HASSIBA BENBOUALI UNIVERSITY OF CHLEF
FACULTY OF EXACTE SCIENCES AND COMPUTER SCIENCE
COMMUN CORE DEPARTEMENT

Physics 1

Prepared by

Dr.HOURIA CHAACHOUA SAMEUT

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Foreword

Kinematics is the subdivision of mechanics whose object is the quantitative study of movement of material bodies, regardless of the causes that produce this movement (position, speed, acceleration, etc.), whereas dynamics is the study of the movement of bodies depending on the forces exerted on them. This work is structured in two main parts. The first part will be devoted to the kinematics, preceded by the definition of mathematical tools that make it possible to understand this part as well as the rest. The second part will be the subject of the study of the dynamics followed a chapter on work and energy, which consists of shedding light on the relationships between movements and their causes. This handout represents a course support of mechanics of the material point. It is intended for students of the first semester of the SM track. We want students to find in this support a good working tool that will allow them to fill any gaps that may take place when taking notes while explaining the course or assignments led by their teachers. This handout is just an add-on to the course. He cannot, in any way, exempt the student from his presence in class.

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Mathematics reminder

1.1 Error calculations

1.1.1 Definitions

For any measurable quantity A, it is possible to define :

- its measured value a
- its exact value a_0 which cannot be reached

1.1.1.1 Absolute error

It is then defined by : $\delta a = a - a_0$ This error is the result of several errors (systematic, incidental...)

1.1.1.2 Absolute incertitude

Since the absolute error δa is not known, we simply give an upper limit Δa called the absolute uncertainty such that :
$|\delta a| \leq \Delta a \Rightarrow \Delta a > 0$ (the absolute uncertainty is always > 0) This means that the absolute uncertainty is the maximum value that the absolute error can reach.

1.1.1.2.1 Addition and subtraction Let is suppose that the quantity we are looking for R is the sum of two measurements A and B
$R = A + B$
In this case the uncertainty about the result is
$\Delta R = \Delta A + \Delta B$
It is the same for : $R = A - B$
The absolute uncertainty about a sum or a difference is the sum of absolute uncertainties of each term.

Example
A container has mass $m = (50 \pm 1)g$.Filled with water, its mass is: $M = (200 \pm 1)g$.

The mass of water it contains is therefore:
$m_{eau} = M - m$
By applying the rule above: $\Delta m_{eau} = \Delta M + \Delta m = 1 + 1 = 2g$,
it follows that : $m_{eau} = (150 \pm 2)g$

1.1.1.3 Relative uncertainty

It is defined by the ratio $\frac{\Delta a}{a}$.

1.1.1.4 Error calculations

Let a quantity $g = f(x, y, z)$, sa différentielle totale s'écrit:
$dg = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$
L 'incertitude absolue sur la variable g s'obtient en passant aux variations des variables qui la compose, soit:
$\Delta g = |\frac{\partial f}{\partial x}|\Delta x + |\frac{\partial f}{\partial y}|\Delta y + |\frac{\partial f}{\partial z}|\Delta z$

- **Example**
  Either to determine the relative uncertainty $\frac{\Delta \rho}{\rho}$ ddensity of the substance of à homogeneous cube from the measurement of its mass m and of its edge a
  Solution

  If m and a are the approximate values for the mass and the edge of the cube, we can write :
  $\frac{masse}{volume} = \frac{m}{a^3} \Rightarrow \rho = ma^{-3}$

  Deriving volume from mass and edge.This gives:
  $d\rho = a^{-3}dm - 3a^{-4}mda$
  By approximating small variations **d** with great variations Δ, it comes:
  $\Delta\rho = a^{-3}\Delta m - 3a^{-4}m\Delta a \Rightarrow \Delta\rho = \frac{1}{a^3}\Delta m - 3\frac{1}{a^4}m\Delta a$
  Hence
  $\frac{\Delta\rho}{\rho} = \frac{\Delta m}{m} + 3\frac{m}{a4}\Delta a$

- **Remarks**:
  We find the same thing by taking $Ln(\rho)$ and differentiating the new equation.
  Ln:is the natural function.

