phi} = \begin{pmatrix} \psi_1(x,t) & \psi_2(x,t) & \psi_3(x,t) \\ \phi_1(x,t) & \phi_2(x,t) & \phi_3(x,t) \\ \chi_1(x,t) & \chi_2(x,t) & \chi_3(x,t) \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} 0 & Q_1(x,t) & Q_2(x,t) \\ -Q_1^*(x,t) & 0 & 0 \\ -Q_2^*(x,t) & 0 & 0 \end{pmatrix},$$

$$\mathbf{U}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$\mathbf{V}_0 = \frac{\mathrm{i}}{2}\begin{pmatrix} Q_1(x,t)Q_1^*(x,t) + Q_2(x,t)Q_2^*(x,t) & Q_{1x}(x,t) & Q_{2x}(x,t) \\ Q_{1x}^*(x,t) & -Q_1(x,t)Q_1^*(x,t) & -Q_2(x,t)Q_1^*(x,t) \\ Q_{2x}^*(x,t) & -Q_1(x,t)Q_2^*(x,t) & -Q_2(x,t)Q_2^*(x,t) \end{pmatrix},$$

$$\mathbf{V}_1 = \begin{pmatrix} 0 & -Q_1(x,t) & -Q_2(x,t) \\ Q_1^*(x,t) & 0 & 0 \\ Q_2^*(x,t) & 0 & 0 \end{pmatrix}, \quad \mathbf{V}_2 = \mathrm{i}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $\lambda_{1,2,3}$ is the spectral parameter. The consistency condition $\mathbf{\Phi}_{xt} = \mathbf{\Phi}_{tx}$ leads to $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U},\mathbf{V}] = \mathbf{0}$ which should generate the equations

3.5 Solitonic Solution and its dynamics

- Darboux transformation

After finding Lax pair we apply now the Darboux Transformation by using zero seed for the two components.

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

Where

$$\mathbf{U_{0\,t}} - \mathbf{V_{0\,x}} + [\mathbf{U_0}, \mathbf{V_0}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

$$\mathbf{U_{1\,t}} - \mathbf{V_{1\,x}} + [\mathbf{U_0}, \mathbf{V_1}] + [\mathbf{U_1}, \mathbf{V_0}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.9}$$

$$\mathbf{V_{2\,x}} + [\mathbf{V_2}, \mathbf{U_0}] + [\mathbf{V_1}, \mathbf{U_1}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.10}$$

Here is time-dependent constant and is constants of integration (independent of x and t), The application of the equation (4.21),(4.22)and(4.23).

$$\mathbf{\Phi_x} = \mathbf{U_0} \cdot \mathbf{\Phi} + \mathbf{U_1} \cdot \mathbf{\Phi} \cdot \mathbf{\Lambda}; \tag{3.11}$$

$$\mathbf{\Phi_t} = \mathbf{V_0} \cdot \mathbf{\Phi} + \mathbf{V_1} \cdot \mathbf{\Phi} \cdot \mathbf{\Lambda} + \mathbf{V_2} \cdot \mathbf{\Phi} \cdot \mathbf{\Lambda^2} + \mathbf{V_3} \cdot \mathbf{\Phi} \cdot \mathbf{\Lambda^3}; \tag{3.12}$$

and

$$\mathbf{\Phi} = \begin{pmatrix} Q_1(x,t) & Q_2(x,t) & Q_3(x,t) \\ \phi_1(x,t) & \phi_2(x,t) & \phi_3(x,t) \\ \chi_1(x,t) & \chi_2(x,t) & \chi_3(x,t) \end{pmatrix} \tag{3.13}$$

using the seed solution

$$Q_1(x,t) = 0, \tag{3.14a}$$
$$Q_2(x,t) = 0 \tag{3.14b}$$

We find system of equations raeds:

$$Q_{1x}(x,t) - i\lambda_1 Q_1(x,t) = 0 \tag{3.15}$$

$$Q_{2x}(x,t) - i\lambda_2 Q_2(x,t) = 0 \tag{3.16}$$

$$Q_{3x}(x,t) - i\lambda_3 Q_3(x,t) = 0 \tag{3.17}$$

$$\phi_{1x}(x,t) + i\lambda_1 \phi_1(x,t) = 0 \tag{3.18}$$

$$\phi_{2x}(x,t) + i\lambda_2 \phi_2(x,t) = 0 \tag{3.19}$$

$$\phi_{3x}(x,t) + i\lambda_3 \phi_3(x,t) = 0 \tag{3.20}$$

$$\chi_{1x}(x,t) + i\lambda_1 Q_1(x,t) = 0 \tag{3.21}$$

$$\chi_{2x}(x,t) + i\lambda_2 Q_2(x,t) = 0 \tag{3.22}$$

$$\chi_{3x}(x,t) + i\lambda_3 Q_3(x,t) = 0 \tag{3.23}$$

$$Q_{1t}(x,t) + i\lambda_1^2 Q_1(x,t) = 0 \tag{3.24}$$

$$Q_{2t}(x,t) + i\lambda_2^2 Q_2(x,t) = 0 \tag{3.25}$$

$$Q_{3t}(x,t) + i\lambda_3^2 Q_3(x,t) = 0 \tag{3.26}$$

$$\phi_{1t}(x,t) - i\lambda_1^2 \phi_1(x,t) = 0 \tag{3.27}$$

$$\phi_{2t}(x,t) - i\lambda_2^2 \phi_2(x,t) = 0 \tag{3.28}$$

$$\phi_{3t}(x,t) - i\lambda_3^2 \phi_3(x,t) = 0 \tag{3.29}$$

$$\chi_{1t}(x,t) - i\lambda_1^2 Q_1(x,t) = 0 \tag{3.30}$$

$$\chi_{2t}(x,t) - i\lambda_2^2 Q_2(x,t) = 0 \tag{3.31}$$

$$\chi_{3t}(x,t) - i\lambda_3^2 Q_3(x,t) = 0 \tag{3.32}$$

- We found 18 equations depends on space(x) and time (t). We solved the 9 first equations we found the solution a function of x we replaced it in the equations function of time we found the exact solution.

Where, the linear system will have the solution

$$Q_1(x,t) = c_1 \exp(i\lambda_1(x - t\lambda_1)) \tag{3.33}$$

$$Q_2(x,t) = c_2 \exp(i\lambda_2(x - t\lambda_2)) \tag{3.34}$$

$$Q_3(x,t) = c_3 \exp(i\lambda_3(x - t\lambda_3)) \tag{3.35}$$

$$\phi_1(x,t) = c_4 \exp(i\lambda_1(-x + t\lambda_1)) \tag{3.36}$$

$$\phi_2(x,t) = c_5 \exp(i\lambda_2(-x + t\lambda_2)) \tag{3.37}$$

$$\phi_3(x,t) = c_6 \exp(i\lambda_3(-x + t\lambda_3)) \tag{3.38}$$

$$\chi_1(x,t) = c_7 \exp(i\lambda_1(-x + t\lambda_1)) \tag{3.39}$$

$$\chi_2(x,t) = c_8 \exp(i\lambda_2(-x + t\lambda_2))) \tag{3.40}$$

$$\chi_3(x,t) = c_9 \exp(i\lambda_3(-x + t\lambda_3)) \tag{3.41}$$

Where

$$c_1(t) = c_1 \exp(-it\lambda_1^2) \tag{3.42}$$
$$c_2(t) = c_2 \exp(-it\lambda_2^2) \tag{3.43}$$
$$c_3(t) = c_3 \exp(-it\lambda_3^2) \tag{3.44}$$
$$c_4(t) = c_4 \exp(it\lambda_1^2) \tag{3.45}$$
$$c_5(t) = c_5 \exp(it\lambda_2^2) \tag{3.46}$$
$$c_6(t) = c_6 \exp(it\lambda_3^2) \tag{3.47}$$
$$c_7(t) = c_7 \exp(it\lambda_1^2) \tag{3.48}$$
$$c_8(t) = c_8 \exp(it\lambda_2^2) \tag{3.49}$$
$$c_9(t) = c_9 \exp(it\lambda_3^2) \tag{3.50}$$

and considering the following version of DT [243]

$$\mathbf{\Phi}[1] = \mathbf{\Phi\Lambda} - \sigma\mathbf{\Phi}, \tag{3.51}$$

where, $\mathbf{\Phi}[1]$ is the transformed field and $\sigma = \Phi_0\,\Lambda\,\Phi_0^{-1}$. Here $\mathbf{\Phi_0}$ is a known solution of the linear system (4.10). Applying the Darboux Transformation on the linear system, Eqs. (3.6), and requiring the transformed linear system to be covariant with the original one requires

$$\mathbf{U_0}[1] = \mathbf{U_0} + [\mathbf{U_1}, \sigma]. \tag{3.52}$$

The new solution to the nonlinear equations, Eqs. (3.5), in terms of the seed solution is obtained from the last equation.

$$Q_1(x,t) = \frac{-2c_5 \exp(2it) sech(2x)}{c_2} \tag{3.53}$$

$$Q_2(x,t) = \frac{2i\sqrt{c_3^2(c_2^2 + c_5^2)} \exp(2it) sech(2x)}{c_2 c_3} \tag{3.54}$$

with $c_{2,3,5}$ are arbitrary real constants. It should be noted that in order to obtain such a localized solution we have set the following values to the spectral parameters: $\lambda_1 = \lambda_3 = i$ and $\lambda_2 = -i$.

Using the relation between $Q_1(x,t)$,$Q_2(x,t)$,$\psi_1(x,t)$ and $\psi_2(x,t)$ presented above the solitonic solution of the coupled Gross-Piteavskii take the form:

$$\psi_1(x,t) = \frac{-4ie^{i[2\int \xi_1(t)dt - 2\int_1^t \xi_2(t)dt + \xi_3(t) - \frac{x^2 h_1^{'}(t)}{2h_1(t)}]}\sqrt{h_1(t)}}{c_2(1+e^{4(-2\int h_1^2(t)dt + xh_1(t))})} \tag{3.55}$$

$$\psi_2(x,t) = 4i\sqrt{(-1+c_2^2)c_3^2}\frac{e^{i[2\int \xi_1(t)dt - 2\int_1^t \xi_2(t)dt + \xi_3(t) - \frac{x^2 h_1^{'}(t)}{2h_1(t)}]}\sqrt{h_1(t)}}{c_2 c_3(1+e^{4(-2\int h_1^2(t)dt + xh_1(t))})} \tag{3.56}$$

where

$$\xi_1(t) = h_1^2(t) + 2ih_1^2(t) = (1+2i)h_1^2(t)$$
$$\xi_2(t) = h_1^2(t) + h_2^{'}(t)$$
$$\xi_3(t) = (1-i)2xh_1(t) + h_2(t)$$

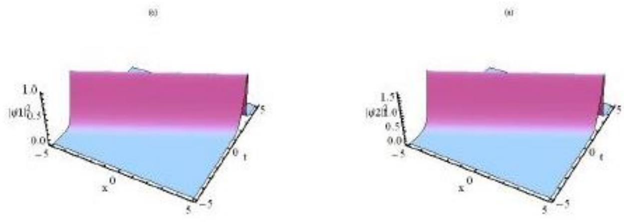

Figure 3.1: Solitonic solutions of (3.55)-(3.56) with parameters $c_2 = 2; c_3 = 1, \omega(t)^2 = 0$.

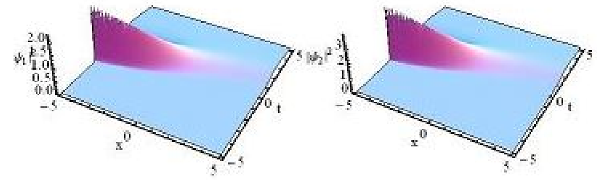

Figure 3.2: Solitonic solutions of (3.55)-(3.56) with parameters $c_2 = 2;c_3 = 1, \omega(t)^2 = 0.5$.

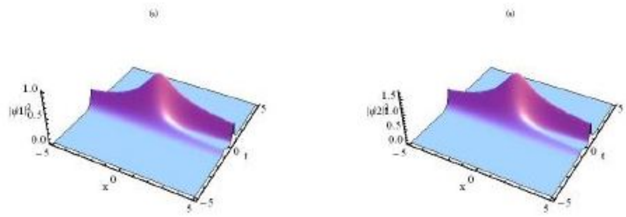

Figure 3.3: Solitonic solutions of (3.55)-(3.56) with parameters $c_2 = 2;c_3 = 1, \omega(t)^2 = 5 + 50t^2$.

The preceding figures depict a special set of solitonic solutions of the CGPE. The spectral parameters are set to $\lambda_1 = -\lambda_2 = \lambda_3 = i$ in order to obtain localized solutions. Notice how their shape is modified upon managaing the trapping frequency from a homogeneous system (fig. 3.1), to a constant frequency (fig. 3.2) and finally to a rapidly growing tight confinement (fig. 3.3).

3.6 Discussion

We present the Lax pair of the two components CGPE in a time dependent harmonic trap by transforming our model to Manakov System via similarity transformation.We employ the Darboux transformation by choosing zero seed for the two components. We find the solitonic solutions and observe that they depend on the external potential i.e, when the trap is set up, a sudden shoot up in the density appears indicating the onset of instability in the dynamical system. This means in turn that the unstable solitonic solution can be effectively controlled by modulating the trap frequency .

Chapter 4

Peregrine Soliton Management of Breathers in Two Coupled Gross- Pitaevskii Equations with External Potential

4.1 Introduction

Freak (or rogue) waves are mainly rare events, but appear in so many different areas of physics, ranging from the large amplitude ocean wave events[195] to optics[196, 197, 198, 199] , and also as solutions of the nonlinear Schrödinger equation. The main feature of these waves is their ability to suddenly emerge from nowwhere with an amplitude significantly larger than that of the surrounding wave crests and disappear without leaving a noticeable trace. They are in many limiting cases described by Peregrine solitons [200] which represent a spatially localized breather with only one oscillation in time. They constitute one of the group of breathers family along with Akhmediev and KuznetsovMa breather. Generally, they are mathematically expressed by rational polynomials [201]. It is a matter of fact that in order to establish a link with observed rare events, the excitations of these solutions should be as random as possible. Otherwise, one is led to describe them as deterministic rogue waves [202, 203]. In this context, one may use special initial configurations to excite higher order rogue waves as solutions of the nonlinear Schrödinger equation[204, 205], the Hirota equation[206, 207], the Sasa Satsuma equation, and the coupled nonlinear Schrödinger equations[203, 208, 209]. In the latter cases, and in particular in BEC experiments, Feshbach resonances [210, 211, 212] allows for a tunability of the interatomic interactions, thus managing the nonlinearity of the underlying equations at will. The control of the trapping fields also provides a powerful tool for manipulating rogue waves. In the case of binary mixtures, the situation is rather original. The appearance of rogue waves in these systems bears an interest of its own, both mathematical and physical. From the mathematical point of view, finding exact solutions can lead to a better understanding of the conditions under which the system can sustain Peregrine solitons. On the other hand, nowadays running experiments may determine whether these solutions can indeed be observed by a fine tuning of the various parameters at hand. In the present work, we are mainly interested in finding and describing analytically the Peregrine soliton solutions of two component BEC described by a set of two coupled GrossPitaevskii equations (CGPE) in one dimension or quasi-one dimensional space. By considering a harmonic trap with time-dependent frequency, we will focus on the formation mechanism of these solutions, which may lead to a kind of controllability of these rogue waves.

It is worthwhile noticing that the recently published paper [213] considers only two coupled nonlinear Schrödinger (CNLS) equations with coherent coupling terms, fixed attractive interactions and without external potentials. Here, we analyze the solutions of two coupled NLS equations with external timedependent harmonic potential. The time-dependence of its frequency will lead to novel phenomena such as the stabilization of the solitons. The presence of the trap breaks the translation invariance of the system and this will have dramatic consequences on the solutions. The various interaction parameters are left free to make the formalism as flexible as possible. In this chapter,by transforming our CGPE into a Manakov system by using a similarity transformation [214, 215, 216], we discuss the corresponding Lax pair and analytical methods which we employ to construct the exact solutions.

Consider the same system as chapter 3, described by the set(3.1,3.2)

4.2 Peregrine Soliton solution and their dynamics

4.2.1 Symmetric Case: Same Seed Solutions

$$Q_1(x,t) = Q_2(x,t) = A \exp^{2iA^2t} \tag{4.1}$$

Consider the following version of the Darboux transformation [243]:

$$\mathbf{\Phi}[1] = \mathbf{\Phi\Lambda} - \sigma\mathbf{\Phi}, \tag{4.2}$$

where $\mathbf{\Phi}[1]$ is the transformed field and $\sigma = \Phi_0\,\Lambda\,\Phi_0^{-1}$, where $\mathbf{\Phi_0}$ is a known solution of the linear system (4.10).

Requiring the transformed linear system to be covariant with the original one

$$\mathbf{U_0}[1] = \mathbf{U_0} + [\mathbf{U_1}, \sigma]. \tag{4.3}$$

which in turn gives the solutions

$$Q_1(x,t) = Q_2(x,t) = -\frac{Ae^{2iA^2t}(4A^2c_1^2 + a(x,t))) + b(x,t))}{4A^2c_1^2 + c(x,t)) + b(x,t)} \tag{4.4}$$

where

$$a(x,t) = c_2^2(8A^2t^2 + 4A^2(x^2 - 2it) + 2\sqrt{2}Ax - 1) \tag{4.5a}$$

$$b(x,t) = 2Ac_1c_2(4Ax + \sqrt{2}) \tag{4.5b}$$

$$c(x,t) = c_2^2(8A^2t^2 + 4A^2x^2 + 2\sqrt{2}Ax + 1) \tag{4.5c}$$

where $c_{1,2}$ are arbitrary real constants. The localized solutions are obtained provided $\lambda_1 = -\lambda_2 = -\lambda_3 = -\sqrt{2}i$ Finally, we get the Peregrine solutions

$$\psi_1(x,t) = \psi_2(x,t) = \frac{A\sqrt{h_1(t)}e^{ik_3(x,t)}k_4(x,t)}{k_5(x,t)} \tag{4.6}$$

where

$$\begin{aligned} k_3(x,t) = 2A^2 \int h_1(t)^2 dt + B_1(x) + B_2(t) \\ - \frac{x^2 h_1'(t)}{2h_1(t)} + 2xh_1(t) + h_2(t) + h_3(x) \end{aligned} \tag{4.7a}$$

$$\begin{aligned} k_4(x,t) = 4A^2c_1^2 + 2c_2^2(4A^2x^2h_1(t)^2 \\ + 8A^2(A^2+2)(\int h_1(t)^2dt)^2 + 2\sqrt{2}Axh_1(t) \\ 4 - 4A(4Axh_1(t) + 2iA + \sqrt{2}) \int h_1(t)^2 dt - 1) + 2Ac_1^2c_2^2 \\ (4Axh_1(t) - 8A \int h_1(t)^2 dt + \sqrt{2}) \end{aligned} \tag{4.7b}$$

$$
\begin{aligned}
k_5(x,t) &= 4A^2c_1^2 + c_2^2(8A^4(\int h_1(t)^2dt)^2 \\
&+ 4A^2(xh_1(t) - 2\int h_1(t)^2dt)^2 + 2\sqrt{2}A(xh_1(t) - 2\int h_1(t)^2dt) \\
&+ 1) + 2Ac_1c_2(4Axh_1(t) - 8A\int h_1(t)^2dt + \sqrt{2})
\end{aligned} \tag{4.7c}
$$

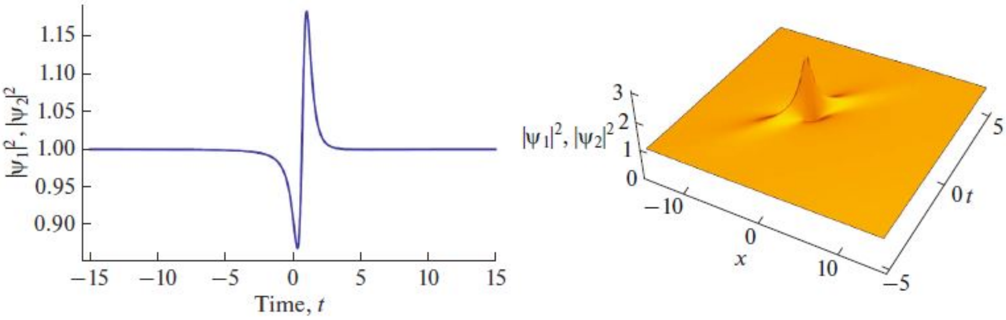

Figure 4.1: Density profile $|\psi_1|^2, |\psi_2|^2$ for bright vector solitons. The parameters are: $A = 1, c_1 = 0.1, c_2 = 0.2, \omega(t) = 0$.[227]

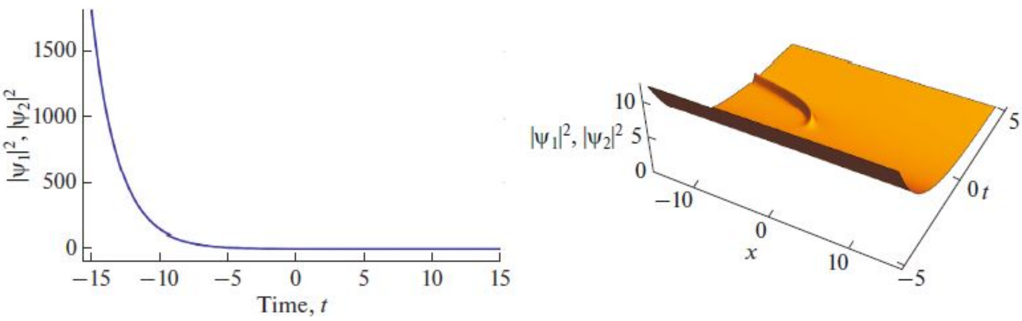

Figure 4.2: The onset of instability in the densities for a static harmonic trap with $A = 1, c_1 = 0.1, c_2 = 0.2, \omega(t)^2 = 0.5$.[227]

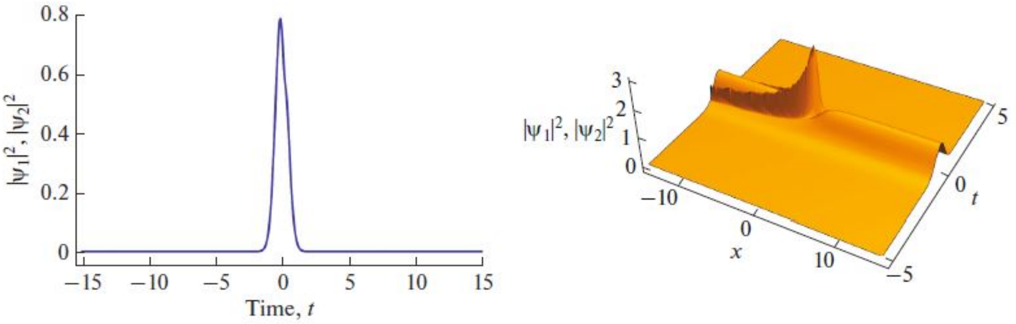

Figure 4.3: Density profile $|\psi_1|^2, |\psi_2|^2$ for the bright vector solitons in the time dependent trap with $A = 1, c_1 = 0.1, c_2 = 0.2, \omega(t)^2 = 5 + 50t^2$.[227]

To understand the dynamics of these solutions,we first consider the homogeneous system by choosing $h_1(t) = 0$, $\omega(t) = 0$ which leads to the well-known Manakov model [218]. The behavior of the Peregrine soliton under the above condition is shown in Figure 4.1. The upper and lower panels in these figures show, respectively, the projected square moduli of the solutions. Consider now a static harmonic trap with $h_1(t) = \exp(-t)$, $\omega(t) = 0.5$ The behavior changes dramatically since the densities grow abruptly as shown in the Figure 4.2. The solitons become highly unstable. In order to overcome this instability and to increase the lifetime of the solitons, a fine tuning of the trap frequency may be helpful [220]. Indeed, choosing arbitrarily $h_1(t) = \exp(\int -10tdt)$, yields a significant stabilization of both modes ψ_1 and ψ_1 as shown in the Figure 4.3. This may be well understood since for this set of parameters, the trap is very tight for all times, as its curvature is rapidly growing. One may wonder whether the reduction of the trap frequency will induce instabilities once more. In fact, as we show in Figure 4.3, even with a very slowly varying frequency, that is with a very flat trap, the system is still stable.

4.2.2 Non-Symmetric Case: Distinct Seed Solutions

$$Q_1(x,t) = \exp^{it}, Q_2(x,t) = 0 \tag{4.8}$$

In order to confirm our findings, the question is whether they depend on the seed solutions. We therefore consider different seed solutions. Following the same procedure as in the previous section, we get

$$\begin{aligned} Q_1(x,t) &= \frac{e^{it}[-1+2c_1^2e^{2x}+2c_1c_2e^{2x}(-1+2x)+c_2^2e^{2x}(-1-4it+2t^2-2x+2x^2)]}{1+2c_1^2e^{2x}+2c_1c_2e^{2x}(-1+2x)+c_2^2e^{2x}(1+2t^2-2x+2x^2} \\ Q_2(x,t) &= \frac{4e^{3it/2+x}[c_1+c_2(it+x)]}{1+2c_1^2e^{2x}+2c_1c_2e^{2x}(-1+2x)+c_2^2e^{2x}(1+2t^2-2x+2x^2} \end{aligned} \tag{4.9}$$

where $c_{1,2}$ are arbitrary real constants. The spectral parameters have been chosen such that $\lambda_1 = \lambda_3 = i$ and $\lambda_2 = -i$. The relations between theQ_i and theψ_i yield the Peregrine solutions:

$$\psi_1(x,t) = \sqrt{h_1(t)}e^{ik_1(x,t)}(-1+\frac{2+c_2^2(2+4i\int h_1(t)^2dt)e^{2xh_1(t)-4\int h_1(t)^2dt}}{G(xh_1(t)-2\int h_1(t)^2dt,\int h_1(t)^2dt)}) \tag{4.10a}$$

$$\psi_2(x,t) = \sqrt{h_1(t)}e^{ik_2(x,t)}(-1+\frac{2+c_2^2(2+4i\int h_1(t)^2dt)e^{2xh_1(t)-4\int h_1(t)^2dt}}{G(xh_1(t)-2\int h_1(t)^2dt,\int h_1(t)^2dt)}) \tag{4.10b}$$

where

$$\begin{aligned} k_1(x,t) &= B_1(x)+B_2(t)-\frac{x^2h_1'(t)}{2h_1(t)}+2xh_1(t)+h_2(t) \\ &+\int h_1(t)^2dt+h_3(x) \end{aligned} \tag{4.11a}$$

$$\begin{aligned} k_2(x,t) &= B_1(x)+B_2(t)-\frac{x^2h_1'(t)}{2h_1(t)}+(2-i)xh_1(t)+h_2(t) \\ &+(\frac{3}{2}+2i)\int h_1(t)^2dt+h_3(x) \end{aligned} \tag{4.11b}$$

$$B_1(x) = -h_3(x), B_2(t) = \int_1^t \frac{1}{2}(-2h_2'(t)-4h_1(t)^2)dt\,\mathrm{d}x \tag{4.11c}$$

$$G(X_1,T_1) = 2c_2^2c_1e^{2X_1}(2X_1-1)+2c_1e^{2X_1}+1+c_2^2e^{2X_1}$$
$$(2X_1-X_1+2T_1^2+1) \tag{4.11d}$$

with

$$X_1 = xh_1(t) - 2\int h_1(t)^2dt, T_1 = \int h_1(t)^2dt$$

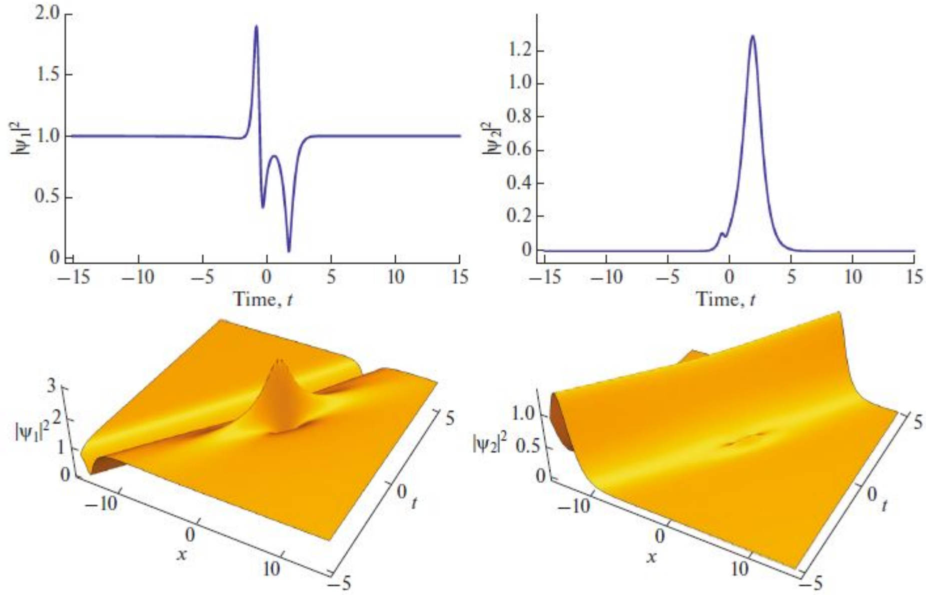

Figure 4.4: (Color online) Trapless bright vector solitons. The parameters are $A = 1, c_1 = -5, c_2 = 5, \omega(t)^2 = 0$.[227]

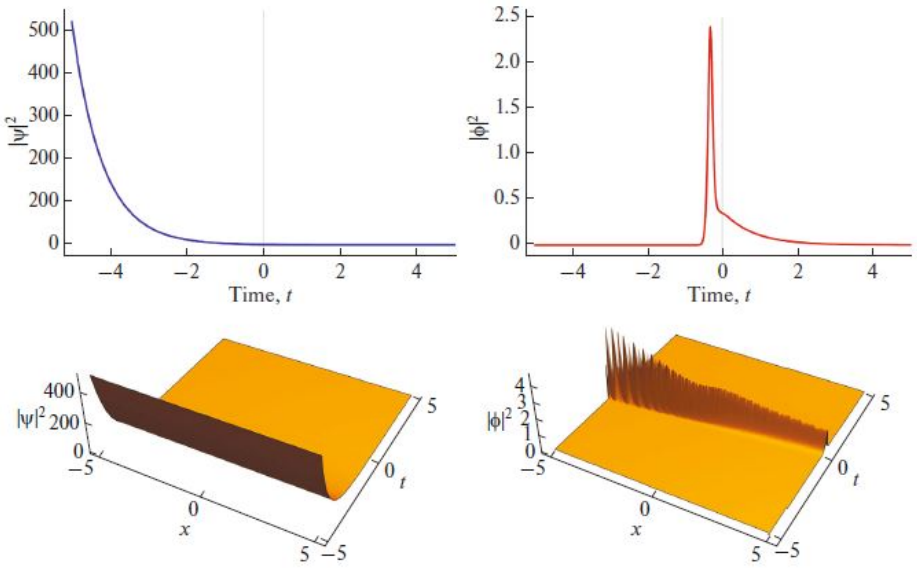

Figure 4.5: (Color online) The onset of instability in the densities for a static harmonic trap with $A = 1, c_1 = -0.1, c_2 = 0.1, \omega(t)^2 = 0.5$.[227]