1.1.1.4.1 Case of a product or a quotient $k\frac{x^a y^b}{(z+t)^c}$

We apply the logarithm function to this equation
$LnF = Ln(k\frac{x^a y^b}{(z+t)^c})$
$LnF = Lnk + Lnx^a y^b - Ln(z+t)^c$
$LnF = Lnk + aLnx + bLny - cLn()z + t$
We move on to the derivative of this equation, with k constant,
$\frac{dF}{F} = a\frac{dx}{x} + b\frac{dy}{y} - c\frac{dz+dt}{z+t}$
Finally, we replace the differential elements by the uncertainties on the associated quantities and transform all the negative signs into positive signs. The result is
$\frac{\Delta F}{F} = a\frac{\Delta x}{x} + b\frac{\Delta y}{y} + c\frac{\Delta z+dt}{z+t}$

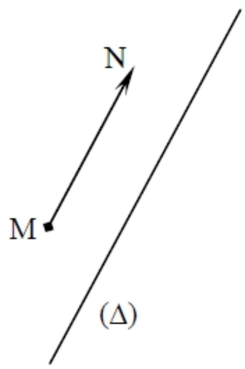
The relative uncertainty on a product or a quotient is the sum of relative uncertainties of each term..

1.2 Vectors

1.2.1 Definition

A vector is directed line segment. MN , having an origin M and an extremity N. It is completely defined if we specify:

1. both its length (Magnitude) and its direction,
2. We can represent a vector graphically by an arrow, showing both its scale length and its direction.
3. A vector is represented graphically by an arrow drawn on a scale as shown in the Figure
4. A vector is denoted with a small arrow over the symbol like $\overrightarrow{MN}$, $\overrightarrow{A}$, $\overrightarrow{a}$, etc.....
5. The magnitude of a vector quantity is referred by simple identifier like MN or as the absolute value of the vector as $|MN|$



- **Unit vector**: Each vector can be associated with a unit vector which has the same direction and a magnitute equal to one . The unit vector is obtained by dividing the initial vector by its magnitude :
$\|\overrightarrow{u}\| = \frac{\overrightarrow{A}}{\|\overrightarrow{A}\|}$

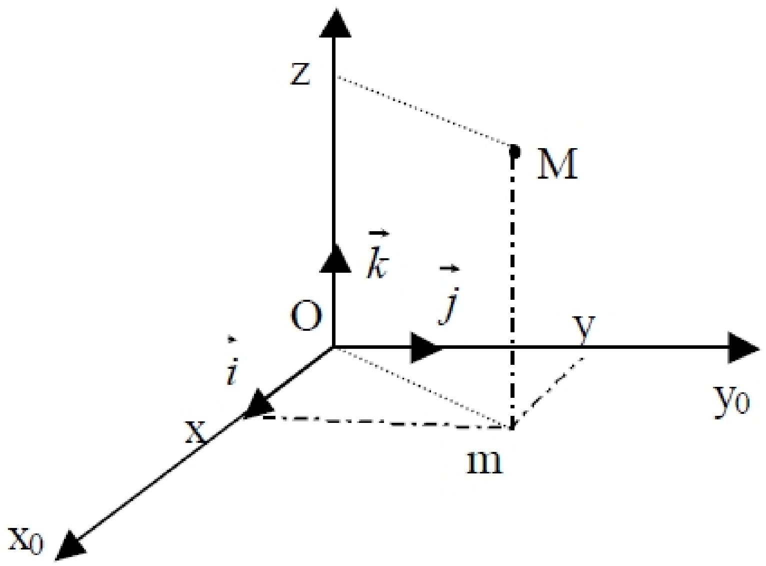
1.2.2 Orthonormal direct basis

1.2.2.0.2 Definition $(B) = (\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is an orthonormal basis if and only if:

- $\overrightarrow{i} \perp \overrightarrow{j} \perp \overrightarrow{k}$ (vectors orthogonal to each other)
- $\|\overrightarrow{i}\| = \|\overrightarrow{j}\| = \|\overrightarrow{k}\| = 1$ unitary vectors
In this course, we will always consider a direct orthonormal basis

1.2.3 Coordinates of a point and components of a vector

Let $(B) = (\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ a direct orthonormal basis associated with a reference frame in space (R)



- A point M is identified by its coordinates in the reference frame (R):$M(x, y, z)$
- A Vector $\overrightarrow{v} = \overrightarrow{OM}$ is defined by its components in the base (B):
  $\overrightarrow{v} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$

1.2.4 The Algebra of Vectors

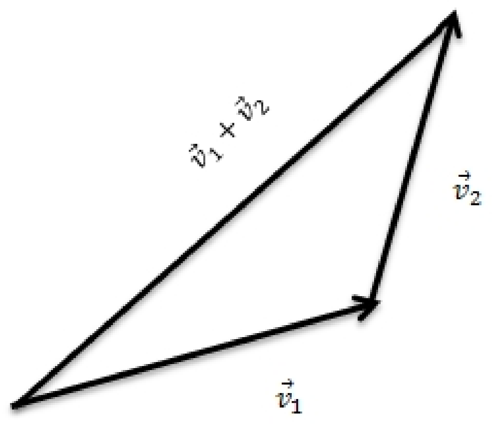
Let $(B) = (\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ a direct orthonormal basis and two vectors:
$\overrightarrow{v}_1 = x_1\overrightarrow{i} + y_1\overrightarrow{j} + z_1\overrightarrow{k}$
$\overrightarrow{v}_2 = x_2\overrightarrow{i} + y_2\overrightarrow{j} + z_2\overrightarrow{k}$
Note $\theta = (\overrightarrow{v}_1, \overrightarrow{v}_2)$ the angle where $0 \leq \theta \leq \pi$

1.2.4.1 Multiplying a Vector by a Scalar

$\forall\mu \in \Re, \mu\,\overrightarrow{v} = \mu\, x\,\overrightarrow{i} + \mu\, y\,\overrightarrow{j} + \mu\, z\,\overrightarrow{k}$

1.2.4.2 Adding Vectors

- graphical methods
  Addition of two vectors has the simple geometrical interpretation shown by the drawing. The rule is: to add $\overrightarrow{v}_2$ to $\overrightarrow{v}_1$, place the tail of $\overrightarrow{v}_2$ at the head of $\overrightarrow{v}_1$ by parallel translation of $\overrightarrow{v}_2$. The sum is a vector from the tail of $\overrightarrow{v}_1$ to the head of $\overrightarrow{v}_2$.



- analytical methods
  $\overrightarrow{v}_1 + \overrightarrow{v}_2 = (x_1 + x_2)\overrightarrow{i} + (y_1 + y_2)\overrightarrow{j} + (z_1 + z_2)\overrightarrow{k}$

1.2.5 Multiplying Vectors

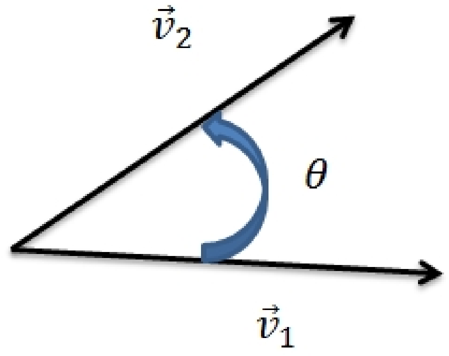
There are two types of vector multiplication are useful in physics.

1.2.6 Scalar Product (Dot Product)

The first type of multiplication is called the scalar product because the result of the multiplication is a scalar.