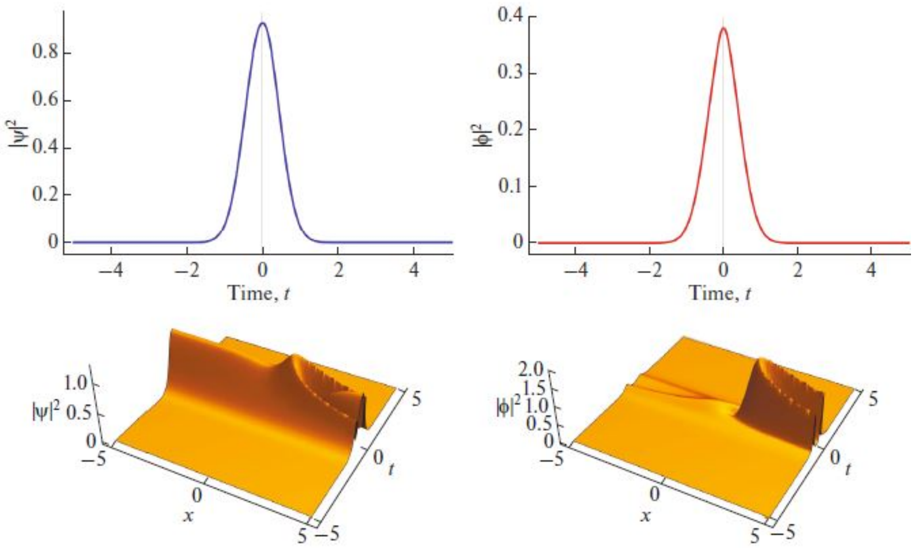

Figure 4.6: (Color online) Trapped bright vector solitons in a time dependent trap $A = 1, c_1 = -0.1, c_2 = 0.1, \omega(t)^2 = 5 + 50t^2$.[227]

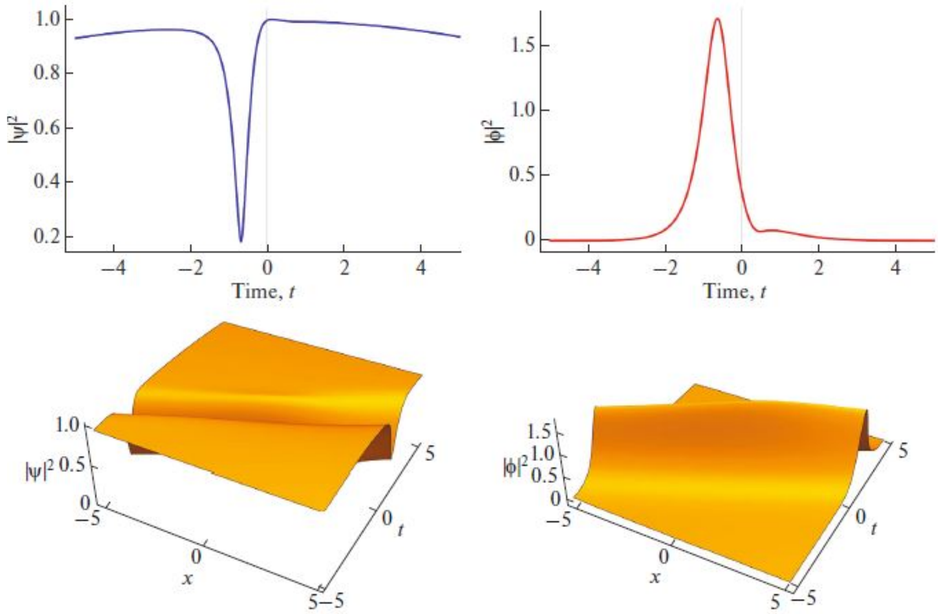

Figure 4.7: (Color online) Trapped dark and bright solitons for $A = 1, c_1 = -0.1, c_2 = 0.1, \omega(t)^2 = 0.05 + 0.01t^2$.[227]

The results are discussed below. Figure 4.4 depicts the densities for a homogeneous system ($h_1(t) = 0$, $\omega(t) = 0$,). For a static harmonic trap ($h_1(t) = exp(-t)$, $\omega(t) = 0.5$), Figure 4.5, the solitons are destabilized. Upon tuning the trap frequency with a rapidly growing curvature ($h_1(t) = exp(\int -10tdt)$, $\omega(t) = 5 + 50t^2$) significantly stabilizes the two modes and as shown in Figure 4.6. Finally, even with a very slowly varying frequency, the system is not only still stable, but a novel phenomenon takes place. The system now sustains the coexistence of both a bright and a dark soliton (Figure 4.7). This result is quite interesting by itself since, to the best of our knowledge, the existence of a stable bright-dark soliton pair in a binary mixture has attracted very little attention.

In this chapter, the spectral parameters have been given special values namely, $\lambda_1 = -\lambda_2 = -\lambda_3 = \sqrt{2}i$, for the same seed solution and $\lambda_1 = -\lambda_2 = \lambda_3 = i$ for distinct seed solutions. In the next chapter, we will see that, upon letting the λ's free, we will discover new families of solitonic solutions.

Chapter 5

New families of breathers in trapped two component condensates

5.1 Introduction

In recent years, great effort has been devoted to explain rogue wave (RW) excitations through nonlinear processes [221, 222, 223, 224, 225, 226]. Among other things, it has been found that the nonlinear Schrödinger equation (NLSE) can describe many of their dynamical features. Indeed, some families of its exact solutions have been considered as describing possible mechanisms for the formation of RW's such as Peregrine solitons [227], time periodic (or Ma) breathers[228] and space periodic (or Akhmediev) breathers[229, 230]. The name 'breather' reflects the behavior of the density profile of the solution which is either periodic in time and localized in space or periodic in space and localized in time.

Bose-Einstein condensates (BECs) constitute by now important experimental and theoretical grounds for the study of such nonlinear structures[231, 232, 233, 234, 235, 236] owing to the ability of fine tuning the interactions by Feshbach resonances[237, 238] and to the manageability of trapping fields[239]. Moreover, mixing several condensate species or several components of the same species yields phenomena that cannot be observed elsewhere. In this context, multi-solitons and multi-rogue waves have also been predicted in the two-component BEC[240, 241]. These macroscopic structures can be described by the 1D coupled Gross-Pitaevskii equations (CGPE), which are particular cases of the NLSE with cubic nonlinearity.

In this chapter, we will extend the results of the preceding chapter by considering the spectral parameters as free parameters of the method.

Upon changing the spectral parameters λ_i, the nature of the solutions changes accordingly from general breathers, passing by Ma and Akhmediev breathers and finally arriving at rogue waves. The Peregrine solitons are also recovered as a particular case of our general formulae.

5.2 Darboux transformation and analytic solutions

Following exactly the same procedure as in the preceding chapter, with a symmetric seed solution $Q_1(x,t) = Q_2(x,t) = A\exp(2iA^2t)$, we find the following solution

$$Q_1(x,t) = Q_2(x,t) = A\exp^{2iA^2t}\left[1 - \frac{F_1(x,t,\lambda_1,\lambda_2)}{F_2(x,t,\lambda_1,\lambda_2) + F_3(x,t,\lambda_1,\lambda_2)}\right] \quad (5.1)$$

where

$$F_1(x,t,\lambda_1,\lambda_2) = 2i(\lambda_1 - \lambda_2)\,\mu\,\cosh\theta_1\sinh\theta_2 \quad (5.2a)$$

$$F_2(x,t,\lambda_1,\lambda_2) = -\mu^2\sinh\theta_1\sinh\theta_2 \quad (5.2b)$$

$$F_3(x,t,\lambda_1,\lambda_2) = \mu\cosh\theta_1(\nu\cosh\theta_2 + i(\lambda_1 - \lambda_2)\sinh\theta_2) \quad (5.2c)$$

and

$$\theta_1 = [(x - \lambda_1 t)\mu] \quad , \quad \theta_2 = [(x - \lambda_2 t)\nu] \quad , \quad \mu = \sqrt{-2A^2 - \lambda_1^2} \quad , \quad \nu = \sqrt{-2A^2 - \lambda_2^2}$$

Using the similarity transformation (chapter 3, section 3), we return back to the wave functions. The result (5.1) is our main finding. Indeed, as we will see, it is a general formula "interpolating" from the well known breather (Akhmediev, Ma, Peregrine) to less known ones, as the rogue waves.

$$\psi_1(x,t) = \psi_2(x,t) = A\exp^{if_1(x,t)}\left[1 - \frac{G_1(x,t,\lambda_1,\lambda_2)}{G_2(x,t,\lambda_1,\lambda_2) + G_3(x,t,\lambda_1,\lambda_2)}\right]\sqrt{h_1(t)} \quad (5.3)$$

where

$$f_1(x,t) = -\int h_2(t)' dt - 2(1-A^2)\int h_1(t)^2 dt + 2xh_1(t) + h_2(t) - x^2 \frac{h_1'(t)}{2h_1(t)} \tag{5.4a}$$

$$G_1(x,t,\lambda_1,\lambda_2) = 2i(\lambda_1-\lambda_2)\mu \cosh\varphi_1 \sinh\varphi_2 \tag{5.4b}$$

$$G_2(x,t,\lambda_1,\lambda_2) = -\mu^2 \sinh\varphi_1 \sinh\varphi_2 \tag{5.4c}$$

$$G_3(x,t,\lambda_1,\lambda_2) = \mu \cosh\varphi_1 (\nu \cosh\varphi_2 + i(\lambda_1-\lambda_2)\sinh\varphi_2) \tag{5.4d}$$

$$\varphi_1 = \mu\big(-(2+\lambda_1)\int h_1(t)^2 dt + xh_1(t)\big)$$

$$\varphi_2 = \nu\left(-(2+\lambda_2)\int h_1(t)^2 dt + xh_1(t)\right)$$

and h_1, h_2 are given in the appendix of ref[227].

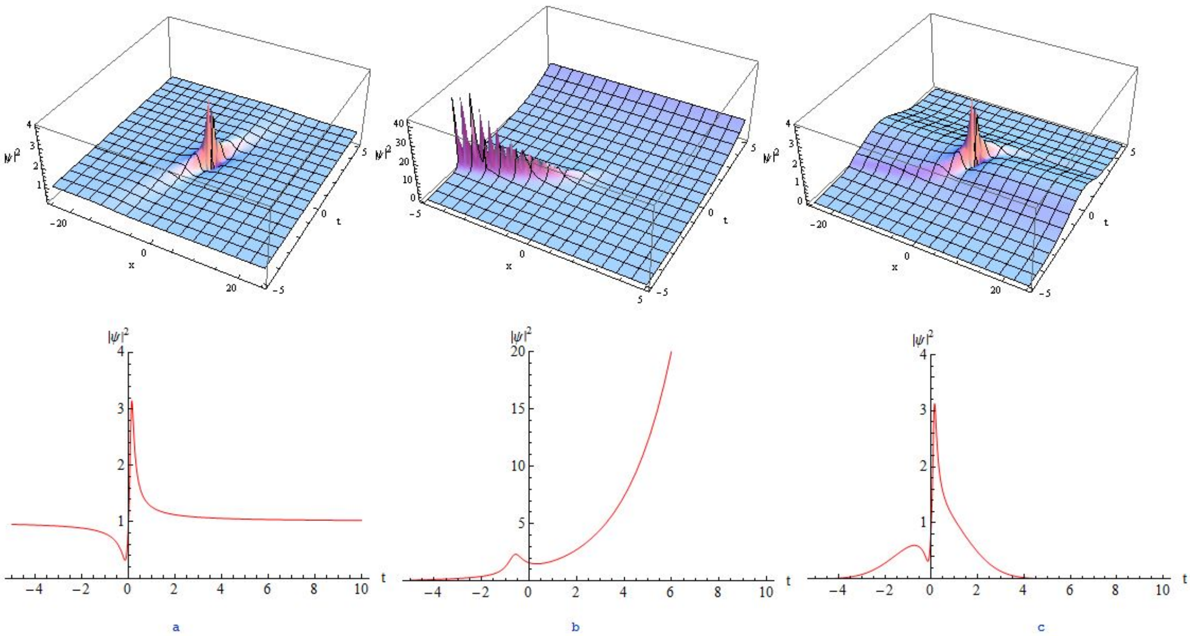

Figure 5.1: Density profile $|\psi|^2$ for a Peregrine soliton (rogue wave). The spectral parameters are $\lambda_1 = \lambda_2^* = i\sqrt{2}$. (a) Homogeneous case: $\omega^2(t) = 0$. (b) Static harmonic confinement: $\omega^2(t) = 0.5$. (c) Growing tight confinement: $\omega^2(t) = 0.5(1 + t^2)$.[242]

As announced previously, upon setting $\lambda_1 = \lambda_2^* = i\sqrt{2}$, we get the Peregrine soliton (a kind of rogue waves) found in ref.[227]. To analyze its behavior under frequency modulation, we plot in figure 5.1 the density profile in the (x, t) plane (upper panel) for various frequencies. In the homogeneous case, we observe a self sustained matter wave which is completely destroyed by a static trap and then arises again for a tight growing confinement. Notice its very short lifetime (Figure 5.1a, lower panel) compared to the trapped case (Figure 5.1c), which may have great applications in the control of rogue waves.

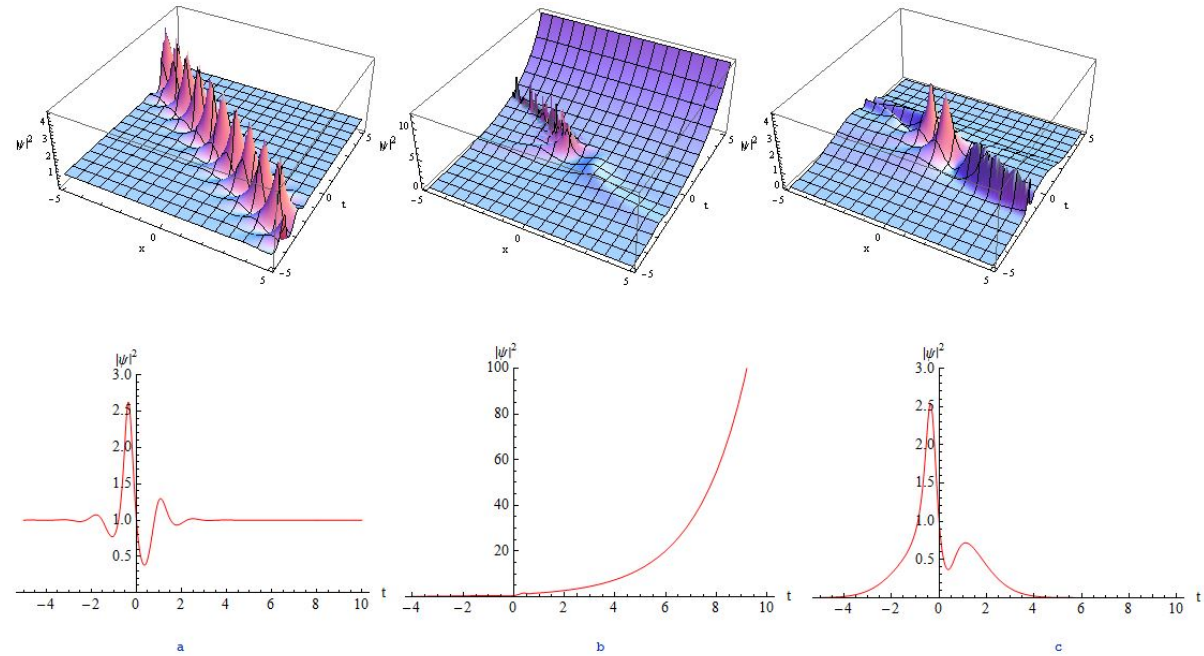

Figure 5.2: Density profile $|\psi|^2$ for a general breather solution. The spectral parameters are $\lambda_1 = \lambda_2^* = i - 1$. (a) Homogeneous case: $\omega^2(t) = 0$. (b) Static harmonic confinement: $\omega^2(t) = 0.5$. (c) Growing tight confinement: $\omega^2(t) = 0.5(1 + t^2)$.[242]

As a second illustration, we set $\lambda_1 = i - 1$ and $\lambda_2 = -i - 1$ (figure 5.2). We now modify the trap frequency from 0 (Figure. 5.2a: no trap), to $1/\sqrt{2}$ (Figure. 5.2b: static trap), and finally to $\sqrt{(1 + t^2)/2}$ (Figure. 5.2c: growing confinement). The (x, t) solutions are represented in the upper panel of Figure 5.2. We observe a general breather profile for the homogeneous system depicting a self sustained matter wave (since there is no trap). When one introduces a static harmonic trap, the solution is destroyed and then reappears when a growing tight confinement is applied. The lifetime of the breather is also larger than its value in the homogeneous case.

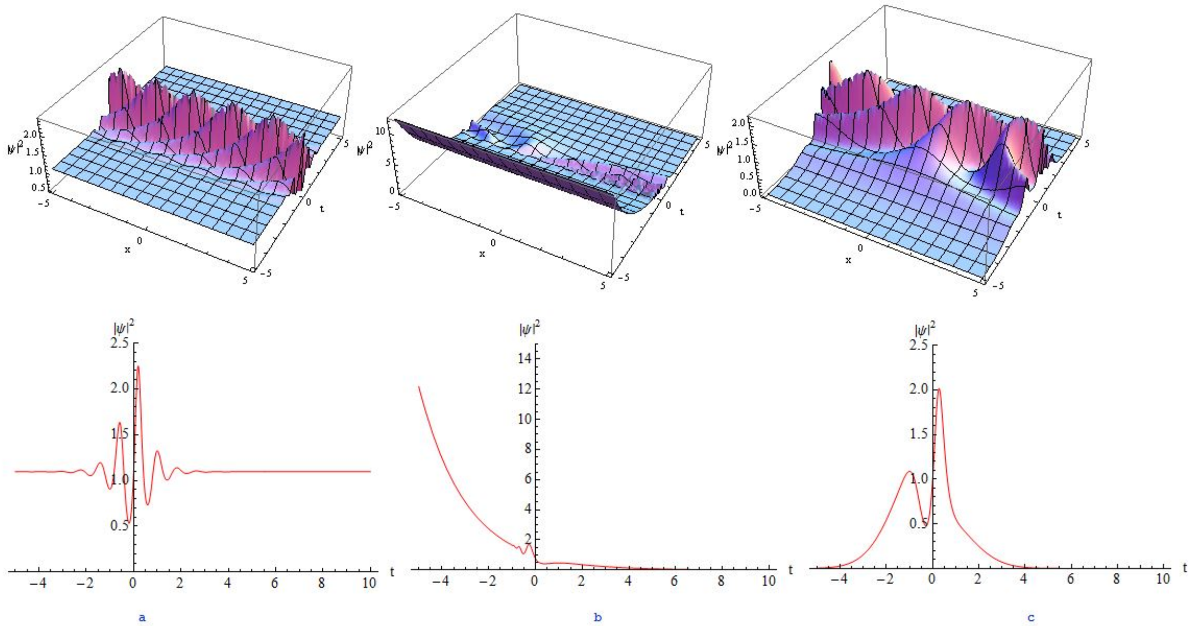

Figure 5.3: Density profile $|\psi|^2$ for Akhmediev breather solution. The spectral parameters are $\lambda_1 = \lambda_2^* = 0.5i$. (a) Homogeneous case: $\omega^2(t) = 0$. (b) Static harmonic confinement: $\omega^2(t) = 0.5$. (c) Growing tight confinement: $\omega^2(t) = 0.5(1 + t^2)$.[242]

In figure 5.3, we select another set of spectral parameters, namely $\lambda_1 = 0.5i$ and $\lambda_2 = -0.5i$, to obtain an Akhmediev breather which has the same behavior as before regarding the trap. Indeed, the homogeneous system sustains a stable solution, which is destroyed by a static trap. The rapidly growing harmonic confinement allows for a revival of this breather and a larger lifetime than the homogeneous case (lower panel).

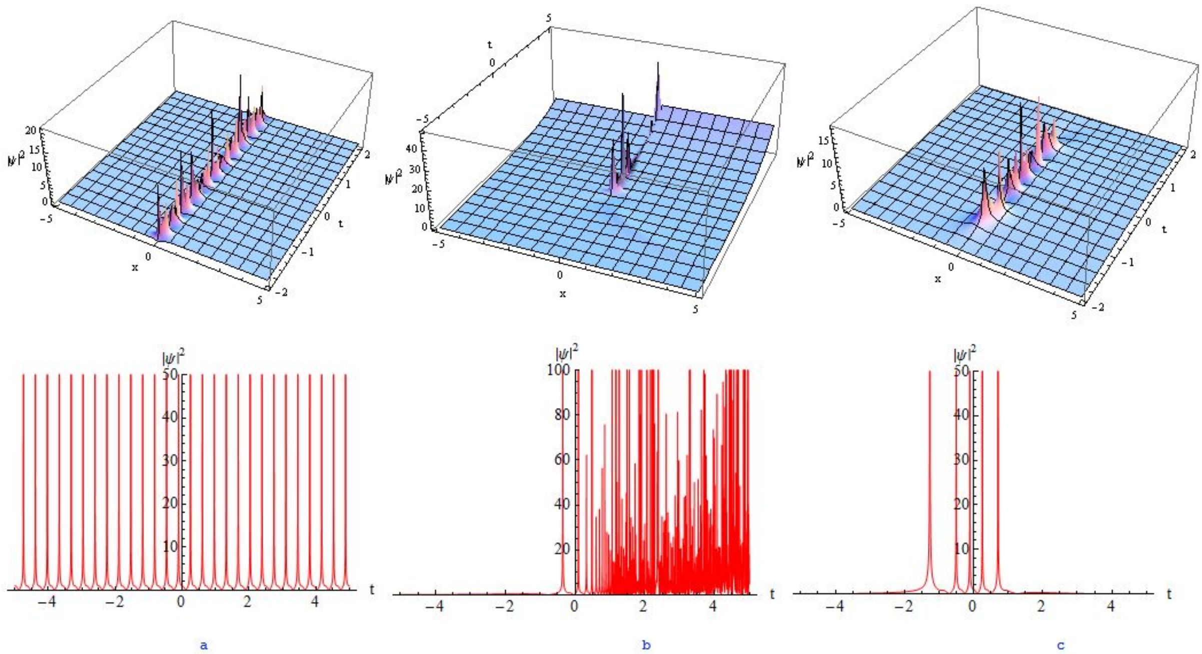

Figure 5.4: Density profile $|\psi|^2$ for Ma breather solution. The spectral parameters are $\lambda_1 = \lambda_2^* = -i\pi - 2$. (a) Homogeneous case: $\omega^2(t) = 0$. (b) Static harmonic confinement: $\omega^2(t) = 0.5$. (c) Growing tight confinement: $\omega^2(t) = 0.5(1 + t^2)$.[242]

This is to be contrasted to the situation presented in figure 5.4, where the spectral parameters are $\lambda_1 = -i\pi - 2$ and $\lambda_2 = i\pi - 2$, and the resulting profile is a Kuznetsov-Ma breather. The latter is also self sustained in the homogeneous case, and is destabilized by a static trap. The growing harmonic confinement allows for a revival of this breather but, unlike the previous cases, the lifetime is smaller than in the homogeneous case which is therefore more able to sustain long lived Ma breathers.

Conclusions and perspectives

In this work, we have considered a bose-bose mixture at zero temperature and developed a systematic method to find analytic solutions.
We used two powerful mathematical fools: Variational method, supplemented with the Darboux transformation and the Lax pair method.
In chapter 2, we present the Balian V*é*n*é*roni variational principle and derive the time dependent mean field equations for a binary mixture.

In chapter2, we focus in $T = 0$ by neglecting thermal and fluctuation effects. This led us to two coupled Gross- Pitaevskii equations.
To solve analytically these equations, we present in the same chapter the Darboux transformation method and illustrate it on a simple example.

In chapters 3, 4 and 5, we exploit these methods for more realistic solutions to find a set of solitonic solutions.

we find solitonic solutions, which depend on the frequency of the trapping field. In chapter 4, the Darboux transformation is used in two cases. The symmetric case with the same seed solutions yields in the homogeneous case bright vector solitons which are destabilized by the introduction of a static harmonic confinement. Upon modulating the frequency of the trap, the solitons are stabilized being a pair of bright solitons for a growing tight confinement. These results are almost independent of the seed solutions. Indeed, if one begins with nonsymmetric seed solutions, the overall behavior does not dramatically change. The system still sustains bright vector solitons which are destabilized (for a static trap), then stabilized by a rapidly growing tight confinement. For an almost static confinement, the solutions consist of a dark-bright soliton pair. In chapter 5, we succeed in finding analytically a universal class of solutions for non fixed values of the spectral parameters. It is worth emphasizing that this is a quite formidable technical task since, for free parameters,we notice that the solutions depend crucially on the spectral parameters and on the trap frequency. The Darboux transformation method requires handling (3 x 3) matrices. The search for general solutions is much more elaborate. Our main result, expressed by expression (5.3), shows that these solutions depict quite different physics, depending on the former parameters, which may be varied continuously. Indeed, choosing four different sets of parameters, we span a large space of solutions ranging from Peregrine solitons, to Akhemdiev and Kuznetsov-Ma breathers and more general breathers as well. The whole solutions bear a common behavior regarding the confinement. They are self sustained in the trapless case, destroyed by a static harmonic trap and arise again when a growing tight confinement is applied. However, unlike the Kuznetsov-Ma breather, the trapped Akhmediev and rogue wave solutions have a larger lifetime than the corresponding homogeneous case. This can be helpful in experimental setups.

Most importantly, in situations where these phenomena are expected to have undesirable effects, we have presented a simple way to avoid them which consists in employing a static confinement.

Moreover, in this work, we have considered contract interaction only. It is natural to ask whether these results remain true for long range interactions, such as dipolar forces. Finally, extensions to finite temperature cases is also of considerable interest. In this case, one has to handle the whole set of equations (2.15-2.16 and 2.17).

Beyond mean field effects such as LHY corrections are also of considerable actual interests. In double condensates, these have been shown to lead to new states of matter known as droplets[244, 245, 246]. A crucial question is the behavior of these droplets under time dependent traps and if analytical approaches such as the ones presented here, may be applied. These and other perspectives will constitute our future field of investigations.

Chapter 6

Appendix

In this Appendix, we show a detailed derivation of the timedependent Coupled Gross Pitaevskii equations, by using the time dependent variational principle of Balian-V*é*n*é*roni

6.1 The Balian-V*é*n*é*roni variational principle

The time-dependent variational principle of Balian and V*é*n*é*roni requires first the choice of a trial density operator. In our case, we will consider a Gaussian time-dependent density operator

$$D(t) = D_0(t) = \exp Q(t) \tag{6.1}$$

when

$$\begin{aligned}
Q(t) = \nu_1 &+ \int_r [\lambda_1(r,t)\psi_1^\dagger(r) + \lambda_2(r,t)\psi_2^\dagger(r) + \lambda_1^*(r,t)\psi_1(r) + \lambda_2^*(r,t)\psi_2(r)] \\
&+ \int_{r,r'} [\bar{\psi}_1(r) s_1(r,r',t)\bar{\psi}_1(r') + \bar{\psi}_1^\dagger(r) s_1^*(r,r',t)\bar{\psi}_1^\dagger(r') + \bar{\psi}_1^\dagger(r) s_2(r,r',t)\bar{\psi}_1(r')] \\
&+ \int_{r,r'} [\bar{\psi}_2(r) s_3(r,r',t)\bar{\psi}_2(r') + \bar{\psi}_2^\dagger(r) s_3^*(r,r',t)\bar{\psi}_2^\dagger(r') + \bar{\psi}_2^\dagger(r) s_4(r,r',t)\bar{\psi}_2(r')] \\
&+ \int_{r,r'} [\bar{\psi}_1(r) s_5(r,r',t)\bar{\psi}_2(r') + \bar{\psi}_1^\dagger(r) s_5^*(r,r',t)\bar{\psi}_2^\dagger(r')] \\
&+ \int_{r,r'} [\bar{\psi}_1^\dagger(r) s_6(r,r',t)\bar{\psi}_2(r') + \bar{\psi}_2^\dagger(r) s_6^*(r,r',t)\bar{\psi}_1(r')]
\end{aligned}$$

$$(6.2)$$

We choose the ansatz

$$
\begin{aligned}
A(t) = A_0(t) = \nu_2(t) + \int_r [u_1(r,t)\psi_1^\dagger(r) + u_2(r,t)\psi_2^\dagger(r) + u_1^*(r,t)\psi_1(r) + u_2^*(r,t)\psi_2(r)] \\
+ \int_{r,r'} [u_3(r,r',t)\bar{\psi}_1(r)\bar{\psi}_1(r') + u_3^*(r,r',t)\bar{\psi_1^\dagger}(r)\bar{\psi_1^\dagger}(r') + u_4(r,r',t)\bar{\psi_1^\dagger}(r)\bar{\psi}_1(r')] \\
+ \int_{r,r'} [u_5(r,r',t)\bar{\psi}_2(r)\bar{\psi}_2(r') + u_5^*(r,r',t)\bar{\psi_2^\dagger}(r)\bar{\psi_2^\dagger}(r') + u_6(r,r',t)\bar{\psi_2^\dagger}(r)\bar{\psi}_2(r')] \\
+ \int_{r,r'} [u_7(r,r',t)\bar{\psi}_1(r)\bar{\psi}_2(r') + u_7^*(r,r',t)\bar{\psi_1^\dagger}(r)\bar{\psi_2^\dagger}(r')] \\
+ \int_{r,r'} [u_8(r,r',t)\overline{\psi_1^\dagger}(r)\bar{\psi}_2(r') + u_8^*(r,r',t)\bar{\psi_2^\dagger}(r)\bar{\psi}_1(r')]
\end{aligned}
\tag{6.3}
$$

The action:

$$I = Tr(AD)_{t_f} - \int_{t_i}^{t_f} dt Tr A(t)(\frac{d}{dt}D(t) + i[H, D(t)]) \tag{6.4}$$

6.2 Coupled Gross- Pitaevskii Equation (CGPE)

The second order quantized Hamiltonian for condensate mixtures is written in terms of the Bose field operators $\hat{\Psi}(\mathbf{r},t)(\hat{\Psi}(\mathbf{r},t)^\dagger)$ for annihilation (creation) of particle in species i at position r and time $t$by

$$
\begin{aligned}
H = \int d\mathbf{r}\hat{\psi_1}^\dagger(\mathbf{r},t)\hat{h_1}\hat{\psi_1}(\mathbf{r},t) \\
+ \frac{1}{2}\int\int d\mathbf{r}d\mathbf{r}'\hat{\psi_1}^\dagger(\mathbf{r},t)\hat{\psi_1}^\dagger(\mathbf{r}',t)V_1(\mathbf{r}-\mathbf{r}')\hat{\psi_1}(\mathbf{r}',t)\hat{\psi_1}(\mathbf{r},t) \\
+ \int d\mathbf{r}\hat{\psi_2}^\dagger(\mathbf{r},t)\hat{h_2}\hat{\psi_2}(\mathbf{r},t) \\
+ \frac{1}{2}\int\int d\mathbf{r}d\mathbf{r}'\hat{\psi_2}^\dagger(\mathbf{r},t)\hat{\psi_2}^\dagger(\mathbf{r}',t)V_2(\mathbf{r}-\mathbf{r}')\hat{\Psi}_2(\mathbf{r}',t)\hat{\psi_2}(\mathbf{r},t) \\
+ \int\int d\mathbf{r}d\mathbf{r}'\hat{\psi_1}^\dagger(\mathbf{r}',t)\hat{\psi_2}^\dagger(\mathbf{r},t)V_{12}(\mathbf{r}-\mathbf{r}')\hat{\psi_1}(\mathbf{r},t)\hat{\psi_2}(\mathbf{r}',t)
\end{aligned}
\tag{6.5}
$$

where V_1, V_2 and V_{12} are the contact interactions acting between the bosons of species one, species two and between each species respectively.