- the Scalar Product of $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ is the Scalar noted by $\overrightarrow{v}_1.\overrightarrow{v}_2$ defined by:
  $\overrightarrow{v}_1.\overrightarrow{v}_2 = \|\overrightarrow{v}_1\|\,\|\overrightarrow{v}_1\|\cos\theta$
- Analytical expression of this product is done by:
  $\overrightarrow{v}_1.\overrightarrow{v}_2 = x_1x_2 + y_1y_2 + z_1z_2.$

- Properties
  1. Commutative law
     The scalar product is commutative
     $\overrightarrow{v}_1.\overrightarrow{v}_2 = \overrightarrow{v}_2.\overrightarrow{v}_1$
  2. Distributive law

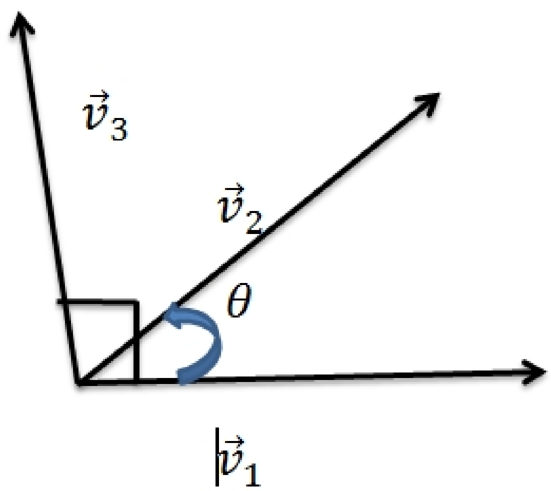


the scalar product is distributive
$\overrightarrow{v}_1.(\overrightarrow{v}_2 + \overrightarrow{v}_2) = \overrightarrow{v}_1.\overrightarrow{v}_2 + \overrightarrow{v}_1.\overrightarrow{v}_3$

3. Consequence
If $\overrightarrow{v}_1.\overrightarrow{v}_2 = \overrightarrow{0}$, with $\overrightarrow{v}_1 \neq 0$ and $\overrightarrow{v}_2 \neq 0$ So $\overrightarrow{v}_1 \perp \overrightarrow{v}_2$

1.2.7 Vector Product (Cross Product)

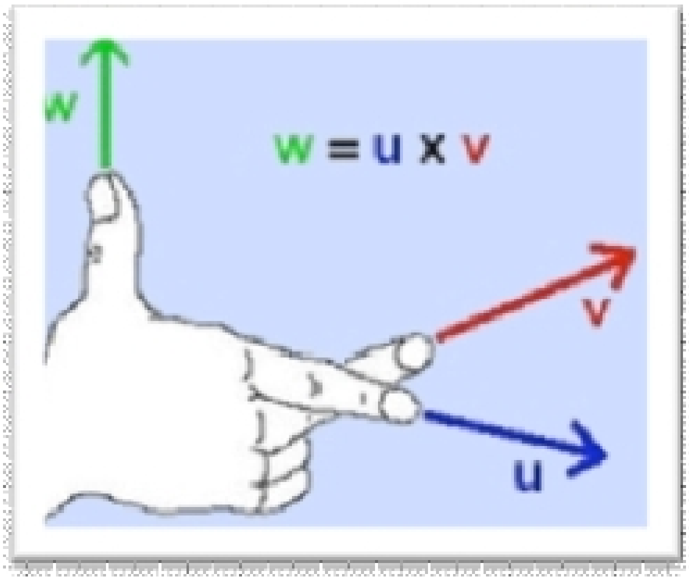
The second type of product useful in physics is the vector product, in which two vectors $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ are combined to form a third vector $\overrightarrow{v}_3$.