$\hat{h_1} = -\frac{-\hbar^2}{2m}\nabla^2 + V_{exti}(\mathbf{r})$ is the single particle Hamiltonian where m_i is the mass and $V_{ext(i)}(\mathbf{r})$ the external potential acting on each species. we can approximate the contact interactions to

$$V_1(\mathbf{r}-\mathbf{r}^{'}) = g_{11}\delta(\mathbf{r}-\mathbf{r}^{'}) \tag{6.6}$$

$$V_2(\mathbf{r}-\mathbf{r}^{'}) = g_{22}\delta(\mathbf{r}-\mathbf{r}^{'}) \tag{6.7}$$

$$V_{12}(\mathbf{r}-\mathbf{r}^{'}) = g_{12}\delta(\mathbf{r}-\mathbf{r}^{'}) \tag{6.8}$$

where

$$g_{ii} = \frac{4\pi\hbar^2 a_{ii}}{2m_i}$$

for $i = 1, 2$ and

$$g_{12} = (g_{21}) = \frac{2\pi\hbar^2(m_1+m_2)a_{12}}{m_1 m_2}$$

The Bose field operators obey the following commutation relations:

$$[\hat{\psi}_i(\mathbf{r},t), \hat{\psi_i}^\dagger(\mathbf{r}^{'},t)] = \delta(\mathbf{r}-\mathbf{r}^{'}) \tag{6.9}$$

$$[\hat{\psi}_i(\mathbf{r},t), \hat{\psi}_i(\mathbf{r}^{'},t)] = \hat{\psi_i}^\dagger(\mathbf{r},t)], \hat{\psi_i}^\dagger(\mathbf{r}^{'},t)] = 0 \tag{6.10}$$

$$[\hat{\psi}_i(\mathbf{r},t), \hat{\psi_j}^\dagger(\mathbf{r}^{'},t)] = [\hat{\psi}_i(\mathbf{r},t), \hat{\psi}_j(\mathbf{r}^{'},t)] = [\hat{\psi_i}^\dagger(\mathbf{r},t), \hat{\psi_j}^\dagger(\mathbf{r}^{'},t)] = 0 \tag{6.11}$$

and

$$\psi_i(r) = \bar{\psi}_i(r) + < \psi_i(r) > = \bar{\psi}_i(r) + \phi_i$$

Inserting the contact interaction assumption into Equation (3.4) and integrating out the dependence on $\mathbf{r}^{'}$ leads to

$$\begin{aligned} H = & \int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r},t) \\ & + \frac{g_{11}}{2}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\psi_1^\dagger(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t) \\ & + \int d\mathbf{r}\psi_2^\dagger(\mathbf{r},t)\hat{h_2}\psi_2(\mathbf{r},t) \\ & + \frac{g_{22}}{2}\int d\mathbf{r}\psi_2^\dagger(\mathbf{r},t)\psi_2^\dagger(\mathbf{r},t)\psi_2(\mathbf{r},t)\psi_2(\mathbf{r},t) \\ & + g_{12}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\psi_2^\dagger(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t) \end{aligned} \tag{6.12}$$

The terms of Hamiltonian labelled by:

$$H_1 = \int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r},t)$$

$$H_2 = \frac{g_{11}}{2}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\psi_1^\dagger(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)$$

$$H_3 = \int d\mathbf{r}\psi_2^\dagger(\mathbf{r},t)\hat{h_2}\psi_2(\mathbf{r},t)$$

$$H_4 = \frac{g_{22}}{2}\int d\mathbf{r}\psi_2^\dagger(\mathbf{r},t)\psi_2^\dagger(\mathbf{r},t)\psi_2(\mathbf{r},t)\psi_2(\mathbf{r},t)$$

$$H_5 = g_{12}\int d\mathbf{r}\psi_1^\dagger(\mathbf{r},t)\psi_2^\dagger(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)$$

we need to calculated the terms: $\langle[\psi_1(r^{'}),H]\rangle$,$\langle[\psi_2(r^{'}),H]\rangle$ $\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H]\rangle$, $\langle[\bar{\psi_1^\dagger}(r^{'})\bar{\psi_1}(r^{''}),H]\rangle$ $\langle[\bar{\psi_2}(r^{'})\bar{\psi_2}(r^{''}),H]\rangle$, and $\langle[\bar{\psi_2^\dagger}(r^{'})\bar{\psi_2}(r^{''}),H]\rangle$
when

$$\langle[\psi_1(r^{'}),H]\rangle = \langle[\psi_1(r^{'}),H_1]\rangle + \langle[\psi_1(r^{'}),H_2]\rangle + \langle[\psi_1(r^{'}),H_5]\rangle$$

$$\langle[\psi_2(r^{'}),H]\rangle = \langle[\psi_2(r^{'}),H_3]\rangle + \langle[\psi_2(r^{'}),H_4]\rangle + \langle[\psi_2(r^{'}),H_5]\rangle$$

$$\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H]\rangle = \langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_1]\rangle+\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_2]\rangle+\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_5]\rangle$$

$$\langle[\bar{\psi_1^\dagger}(r')\bar{\psi_1}(r^{''}),H]\rangle = \langle[\bar{\psi_1^\dagger}(r^{'})\bar{\psi_1}(r^{''}),H_1]\rangle+\langle[\bar{\psi_1^\dagger}(r^{'})\bar{\psi_1}(r^{''}),H_2]\rangle+\langle[\bar{\psi_1^\dagger}(r^{'})\bar{\psi_1}(r^{''}),H_5]\rangle$$

$$\langle[\bar{\psi_2}(r)\bar{\psi_2}(r^{'}),H]\rangle = \langle[\bar{\psi_2}(r)\bar{\psi_2}(r^{'}),H_3]\rangle+\langle[\bar{\psi_2}(r)\bar{\psi_2}(r^{'}),H_4]\rangle+\langle[\bar{\psi_2}(r)\bar{\psi_2}(r^{'}),H_5]\rangle$$

$$\langle[\bar{\psi_2^\dagger}(r^{'})\bar{\psi_2}(r^{''}),H]\rangle = \langle[\bar{\psi_2^\dagger}(r^{'})\bar{\psi_2}(r^{''}),H_3]\rangle+\langle[\bar{\psi_2^\dagger}(r^{'})\bar{\psi_2}(r^{''}),H_4]\rangle+\langle[\bar{\psi_2^\dagger}(r^{'})\bar{\psi_2}(r^{''}),H_5]\rangle$$

Taking the term by term

$$
\begin{aligned}
[\psi_1(r^{'}), H_1] &= [\psi_1(r^{'}), \int d\mathbf{r}\hat{\psi_1}^{\dagger}(\mathbf{r},t)\hat{h_1}\hat{\psi_1}(\mathbf{r},t)] \\
&= \int d\mathbf{r}(\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r},t) - \psi_1^{\dagger}(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t)) \\
&= \int d\mathbf{r}(\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r},t) - \psi_1^{\dagger}(\mathbf{r},t)\hat{h_1}\psi_1(\mathbf{r}^{'},t)\psi_1(\mathbf{r},t)) \\
&= \int d\mathbf{r}(\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\hat{h_1}\Psi_1(\mathbf{r},t) - \psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t)\hat{h_1}\psi_1(\mathbf{r},t)) \\
&= \int d\mathbf{r}(\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t) - \psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t))\hat{h_1}\psi_1(\mathbf{r},t) \\
&= \int d\mathbf{r}(\delta(\mathbf{r}^{'} - \mathbf{r}))\hat{h_1}\psi_1(\mathbf{r},t) \\
&= \hat{h_1}\psi_1(\mathbf{r},t)
\end{aligned} \tag{6.13}
$$

Taking the terms

$$
\begin{aligned}
[\psi_1(r^{'},t), H_2] &= [\psi_1(\mathbf{r}^{'},t), \frac{g_{11}}{2}\int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)] \\
&= \frac{g_{11}}{2}\int d\mathbf{r}\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t) \\
&- \frac{g_{11}}{2}\int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t) \\
&= \frac{g_{11}}{2}[\int d\mathbf{r}\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t) \\
&- \int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)] \\
&= \frac{g_{11}}{2}(\int d\mathbf{r}\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t) - \psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t))\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t) \\
&= g_{11}\int d\mathbf{r}(2\delta(\mathbf{r} - \mathbf{r}^{'})\psi_1^{\dagger}(\mathbf{r}^{'},t))\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t) \\
&= g_{11}\psi_1^{\dagger}\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)
\end{aligned} \tag{6.14}
$$

as $[\hat{\Psi}_1(\mathbf{r},t), \hat{\Psi}_1(\mathbf{r}^{'},t)] = 0$ and

$$
\begin{aligned}
&[\psi_1(\mathbf{r}^{'},t),\psi_1^{\dagger}(\mathbf{r},t)\psi_1^{\dagger}(\mathbf{r},t)] \\
&=[\psi_1(\mathbf{r}^{'},t),\psi_1^{\dagger}(\mathbf{r},t)]\psi_1^{\dagger}(\mathbf{r},t)+\psi_1^{\dagger}\mathbf{r},t)[\psi_1(\mathbf{r}^{'},t),\psi_1^{\dagger}(\mathbf{r},t)] \\
&=\delta(\mathbf{r}^{'}-\mathbf{r})\psi_1^{\dagger}(\mathbf{r},t)+\psi_1^{\dagger}(\mathbf{r},t)\delta(\mathbf{r}^{'}-\mathbf{r}) \\
&=2\delta(\mathbf{r}^{'}-\mathbf{r})\psi_1^{\dagger}(\mathbf{r},t)
\end{aligned}
\tag{6.15}
$$

The final terms

$$
\begin{aligned}
[\psi_1(r^{'},t),H_5] &= [\psi_1(r^{'},t),g_{12}\int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)] \\
&= g_{12}\int d\mathbf{r}\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\hat{\psi}_1(\mathbf{r},t)\hat{\psi}_2(\mathbf{r},t) \\
&- g_{12}\int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t) \\
&= g_{12}[\int d\mathbf{r}\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\hat{\psi}_1(\mathbf{r},t)\psi_2(\mathbf{r},t) \\
&- \int d\mathbf{r}\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)] \\
&= g_{12}\int d\mathbf{r}(\psi_1(\mathbf{r}^{'},t)\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)-\hat{\psi}_1{}^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r}^{'},t))\hat{\Psi}_1(\mathbf{r},t)\psi_2(\mathbf{r},t) \\
&= g_{12}\int d\mathbf{r}[\psi_1(\mathbf{r}^{'},t),\psi_1^{\dagger}(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)]\psi_1(\mathbf{r},t)\hat{\Psi}_2(\mathbf{r},t) \\
&= g_{12}\int d\mathbf{r}\delta(\mathbf{r}^{'}-\mathbf{r})\psi_2^{\dagger}(\mathbf{r}^{'},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t) \\
&= g_{12}\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)
\end{aligned}
\tag{6.16}
$$

$$
\begin{aligned}
&[\psi_1(\mathbf{r}^{'},t),\psi_1(\mathbf{r},t)\psi_2^{\dagger}(\mathbf{r},t)] \\
&=[\psi_1(\mathbf{r}^{'},t),\psi_1^{\dagger}(\mathbf{r},t)]\psi_2^{\dagger}(\mathbf{r},t)-\psi_1(\mathbf{r},t)[\psi_1(\mathbf{r}^{'},t),\psi_2^{\dagger}(\mathbf{r},t)] \\
&=\delta(\mathbf{r}^{'}-\mathbf{r})\psi_2^{\dagger}(\mathbf{r},t)
\end{aligned}
\tag{6.17}
$$

Combining these (Eqs. (6.13),Eqs. (6.14)and Eqs. (6.16))leads to

$$
\begin{aligned}
\langle[\psi_1(r^{'}),H]\rangle &= \langle\hat{h_1}\psi_1(\mathbf{r},t)\rangle+\langle g_{11}\psi_1(\mathbf{r},t)\psi_1^{\dagger}\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)\rangle \\
&+\langle g_{12}\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)\rangle \\
&= \hat{h_1}\langle\psi_1(\mathbf{r},t)\rangle+g_{11}\langle\psi_1^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_1(\mathbf{r},t)\rangle \\
&+g_{12}\langle\psi_2^{\dagger}(\mathbf{r},t)\psi_1(\mathbf{r},t)\psi_2(\mathbf{r},t)\rangle
\end{aligned}
\tag{6.18}
$$

We now decompose the Bose field operator $\psi_i(\mathbf{r},t)$ in terms of a macroscopically populated mean field term $\phi_i(r,t) = \langle\psi_i(r,t)\rangle$, and a fluctuation term $\bar{\psi_i}(r,t)$.

$$\begin{aligned}\psi_1(r,t) &= \langle\psi_i(r,t)\rangle + \bar{\psi_i}(r,t) \\ &= \phi_i(r,t) + \bar{\psi_i}(r,t)\end{aligned} \tag{6.19}$$

$$\langle[\psi_1, H_1]\rangle = \langle\hat{h_1}\psi_1\rangle = \hat{h_1}\langle\psi_1\rangle = \hat{h_1}(\langle\bar{\psi_i}(r) > + < \psi_i\rangle) = \hat{h_1}\phi_1$$

because $\langle\bar{\psi_i}\rangle = 0$

We define $\tilde{n_{ii}} = \langle\bar{\psi_i}^\dagger\bar{\psi_i}\rangle$,$\tilde{m_{ii}} = \langle\bar{\psi_i}\bar{\psi_i}\rangle$,as the non-condensate and anomalous densities respectively.

$$\begin{aligned}\psi_1^\dagger\psi_1\psi_1 &= (\phi_1^* + \bar{\psi_1^\dagger})(\phi_1 + \bar{\psi_1})(\phi_1 + \bar{\psi_1}) \\ &= |\phi_1|^2\phi_1 + 2|\phi_1|^2\bar{\psi_1} + 2\phi_1\bar{\psi_1}^\dagger\bar{\psi_1} + \phi_1^*\bar{\psi_1}\bar{\psi_1} + \phi_1^2\bar{\psi_1}^\dagger + \bar{\psi_1}^\dagger\bar{\psi_1}\bar{\psi_1}\end{aligned} \tag{6.20}$$

the other term gives

$$\begin{aligned}\psi_2^\dagger\psi_2\psi_1 &= (\psi_2^* + \bar{\psi_2^\dagger})(\phi_2 + \bar{\psi_2})(\phi_1 + \bar{\psi_1}) \\ &= |\phi_2|^2\phi_1 + |\phi_2|^2\bar{\psi_1} + \phi_2^*\bar{\psi_2}\phi_1 + \phi_2^*\bar{\psi_2}\bar{\psi_1} \\ &+ \bar{\psi_2}^\dagger\phi_2\phi_1 + \bar{\psi_2}^\dagger\phi_2\bar{\psi_1} + \bar{\psi_2}^\dagger\bar{\psi_2}\phi_1\bar{\psi_2}^\dagger\bar{\psi_2}\bar{\psi_1}\end{aligned} \tag{6.21}$$

the values of the product of operators are

$$\langle\psi_1^\dagger\psi_1\psi_1\rangle = |\phi_1|^2\phi_1 + \phi_1^*\langle\bar{\psi_1}\bar{\psi_1}\rangle + 2\phi_1\langle\bar{\psi_1}^\dagger\bar{\psi_1}\rangle + \langle\bar{\psi_1}^\dagger\bar{\psi_1}\bar{\psi_1}\rangle \tag{6.22}$$

$$\langle\psi_2^\dagger\psi_2\psi_1\rangle = |\phi_2|^2\phi_1 + \phi_2^*\langle\bar{\psi_2}\bar{\psi_1}\rangle + \phi_2\langle\bar{\psi_2}^\dagger\bar{\psi_1}\rangle + \phi_1\langle\bar{\psi_2}^\dagger\bar{\psi_2}\rangle\langle\bar{\psi_2}^\dagger\bar{\psi_2}\bar{\psi_1}\rangle \tag{6.23}$$

Considering that the fluctuations of two species are uncorrelated $\langle\bar{\psi_2}\bar{\psi_1}\rangle = \langle\bar{\psi_2}^\dagger\bar{\psi_1}\rangle = 0$[181, 182]

the commutator of (Eqs. (6.18)give

$$\langle[\psi_1, H]\rangle = \hat{h_1}\phi_1 + g_{11}(|\phi_1|^2\phi_1 + \tilde{m}_{11}\phi_1^* + \tilde{n}_{11}\phi_1) + g_{12}|\phi_2|^2\phi_1 \tag{6.24}$$

when$\langle\bar{\psi_2}^\dagger\bar{\psi_2}\bar{\psi_1}\rangle = 0$

Similarly, it can be shown for$\langle[\psi_2(r^{'}),H]\rangle$ that

$$\langle[\psi_2,H]\rangle = \hat{h_2}\phi_2 + g_{22}(|\phi_2|^2\phi_2 + \tilde{m}_{22}\phi_2^* + \tilde{n}_{22}\phi_2) + g_{12}|\phi_1|^2\phi_1 \tag{6.25}$$

Now, we are taking the term $\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H]\rangle$. when

$$\langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H]\rangle = \langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_1]\rangle + \langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_2]\rangle + \langle[\bar{\psi_1}(r^{'})\bar{\psi_1}(r^{''}),H_5]\rangle \tag{6.26}$$

with the same method,we find other terms

$$\langle[\bar{\psi_1}\bar{\psi_1},H]\rangle = 2\hbar_1\langle[\bar{\psi_1}\bar{\psi_1}]\rangle \quad +2g_{11}\langle\psi_1^\dagger\psi_1\psi_1\bar{\psi_1}\rangle + 2g_{12}\langle\bar{\psi_1}\psi_2^\dagger\psi_2\psi_1\rangle \tag{6.27}$$

when

$$\begin{aligned}
&\langle\bar{\psi_1}\bar{\psi_1}\rangle = \tilde{m}_{11}\\
&\langle\psi_1^\dagger\psi_1\psi_1\bar{\psi_1}\rangle = 2|\phi_1|^2\tilde{m}_{11} + \phi_1^2\tilde{n}_{11} + \tilde{n}_{11}\tilde{m}_{11}\\
&\langle\bar{\psi_1}\psi_2^\dagger\psi_2\psi_1\rangle = |\phi_1|^2\tilde{m}_{11}
\end{aligned} \tag{6.28}$$

finally, the term gives

$$\langle[\bar{\psi_1}\bar{\psi_1},H]\rangle = 2h_1\tilde{m}_{11} + 2g_{11}(2|\phi_1|^2\tilde{m}_{11} + \phi_1^2\tilde{n}_{11} + \tilde{n}_{11}\tilde{m}_{11}) + 2g_{12}|\phi_2|^2\tilde{m}_{11} \tag{6.29}$$

for the second species

$$\langle[\bar{\psi_2}\bar{\psi_2},H]\rangle = 2\hbar_2\langle[\bar{\psi_2}\bar{\psi_2}]\rangle \quad +2g_{22}\langle\psi_2^\dagger\psi_2\psi_2\bar{\psi_2}\rangle + 2g_{12}\langle\bar{\psi_2}\psi_1^\dagger\psi_2\psi_1\rangle \tag{6.30}$$

when

$$\begin{aligned}
&\langle\bar{\psi_2}\bar{\psi_2}\rangle = \tilde{m}_{22}\\
&\langle\psi_2^\dagger\psi_2\psi_2\bar{\psi_2}\rangle = 2|\phi_2|^2\tilde{m}_{22} + \phi_2^2\tilde{n}_{22} + \tilde{n}_{22}\tilde{m}_{22}\\
&\langle\bar{\psi_2}\psi_1^\dagger\psi_2\psi_1\rangle = |\phi_1|^2\tilde{m}_{22}
\end{aligned} \tag{6.31}$$

finally, the term gives

$$\langle[\bar{\psi_2}\bar{\psi_2},H]\rangle = 2h_2\tilde{m}_{22} + 2g_{22}(2|\phi_2|^2\tilde{m}_{22} + \phi_2^2\tilde{n}_{22} + \tilde{n}_{22}\tilde{m}_{22}) + 2g_{12}|\phi_1|^2\tilde{m}_{22} \tag{6.32}$$

for the non condensate term, we have

$$\langle[\bar{\psi_1}^\dagger\bar{\psi_1},H]\rangle = \quad g_{11}(\langle\bar{\psi}_1^\dagger\psi_1^\dagger\psi_1\psi_1\rangle - \langle\bar{\psi_1}\psi_1^\dagger\psi_1^\dagger\psi_1\rangle) + g_{12}(\langle\bar{\psi}_1^\dagger\psi_2^\dagger\psi_2\psi_1\rangle - \langle\psi_1^\dagger\psi_2^\dagger\psi_2\bar{\psi}_1^\dagger\rangle)$$

$$(6.33)$$

when

$$\begin{aligned}
&\langle \bar{\psi}_1^\dagger \bar{\psi}_1 \rangle = \tilde{n}_{11} \\
&\langle \bar{\psi}_1^\dagger \psi_1^\dagger \psi_1 \psi_1 \rangle = 2|\phi_1|^2 \tilde{n}_{11} + 2\tilde{n}_{11}\tilde{n}_{11} + \phi_1^2 \tilde{m^*}_{11} + \tilde{m^*}_{11}\tilde{m}_{11} \\
&\langle \bar{\psi}_1 \psi_1^\dagger \psi_1^\dagger \psi_1 \rangle = 2|\phi_1|^2 \tilde{n}_{11} + 2\tilde{n}_{11}\tilde{n}_{11} + \phi_1^{*2} \tilde{m}_{11} + \tilde{m^*}_{11}\tilde{m}_{11} \\
&\langle \bar{\psi}_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \rangle = |\phi_2|^2 \tilde{n}_{11} + \tilde{n}_{11}\tilde{n}_{22} \\
&\langle \psi_1^\dagger \psi_2^\dagger \psi_2 \bar{\psi}_1^\dagger \rangle = |\phi_2|^2 \tilde{n}_{11} + \tilde{n}_{11}\tilde{n}_{22}
\end{aligned} \quad (6.34)$$

back to the equation (Eqs. (6.33)

$$\langle [\bar{\psi}_1^\dagger \bar{\psi}_1, H] \rangle = g_{11}(\phi_1^2 \tilde{m^*}_{11} - \phi_1^{*2} \tilde{m}_{11}) \quad (6.35)$$

the non condensate for the second species

$$\langle [\bar{\psi_2}^\dagger \bar{\psi}_2, H] \rangle = \quad g_{22}(\langle \bar{\psi}_2^\dagger \psi_2^\dagger \psi_2 \psi_2 \rangle - \langle \bar{\psi}_2 \psi_2^\dagger \psi_2^\dagger \psi_2 \rangle) + g_{12}(\langle \bar{\psi}_2^\dagger \psi_1^\dagger \psi_2 \psi_1 \rangle - \langle \psi_1^\dagger \psi_2^\dagger \psi_1 \bar{\psi}_2^\dagger \rangle) \quad (6.36)$$

when

$$\begin{aligned}
&\langle \bar{\psi}_2^\dagger \bar{\psi}_2 \rangle = \tilde{n}_{22} \\
&\langle \bar{\psi}_2^\dagger \psi_2^\dagger \psi_2 \psi_2 \rangle = 2|\phi_2|^2 \tilde{n}_{22} + 2\tilde{n}_{22}\tilde{n}_{22} + \phi_2^2 \tilde{m^*}_{22} + \tilde{m^*}_{22}\tilde{m}_{22} \\
&\langle \bar{\psi}_2 \psi_2^\dagger \psi_2^\dagger \psi_2 \rangle = 2|\phi_2|^2 \tilde{n}_{22} + 2\tilde{n}_{22}\tilde{n}_{22} + \phi_2^{*2} \tilde{m}_{22} + \tilde{m^*}_{22}\tilde{m}_{22} \\
&\langle \bar{\psi}_2^\dagger \psi_1^\dagger \psi_2 \psi_1 \rangle = |\phi_2|^2 \tilde{n}_{22} + \tilde{n}_{11}\tilde{n}_{22} \\
&\langle \psi_1^\dagger \psi_2^\dagger \psi_1 \bar{\psi}_2^\dagger \rangle = |\phi_2|^2 \tilde{n}_{22} + \tilde{n}_{11}\tilde{n}_{22}
\end{aligned} \quad (6.37)$$

back to the equation (Eqs. (6.36)

$$\langle [\bar{\psi}_2^\dagger \bar{\psi}_2, H] \rangle = g_{22}(\phi_2^2 \tilde{m^*}_{22} - \phi_2^{*2} \tilde{m}_{22}) \quad (6.38)$$

Bibliography

[1] . Ablowitz, MJ, Segur, H: Solitons and the Inverse Scattering Transform. SIAM, Philadelphia (1981)

[2] . Matveev, VB, Salle, MA: Darboux Transformations and Solitons. Springer, Berlin (1991)

[3] . Wang, DS, Chen, F, Wen, XY: Darboux transformation of the general Hirota equation: multisoliton solutions, breather solutions, and rogue wave solutions. Adv. Differ. Equ. 2016, 67 (2016). doi:10.1186/s13662-016-0780-z

[4] . Vekslerchik, VE: Backlund transformations for the Nizhnik-Novikov-Veselov equation. J. Phys. A 37, 5667-5678 (2004)

[5] . Wang, DS, Wei, X: Integrability and exact solutions of a two-component Korteweg-de Vries system. Appl. Math. Lett. 51, 60-67 (2016)

[6] . Wang, DS, Liu, J, Zhang, ZF: Integrability and equivalence relationships of six integrable coupled Korteweg-de Vries equations. Math. Methods Appl. Sci. 39, 3516-3530 (2016)

[7] . Wang, DS, Zhang, DJ, Yang, JK: Integrable properties of the general coupled nonlinear Schrödinger equations. J. Math. Phys. 51, 023510 (2010)

[8] . Wang, DS, Yin, SJ, Tian, Y, Liu, Y: Integrability and bright soliton solutions to the coupled nonlinear Schrödinger equation with higher-order effects. Appl. Math. Comput. 229, 296-309 (2014)

[9] . Wang, DS, Ma, YQ, Li, XG: Prolongation structures and matter-wave solitons in F = 1 spinor Bose-Einstein condensate with time-dependent atomic scattering lengths in an expulsive harmonic potential. Commun. Nonlinear Sci. Numer. Simul. 19, 3556-3569 (2014)

[10] . Hirota, R: Exact solution of the Korteweg de Vries equation for multiple collisions of solitons. Phys. Rev. Lett. 27, 1192-1194 (1971)

[11] . Wang, DS, Li, XG, Chan, CK, Zhou, J: Double Wronskian solution and soliton properties of the nonisospectral BKP equation. Commun. Theor. Phys. 65, 259-265 (2016)

[12] . Wang, DS, Yin, YB: Symmetry analysis and reductions of the two-dimensional generalized Benney system via geometric approach. Comput. Math. Appl. 71, 748-757 (2016)

[13] . Wang, DS, Hu, XH, Hu, J, Liu, WM: Quantized quasi-two-dimensional Bose-Einstein condensates with spatially modulated nonlinearity. Phys. Rev. A 81, 025604 (2010)

[14] . Wang, DS, Song, SW, Xiong, B, Liu, WM: Quantized vortices in a rotating Bose-Einstein condensate with spatiotemporally modulated interaction. Phys. Rev. A 84, 053607 (2011)

[15] . Wang, DS, Xue, YS, Zhang, ZF: Localized nonlinear matter waves in one-dimensional Bose-Einstein condensates with spatiotemporally modulated two- and three-body interactions. Rom. J. Phys. 61(5-6), 827-841 (2016)

[16] . Wang, DS, Shi, YR, Chow, KW, Yu, ZX, Li, XG: Matter-wave solitons in a spin-1 Bose-Einstein condensate with time-modulated external potential and scattering lengths. Eur. Phys. J. D 67, 242 (2013)

[17] N. Akhmediev, V. I. Korneev, Modulation instability and periodic solutions of the nonlinear Schrödinger equation, Theor. Math. Phys. 69 (1986) 1089-1093.

[18] N. Akhmediev, J. M. Soto-Crespoand, A. Ankiewicz, How to excite a rogue wave, Phys. Rev. A 80 (2009) 043818.

[19] Y. C. Ma, The perturbed plane-wave solutions of the cubic Schrödinger equation, Stud. Appl. Math. 64 (1979) 43-58.

[20] M. Shats, H. Punzmann, H. Xia, Capillary Rogue Waves, Phys. Rev. Lett. 104 (2010) 104503.

[21] A. Chabchoub, N. P. Hoffmann, N. Akhmediev, Rogue Wave Observation in a Water Wave Tank, Phys. Rev. Lett. 106(2011) 204502.

[22] M. S. V. Matveev, Darboux Transformations and Solitons, Springer Series in Nonlinear Dynamics (Springer, 1990).

[23] Bose-Einstein Condensates and Atom Lasers Edited by Martcllucci et al., Kluwer Academic/Plenum Publishers, 2000.

[24] A. Einstein Nature 141, 643 (1938).

[25] E. P. Gross Phys. Rev. 106, 161 (1957).

[26] E. P. Gross, Structure of Quantized Vortex in Boson Systems Il Nuovo Cimento 20, 454 (1961).

[27] E. P. Gross, Structure of Quantized Vortex in Boson Systems Journal of Mathematical Physics 4, 195 (1963).

[28] V. L. Ginzburg and L. P. Pitaevskii Sov. Phys. JETP 7, 858 (1958).

[29] L. P. Pitaevskii, Vortex lines in an imperfect Bose gas Sov. Phys. JETP 13, 451 (1961).

[30] C. Pethick and H. Smith, Bose-Einstein condensation in dilute gases (Cambridge University Press, 2002).

[31] C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dorscher, M. Baumert, E. M. Richter, J. Kronjager, K. Bongs, and K. Sengstock, Oscillations and interactions of dark and dark-bright solitons in Bose-Einstein condensates Nature Physics 4, 9 (2008).

[32] N. P. Proukakis and B. Jackson, Finite-temperature models of BoseEinstein condensation Journal of Physics B: Atomic, Molecular and Optical Physics 41, 203002 (2008).

[33] E. P. Gross Phys. Rev. 106, 161 (1957).

[34] E. P. Gross, Structure of Quantized Vortex in Boson Systems Il Nuovo Cimento 20, 454 (1961).

[35] E. P. Gross, Structure of Quantized Vortex in Boson Systems Journal of Mathematical Physics 4, 195 (1963).

[36] V. L. Ginzburg and L. P. Pitaevskii Sov. Phys. JETP 7, 858 (1958).

[37] L. P. Pitaevskii, Vortex lines in an imperfect Bose gas Sov. Phys. JETP 13, 451 (1961).

[38] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases Rev. Mod. Phys. 71, 463 (1999).

[39] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Dark Solitons in Bose-Einstein Condensates Phys. Rev. Lett. 83, 5198 (1999).

[40] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, et al., Generating Solitons by Phase Engineering of a Bose-Einstein Condensate Science 287, 97 (2000).

[41] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Watching Dark Solitons Decay into Vortex Rings in a Bose- Einstein Condensate Phys. Rev. Lett. 86, 2926 (2001).

[42] Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, Observation of Quantum Shock Waves Created with Ultra- Compressed Slow Light Pulses in a Bose-Einstein Condensate Science 293, 663 (2001).

[43] K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, Formation and propagation of matter-wave soliton trains Nature 417, 150 (2002).

[44] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Formation of a Matter-Wave Bright Soliton Science 296, 1290 (2002).

[45] U. A. Khawaja, H. T. C. Stoof, R. G. Hulet, K. E. Strecker, and G. B. Partridge, Bright Soliton Trains of Trapped Bose-Einstein Condensates Phys. Rev. Lett. 89, 200404 (2002).

[46] L. Salasnich, A. Parola, and L. Reatto, Condensate bright solitons under transverse confinement Phys. Rev. A 66, 043603 (2002).

[47] L. Salasnich, A. Parola, and L. Reatto, Modulational Instability and Complex Dynamics of Confined Matter-Wave Solitons Phys. Rev. Lett. 91, 080405 (2003).

[48] L. D. Carr and J. Brand, Spontaneous Soliton Formation and Modulational Instability in Bose-Einstein Condensates Phys. Rev. Lett. 92, 040401 (2004).

[49] L. Guttman and J. R. Arnold, The Nonparticipation ofHe_6 in the Superfluidity of He_4, Phys. Rev. 92, 547 (1953).

[50] Law C K, Pu H, Bigelow N P and Eberly J H 1998 Phys. Rev. A 58 531

[51] Law C K, Pu H, Bigelow N P and Eberly J H 1997 Phys. Rev. Lett. 79 3105

[52] C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, Production of Two Overlapping Bose-Einstein Condensates by Sympathetic Cooling Phys. Rev. Lett. 78, 586 (1997).

[53] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Dynamics of Component Separation in a Binary Mixture of Bose-Einstein Condensates Phys. Rev. Lett. 81, 1539 (1998).

[54] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Vortices in a Bose-Einstein Condensate Phys. Rev. Lett. 83, 2498 (1999).

[55] P. Maddaloni, M. Modugno, C. Fort, F. Minardi, and M. Inguscio, Collective Oscillations of Two Colliding Bose-Einstein Condensates Phys. Rev. Lett. 85, 2413 (2000).

[56] G. Delannoy, S. G. Murdoch, V. Boyer, V. Josse, P. Bouyer, and A. Aspect, Understanding the production of dual Bose-Einstein condensation with sympathetic cooling Phys. Rev. A 63, 051602 (2001).

[57] V. Schweikhard, I. Coddington, P. Engels, S. Tung, and E. A. Cornell, Vortex- Lattice Dynamics in Rotating Spinor Bose-Einstein Condensates Phys. Rev. Lett. 93, 210403 (2004).

[58] K. M. Mertes, J. W. Merrill, R. Carretero-Gonzalez, D. J. Frantzeskakis, P. G. Kevrekidis, and D. S. Hall, Nonequilibrium Dynamics and Superfluid Ring Excitations in Binary Bose-Einstein Condensates Phys. Rev. Lett. 99, 190402 (2007).

[59] R. P. Anderson, C. Ticknor, A. I. Sidorov, and B. V. Hall, Spatially inhomogeneous phase evolution of a two-component Bose-Einstein condensate Phys. Rev. A 80, 023603 (2009).

[60] S. Tojo, Y. Taguchi, Y. Masuyama, T. Hayashi, H. Saito, and T. Hirano, Controlling phase separation of binary Bose-Einstein condensates via mixed-spin-channel Feshbach resonance Phys. Rev. A 82, 033609 (2010).

[61] T. Busch, J. I. Cirac, V. M. Perez-Garca, and P. Zoller, Stability and collective excitations of a two-component Bose-Einstein condensed gas: A moment approach Phys. Rev. A 56, 2978 (1997).

[62] H.-J. Miesner, D. M. Stamper-Kurn, J. Stenger, S. Inouye, A. P. Chikkatur, and W. Ketterle, Observation of Metastable States in Spinor Bose-Einstein Condensates Phys. Rev. Lett. 82, 2228 (1999).

[63] S. B. Papp, J. M. Pino, and C. E. Wieman, Tunable Miscibility in a Dual-Species Bose-Einstein Condensate Phys. Rev. Lett. 101, 040402 (2008).

[64] G. Ferrari, M. Inguscio, W. Jastrzebski, G. Modugno, G. Roati, and A. Simoni, Collisional Properties of Ultracold K-Rb Mixtures Phys. Rev. Lett. 89, 053202 (2002).

[65] G. Modugno, M. Modugno, F. Riboli, G. Roati, and M. Inguscio, Two Atomic Species Superfluid Phys. Rev. Lett. 89, 190404 (2002).

[66] G. Thalhammer, G. Barontini, L. De Sarlo, J. Catani, F. Minardi, and M. Inguscio, Double Species Bose-Einstein Condensate with Tunable Interspecies Interactions Phys. Rev. Lett. 100, 210402 (2008).

[67] D. J. McCarron, H. W. Cho, D. L. Jenkin, M. P. Koppinger, and S. L. Cornish, Dual-species Bose-Einstein condensate of ^{87}Rb and ^{133}CsPhys. Rev. A 84, 011603 (2011).

[68] J. Scott Russell, Report on Waves, in Rept. Fourteenth Meeting of the British Association for the Advancement of Science, (J. Murray, London, 1844).

[69] Pethick C. J. and Smith H. Bose-Einstein Condensation in dilute Gases. Cambridge university press. Second edition 2008.

[70] Dalfovo F., Giorgini S., Pitaevskii L., Stringari S. Rev. Mod. Phys. 71, 463 (1999).

[71] Pitaevskii and Stringari. Bose-Einstein Condensation. Oxford University. 2003

[72] A. Mourachkine, arXiv:cond-mat/0411452v1, 2004.

[73] D. W. Jordan and P. Smith, Nonlinear Ordinary Differential Equations, June 2007.

[74] Nakahara and Mikio, Geometry, Topology and Physics, 2003.

[75] Mermin, N. D. "The topological theory of defects in ordered media". Reviews of Modern Physics 51 (3), (1979).

[76] N. N. Akhmediev; A. Ankiewicz, Solitons, non-linear pulses and beams. Springer(1997).

[77] Y.Yousefi and Khikmat Kh. Muminov, A Simple Classification of Solitons. arXiv:1206.1294v2,Jun2012.

[78] Korteweg, D.J. and de Vries, H. 1895. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philosophical Magazine, 39: 422 43.

[79] H.Chachou Samet, Doctoral Thesis in Physical Science,"DYNAMICS OF A TRAPPED BOSE GAS AT FINITE TEMPERATURE", Chlef University Algeria.2014.

[80] M. H. Anderson et al., Science 269, 198 (1995); K. B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995); C. C. Bradley et al., Phys. Rev. Lett. 78, 985 (1997).

[81] See, e.g., Y. S. Kivshar and B. Luther-Davies, Phys. Rep. 298, 81 (1998).

[82] P. Emplit, J. P. Hamaide, F. Reynaud, G. Froehly, and A. Barthelemy Opt. Commun.62, 374 (1987).

[83] A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, and P. G. Kevrekidis, Experimental Observation of Oscillating and Interacting Matter Wave Dark Solitons Phys. Rev. Lett. 101, 130401

[84] Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, Observation of Quantum Shock Waves Created with Ultra- Compressed Slow Light Pulses in a Bose-Einstein Condensate Science 293, 663 (2001).

[85] P. Engels and C. Atherton, Stationary and Nonstationary Fluid Flow of a Bose-Einstein Condensate Through a Penetrable Barrier Phys. Rev. Lett. 99, 160405 (2007).

[86] Robert William Pattinson,doctor thesis of Philosophy June 2014

[87] Strecker K., Partridge G., Truscott A., and Hulet R. 2002 Nature 417 150.

[88] Khaykovich L., Schreck F., Ferrari G., Bourdel T., Cubizolles J., Carr L., Castin Y., and Salomon C. 2002 Science 296 1290.

[89] Cornish S., Thompson S. and Wieman C. Phys. Rev. Lett. 96 170401 (2006).

[90] C. Kharif, E. Pelinovsky, Eur. J. Mech. B 22 (2003) 603.

[91] K. Dysthe, H. E. Krogstad, P. Muller, Annu. Rev. Fluid Mech. 40 (2008) 287.

[92] E. Pelinovsky, C. Kharif, Extreme Ocean Waves (Springer, New York, 2008)

[93] A. R. Osborne, Nonlinear Ocean Waves (Academic Press, New York, 2009).

[94] C. Kharif, E. Pelinovsky, A. Slunyaev, Rogue Waves in the Ocean (Springer, New York, 2009).

[95] I. Didenkulova, E. Pelinovsky, Nonlinearity 24 (2011) R1.

[96] L. Draper, Oceanus 10 (1964) 13.

[97] Z. Y. Yan, Phys. Lett. A 374 (2010) 672.

[98] Akhmediev N and Pelinovsky E (Eds) 2010 Rogue wavestowards a unifying concept Eur. Phys. J. Spe. Top. 185 (Special issue)

[99] Solli D R, Ropers C, Koonath P and Jalali B 2007 Optical rogue waves Nature 450 1054

[100] Garrett C and Gemmrich J 2009 Rogue waves Phys. Today 62 623

[101] Wabnitz S, Finot Ch, Fatome J and Millot G 2013 Shallow water rogue wavetrains in nonlinear optical fibers Phys. Lett. A 377 9329

[102] B. Wu, J. Liu, and Q. Niu , Controlled generation of dark solitons with phase imprinting, Phys. Rev. Lett. 88, 034101 (2002).

[103] N. Akhmedievand E. Pelinovsky, Eur. Phys. J. Special Topics185, 1 (2010).

[104] K. Dysthe, H.E. Krogstad, and P. Muller, Annu. Rev. Fluid Mech. 40, 287 (2008).

[105] Yu. V. Bludov, V. V. Konotop, and N. Akhmediev, Phys. Rev. A 80, 033610 (2009).

[106] K. Manikandan, P. Muruganandam, M. Senthilvelan, and M. Lakshmanan Phys. Rev. E 90, 062905 (2014).

[107] M. H. Anderson et al., Science 269, 198 (1995).

[108] C. C. Bradley et al., Phys. Rev. Lett. 75, 1687 (1995).

[109] K. B. Davis et al., Phys. Rev. Lett. 75, 3969 (1995).

[110] S. Burger et al., Phys. Rev. Lett. 83, 5198 (1999).[19] J. Denschlag et al., Science 287, 97 (2000).

[111] K. E. Strecker et al, Nature 417, 150 (2002).

[112] L. Khaykovich et al. Science 296, 1290 (2002).

[113] F. Dalfovo et al., Rev. Mod. Phys. 71, 463 (1999).

[114] A. J. Legett, Rev. Mod. Phys. 73, 307 (2001).

[115] C. J. Pethick and H. Smith, Bose-Einstein condensation in dilute gases (Cambridge University Press, Cambridge, UK, 2002).

[116] L. P. Pitaevski and S. Stringari, Bose-Einstein condensation (Oxford University Press, Oxford, UK, 2003).

[117] B. A. Malomed, Soliton management in periodic systems (Springer, New York, 2006).

[118] J. L. Roberts et al., Phys. Rev. Lett. 86, 4211 (2001).

[119] S. Inouye, Nature 392, 151 (1998).

[120] E. Timmermans et al., Phys. Rep. 315, 199 (1999).

[121] P.G. Kevrekidis et al., Phys. Lett. 90, 230401 (2003).

[122] G. D. Montesinos, V. M. Perez-Garca, P. J. Torres, Physica D 191, 193 (2004).

[123] V. M. Perez-Garca, P. J. Torres, V. V. Konotop, Physica D 221, 31 (2006).

[124] G. Herring et al., Phys. Lett. A 367, 140 (2007).

[125] M. Matuszewski et al., Phys. Rev. Lett. 95, 050403 (2005).

[126] H. Sakaguchi and B. A. Malomed, Phys. Rev. E 70, 066613 (2004).

[127] H. Yanay, L. Khaykovich, and B. A. Malomed, CHAOS 19,033145 (2009).

[128] Kuznetsov, E.A., Solitons in parametrically unstable plasma. Doklady Akademii Nauk Sssr, 1977. 236(3): p. 575-577.

[129] Kawata, T. and H. Inoue, Inverse scattering method for non-linear evolution equations under nonvanishing conditions. J. Phys. Soc. Jpn., 1978. 44(5): p. 1722-1729.

[130] 276. Ma, Y.-C., The perturbed plane-wave solutions of the cubic Schrdinger equation. Stud. Appl. Math., 1979. 60: p. 43-58.

[131] Zakharov, V.E. and A.B. Shabat, Interaction between solutions in a stable medium. Zhurnal Eksperimentalnoi Teor. Fiz., 1973. 64(5): p. 1627-1639.

[132] Akhmediev, N.N. and V.I. Korneev, Modulation instability and periodic-solutions of the nonlinear Schrödinger equation. Theor. Math. Phys., 1986. 69(2): p. 1089-1093.

[133] Akhmediev, N.N., V.M. Eleonskii, and N.E. Kulagin, Exact first-order solutions of the nonlinear Schrdinger equation. Theor. Math. Phys., 1987.72(2): p. 809-818.

[134] Peregrine, D.H., Water waves, nonlinear Schrödinger equations and their solutions. J. Aust. Math. Soc. Ser. B, 1983. 25: p. 16-43.

[135] Kristian B, Dysthe and Karsten Trulsen, Note on Breather type solutions of the NLS and Models for Freak-Waves. physica scripta. Vol. T82,48-52,1999.

[136] Akhmediev, N., A. Ankiewicz, and M. Taki, Waves that appear from nowhere and disappear without a trace. Phys. Lett. A, 2009. 373(6): p. 675-678.

[137] Akhmediev, N., A. Ankiewicz, and J.M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation. Phys. Rev. E, 2009. 80(2): p. 9.

[138] Gross E P 1961 Nuovo Cimento 20 454

[139] Pitaevskii L P 1961 Soviet Phys JETP 40 646

[140] Ho Tin-Lun and Shenoy V B 1996 Phys. Rev. Lett. 77 32769

[141] Ballagh R J, Burnett K and Scott T F 1997 Phys. Rev. Lett. 78 160711

[142] Esry B D, Greene C H, Burke J P Jr. and Bohn J L 1997 Phys. Rev. Lett. 78 35947

[143] Hisakado M 1997 Preprint solv-int/9701022v2

[144] Hall D S, Matthews M R, Ensher J R, Wieman C E and Cornell E A 1998 Phys. Rev. Lett. 81 153942

[145] Hall D S, Matthews M R, Wieman C E and Cornell E A 1998 Phys. Rev. Lett. 81 15436

[146] Villain P and Lewenstein M 1998 Preprint quant-ph/9808017

[147] Williams J, Walser R, Cooper J, Cornell E and Holland M 1999 Phys. Rev. A. 59 R31-4

[148] Raghavan S, Smerzi A and Kenkre V M 1999 Phys. Rev. A. 60 R1787-90

[149] Ho T L 1998 Phys. Rev. Lett. 81 742-5

[150] Law C K, Pu H and Bigelow N P 1998 Phys. Rev. Lett. 81 5257-61

[151] Pu H, Raghavan S and Bigelow N P 2000 61 023602

[152] Pu H, Law C K, Raghavan S, Eberly J H and Bigelow N P 1999 Phys. Rev. A. 60 1463-70

[153] Hasegawa A 1989 Optical Solitons in Fibers (Springer-Verlag, Heidelberg)

[154] Ruprecht P A, Holland M J and Burnett K 1995 Phys. Rev. A 51 470411

[155] Heinzen D J, Wynar R, Drummond P D and Kheruntsyan K V 2000 Phys. Rev. Lett. 84 502933

[156] S. V. Tyablikov, Methods in the Quantum Theory of Magnetism (Plenum Press, New York,1967).

[157] M. D. Girardeau, Variational method for the quantum statistics of interacting particles. J. Math. Phys. 3, 131 (1962).

[158] M. D. Girardeau, Variational principle of Gibbs and Bogoliubov. J. Chem. Phys. 41, 2945 (1964).

[159] A. Ishihara, The Gibbs-Bogoliubov inequality. J. Phys. A: Math. Gen. 1, 539 (1968).

[160] S. Okubo. Some general inequalities in quantum statistical mechanics. J. Math. Phys. 12, 1123 (1971).

[161] A. Huber, Variational principles in quantum statistical mechanics. in: Mathematical Methods in Solid and Superfluid Theory, R. C. Clark and G. N. Derrick, eds., (Oliver and Boyd, Edinburgh, 1968), p. 364.

[162] A. Huber, An inequality for traces and its application to extremum principles and related theorems in quantum statistical mechanics. in: Methods and Problems of Theoretical Physics, J. E. Bowcock, ed., (North-Holland, Amsterdam, 1973), p. 37.

[163] M. D. Girardeau, Variational method for the quantum statistics of many-particle systems. Phys. Rev. A 42, 3303 (1990).

[164] D. Ya. Petrina, Mathematical Foundations of Quantum Statistical Mechanics. Continuous Systems (Kluwer Academic Publ., Dordrecht, 1995).

[165] B. Fultz, Phase Transitions in Materials (Cambridge University Press, New York, 2014).

[166] A. L. Kuzemsky, Statistical mechanics and the physics of many-particle model systems. Physics of Particles and Nuclei, 40, 949 (2009).

[167] A.L. Kuzemsky, Bogoliubovs vision: quasiaverages and broken symmetry to quantum protectorate and emergence. Int. J. Mod. Phys. B 24, 835 (2010).

[168] W. Yurgrau and S. Mandelstam, Variational Principles in Dynamics and Quantum Theory (Dover Publ., New York, 1968).

[169] P. Blanchard and E. Bruning, Variational Methods in Mathematical Physics: A Unified Approach (Springer, Berlin, 1992).

[170] S. H. Gould, Variational Methods for Eigenvalue Problems: An Introduction to the Methods of Rayleigh, Ritz, Weinstein, and Aronszajn (Dover, New York, 1995).

[171] J. L. Basdevant, Variational Principles in Physics (Springer, Berlin, 2007).

[172] M. Masujima, Applied Mathematical Methods in Theoretical Physics, 2nd ed. (Wiley-VCH, New York, 2009).

[173] V. L. Berdichevsky, Variational Principles of Continuum Mechanics. I. Fundamentals (Springer, Berlin, 2009). paru incompatible. Memoires de lAcademie Royale de Sciences (Paris, 1774), p.417.

[174] G. C. Pomraning, Variational principle for eigenvalue equations. J. Math. Phys. 8, 149 (1967).

[175] S. T. Epstein, The Variation Method in Quantum Chemistry, (Academic Press, New York, 1974).

[176] R. K. Nesbet, Variational Principles and Methods in Theoretical Physics and Chemistry,(Cambridge University Press, Cambridge, 2004).

[177] A. L. Kuzemsky, Theory of transport processes and the method of the nonequilibrium statistical operator. Int. J. Mod. Phys. B 21, 2821 (2007).

[178] A. L. Kuzemsky, Electronic transport in metallic systems and generalized kinetic equations. Int. J. Mod. Phys. B 25, 3071 (2011).

[179] R. Balian and M. V*é*n*é*roni, Ann. of Phys. (N.Y.) 187 (1988), 29; Ann. of Phys. (N.Y.) 195 (1989), 324.

[180] M. Benarous, Extensions variationnelles de la methode du champ moyen dependant du temps.PhD thesis, IPN-ORSAY-France, 1991.

[181] Arko Roy , Sukla Pal , S. Gautam , D. Angom , P. Muruganandam ,FACt: FORTRAN toolbox for calculating fluctuations in atomic condensates,2020

[182] M,Ota and S, Giorgini,Thermodynamics of dilute Bose gases: Beyond mean-field theory for binary mixtures of Bose-Einstein condensates,2020

[183] Matveev V B and Salle M A 1991 Dardoux Transformations and Solitons (Springer Series in Nonlinear Dynamics) (Berlin: Springer).

[184] Usama Al Khawaja J. Phys. A: Math. Gen. 39 (2006) 96799691

[185] C. Rogers and W. Schief; Bncklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory, Cambridge Texts in Applied Mathematics, 2002;

[186] Maciej Dunajski; Solitons, Instantons, and Twistors, Oxford University Press, 2010;

[187] Ashok Das; Integrable models, World Scientific Publishing, 1989;

[188] Agrawal G P 1989 Nonlinear Fiber Optics (San Diego: Academic)

[189] Dalfovo F, Giorgini S, Pitaevskii L P and Staingari S 1999 Theory of BoseEinstein condensation in trapped gases Rev. Mod. Phys. 71 463512

[190] Ablowitz M J and Segure H 1981 Solitons and the Inverse Scattering Transform (Philadelphia:SIAM)

[191] Agrawal G P. Nonlinear Fibre optics.Academic Press; New york: 2006.

[192] Manakov S V. Zh. Eksp. Teor. Fiz.1973; 65: 505.

[193] Kaup D J, Malomed B A. Soliton trapping and daughter waves in the Manakov model. Phys. Rev. A. 1993; 48: 599.

[194] Matveev V B and Salle M A, Darboux Transformations and Solitons (Springer, Berlin, 1991).

[195] S. Wabnitz, Ch. Finot, J. Fatome, and G. Millot, Shallow water rogue wavetrains in nonlinear optical fibers, Phys. Lett. A. 377 (12), 932939 (2013).

[196] J. M. Dudley, G. Genty, and B. J. Eggleton, Harnessing and control of optical rogue waves in supercontinuum generation, Opt. Express. 16 (6), 36443651 (2008).

[197] R. Driben and I. Babushkin, Accelerated rogue waves generated by soliton fusion at the advanced stage of supercontinuum formation in photonic-crystal fibers,Opt. Lett. 37 (24), 51575159 (2012).

[198] A. Montina, U. Bortolozzo, S. Residori, and F. T. Arecchi, Non-Gaussian statistics and extreme waves in a nonlinear optical cavity, Phys. Rev. Lett. 103 (17), 173901 (2009).

[199] S. Residori, U. Bortolozzo, A. Montina, F. Lenzini, and F. T. Arecchi, Rogue waves in spatially extended optical systems, Fluctuation Noise Lett. 11 (1), 1240014(2012)

[200] D. H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B. 25 (1), 1643 (1983).

[201] N. Akhmediev, A. Ankiewicz, and M. Taki, Waves that appear from nowhere and disappear without a trace, Phys. Lett. A. 373 (6), 675678 (2009).

[202] C. Bonatto, M. Feyereisen, S. Barland, M. Giudici, C. Masoller, J. R. Rios Leite, and J. R. Tredicce, Deterministic optical rogue waves, Phys. Rev. Lett. 107 (5), 053901 (2011).

[203] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, Solutions of the vector nonlinear Schrödinger equations: Evidence for deterministic rogue waves, Phys. Rev. Lett. 109 (4), 044102 (2012).

[204] D. J. Kedziora, A. Ankiewicz, and N. Akhmediev, Second-order nonlinear Schrödinger equation breather solutions in the degenerate and rogue wave limits, Phys. Rev. E. 85 (6), 066601 (2012).

[205] P. Gaillard, Degenerate determinant representation of solutions of the nonlinear Schrödinger equation, higher order Peregrine breathers and multi-rogue waves, J. Math. Phys. 54 (1), 013504 (2013).

[206] A. Ankiewicz, J. M. Soto-Crespo, and N. Akhmediev, Rogue waves and rational solutions of the Hirota equation, Phys. Rev. E. 81 (4), 046602 (2010).

[207] Y. Tao and J. He, Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation, Phys. Rev. E. 85 (2), 026601 (2012).

[208] Li-Chen Zhao and Jie Liu, Localized nonlinear waves in a two-mode nonlinear fiber, J. Opt. Soc. Am. B: Opt. Phys. 29 (11), 31193127 (2012).

[209] Li-Chen Zhao and Jie Liu, Rogue-wave solutions of a three-component coupled nonlinear Schrödinger equation, Phys. Rev. E. 87 (1), 013201 (2013).

[210] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of BoseEinstein condensation in trapped gases, Rev. Mod. Phys. 71 (3), 463512 (1999).

[211] K. Staliunas, S. Longhi, and D. J. de Valcrcel, Faraday patterns in BoseEinstein condensates, Phys. Rev. Lett. 89 (21), 210406 (2002).

[212] F. Kh. Abdullaev, J. G. Caputo, R. A. Kraenkel, and B. A. Malomed, Controlling collapse in BoseEinstein condensates by temporal modulation of the scattering length, Phys. Rev. A. 67 (1), 013605 (2003).

[213] Zai-Dong Li, Cong-Zhe Huo, Qiu-Yan Li, Peng-Bin He, and Tian-Fu Xu, Symmetry and asymmetry rogue waves in two-component coupled nonlinear Schrdinger equations, Chin. Phys. B. 27 (4), 040505 (2018).

[214] G. P. Agrawal, Nonlinear Fibre Optics (Academic, New York, 2006).

[215] S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Sov. Phys.-JETP. 38 (2), 248253 (1974).

[216] D. J. Kaup and B. A. Malomed, Soliton trapping and daughter waves in the Manakov model, Phys. Rev. A. 48 (1), 599604 (1993).

[217] V. B. Matveev and M. A. Salle, Darboux Transformations and Solitons (Springer, Berlin, 1991).

[218] N. Vishnupriya, M. Senthilvelan, and M. Lakshmanan, Breathers and rogue waves: Demonstration with coupled nonlinear Schrödinger family of equations, Pramana J. Phys. 84 (3), 339352 (2015).

[219] Chaachoua Sameut H, Sakthivinayagam Pattu, Al Khawaja U, Benarous M and Belkroukra H, Peregrine Soliton Management of Breathers in Two Coupled Gross-Pitaevskii Equations with External Potential, Physics of Wave Phenomena, Vol. 28.2020.

[220] . R. Radha and P. S. Vinayagam, Stabilization of matter wave solitons in weakly coupled atomic condensates, Phys. Lett. A. 376 (8-9), 944949 (2012).

[221] D.H. Peregrine, J. Austral. Math. Soc. Ser. B 25, 16 (1983)

[222] Y.C. Ma, Stud. Appl. Math. 60, 43 (1979)

[223] E.A. Kuznetsov, Akad. Nauk SSSR Dokl. 236, 575 (1977)

[224] A. Chabchoub, N. Hoffmann, M. Onorato, N. Akhmediev, Phys. Rev. X 2, 011015 (2012)

[225] N. Akhmediev, V.M. Eleonskii, N. Kulagin, Zh. Eksp. Teor. Fiz. 89, 1542 (1985)

[226] N. Akhmediev, V.I. Korneev, Theor. Math. Phys. 69, 1089 (1986)

[227] Chaachoua Sameut H, Sakthivinayagam Pattu, Al Khawaja U, Benarous M and Belkroukra H, Peregrine Soliton Management of Breathers in Two Coupled Gross-Pitaevskii Equations with External Potential, Physics of Wave Phenomena, Vol. 28.2020.

[228] E. Kuznetsov, Sov. Phys. Dokl. 22, 507-508 (1977); Y.C. Ma, Stud. Appl. Math. 60, 43-58 (1979).

[229] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, A. Akhmediev and J. Dudely, Nature 6, 790 (2010).

[230] B. Kibler, J. Fatome, C. Finot, G. Millot, G. Genty, B. Wetzel, A. Akhmediev, F. Dias and J. Dudely, Sci.Rep. 2, 463 (2012).

[231] Pethick C J and Smith H, Bose-Einstein Condensation in Dilute Gases (Cambridge University Press, 2002); Kevrekidis P G, Frantzeskakis D J and Carretero-Gonzalez R, Emergent Nonlinear Phenomena in Bose-Einstein Condensates: Theory and Experiment (Springer-Verlag, Heidelberg, 2008).

[232] Denschlag J, Simsarian J E, Feder D L, Clark C W, Collins L A, Cubizolles J, Deng L, Hagley E W, Helmerson K, Reinhardt W P, Rolston S L, Schneider B I and Phillips W D 2000 Science 287 97; Khaykovich L, Schreck F, Ferrari G, Bourdel T, Cubizolles J, Carr L D, Castin Y and Salomon C 2002 Science 296 1290.

[233] Burger S, Bongs K, Dettmer S, Ertmer W, Sengstock K, Sanpera A, Shlyapnikov G V and Lewenstein M 1999 Phys. Rev. Lett. 83 5198.

[234] Busch Th and Anglin J R 2001 Phys. Rev. Lett. 87 010401.

[235] Matthews M R, Anderson B P, Haljan P C, Hall D S, Wieman C E and Cornell E A 1999 Phys. Rev. Lett. 83 2498; Svidzinsky A A and Fetter A L 2000 Phys. Rev. A 62 063617.

[236] He J S, Charalampidis E G, Kevrekidis P G and Frantzeskakis D J 2014 Phys. Lett. A 378 577; Bludov Yu V, Konotop V V and Akhmediev N 2009 Phys. Rev. A 80 033610.

[237] N. Vishnupriya, M. Senthilvelan, and M. Lakshmanan, Breathers and rogue waves: Demonstration with coupled nonlinear Schrödinger family of equations, Pramana J. Phys. 84 (3), 339352 (2015).

[238] Zhao L C, He S L 2011 Phys. Lett. A 375 3017 Liu X X et al 2010 Chin. Phys. Lett. 27 070306

[239] A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, J. R. Abo-Shaeer, A. P. Chikkatur, S. Gupta, S. Inouye, T. Rosenband, and W. Ketterle, Phys. Rev. Lett. 87, 130402 (2001).

[240] Liu C Yang Z Y. Zhao L C Yang and W L Yue R H 2013 Phys. Lett. A 30 040304.

[241] Yu. V. Bludov, V. V. Konotop, and N. Akhmediev, Phys. Rev. A 80, 033610 (2009).

[242] Belkroukra H, Chaachoua Sameut H and Benarous M, New Families of Breathers in Trapped Two-Component Condensates, Physics of Wave Phenomena, Vol. 30.2022.

[243] Matveev V B and Salle M A, Darboux Transformations and Solitons (Springer, Berlin, 1991).

[244] Petrov D, Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture, Phys. Rev. Lett. 115, 155302 (2015).

[245] Astrakharchik G E and Malomed B A, Dynamics of one-dimensional quantum droplets, Phys. Rev. A 98, 013631 (2018).

[246] P.Öhberg and S. Stenholm, Hartree-Fock treatment of the two-component Bose- Einstein condensate Phys. Rev. A 57, 1272 (1998).