- The symbol for vector product is a cross, so it is often called the cross product:

$\overrightarrow{v}_1 \wedge \overrightarrow{v}_2 = \overrightarrow{v}_3$ with:

1. Magnitude equal $\|\overrightarrow{v}_1\|.\|\overrightarrow{v}_2\| \sin\theta$
2. We define the direction of $\overrightarrow{v}_3$ to be perpendicular to the plane of $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$. The three vectors $\overrightarrow{v}_1$, $\overrightarrow{v}$, and $\overrightarrow{v}_3$ form what is called a right-hand triple.



- Properties
  1. Commutative law
     The vector product is not commutative
     $\overrightarrow{v}_1 \wedge \overrightarrow{v}_2 = -\overrightarrow{v}_2 \wedge \overrightarrow{v}_1$
  2. Distributive law
     the vector product is distributive
     $\overrightarrow{v}_1 \wedge (\overrightarrow{v}_2 + \overrightarrow{v}_2) = \overrightarrow{v}_1 \wedge \overrightarrow{v}_2 + \overrightarrow{v}_1 \wedge \overrightarrow{v}_3$
  3. Consequence
     If $\overrightarrow{v}_1 \wedge \overrightarrow{v}_2 = \overrightarrow{0}$, with $\overrightarrow{v}_1 \neq 0$ and $\overrightarrow{v}_2 \neq 0$ So $\overrightarrow{v}_1 \parallel \overrightarrow{v}_2$
  4. The magnitude of the cross product gives us the area of the parallelogram formed by the vectors $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$
- Analytical expression of this product is done by

$$\overrightarrow{v}_1 \wedge \overrightarrow{v}_2 = \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = (y_1 z_2 - y_2 z_1)\overrightarrow{i} - \overrightarrow{j}(x_1 z_2 - x_2 z_1) + \overrightarrow{k}(x_1 y_2 - x_2 y_1)$$

1.2.8 Mixed Product (Scalar triple product)

- the mixed product is the scalar product of the first vector with the vector product of the two other vectors denoted as $\overrightarrow{v}_1.(\overrightarrow{v}_2 \wedge \overrightarrow{v}_3)$
- Geometrically, the mixed product is the volume of a parallelepiped formed by the tree vectors $\overrightarrow{v}_1$,$\overrightarrow{v}_2$ and $\overrightarrow{v}_3$.
- properties
  1. $\overrightarrow{v}_1.(\overrightarrow{v}_2 \wedge \overrightarrow{v}_3) = \overrightarrow{v}_2.(\overrightarrow{v}_3 \wedge \overrightarrow{v}_1) = \overrightarrow{v}_3.(\overrightarrow{v}_1 \wedge \overrightarrow{v}_2) = (\overrightarrow{v}_2 \wedge \overrightarrow{v}_3).\overrightarrow{v}_1 = (\overrightarrow{v}_3 \wedge \overrightarrow{v}_1).\overrightarrow{v}_2 = (\overrightarrow{v}_1 \wedge \overrightarrow{v}_2).\overrightarrow{v}_3$
  2. $\overrightarrow{v}_1.(\overrightarrow{v}_2 \wedge \overrightarrow{v}_3) = -\overrightarrow{v}_1.(\overrightarrow{v}_3 \wedge \overrightarrow{v}_2) = -\overrightarrow{v}_2.(\overrightarrow{v}_1 \wedge \overrightarrow{v}_3) = -\overrightarrow{v}_3.(\overrightarrow{v}_2 \wedge \overrightarrow{v}_1)$
- The analytical expression is done by:

$$\overrightarrow{v}_1.(\overrightarrow{v}_2 \wedge \overrightarrow{v}_3) = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = x_1(y_2 z_3 - y_3 z_2) + y_1(x_1 z_3 - x_3 z_1) + z_1(x_2 y_3 - x_3 y_2)$$

1.2.9 Double cross product

1.2.10 Vector derivation

Let $M(x(t), y(t), z(t))$ of the reference $R(O, xyz)$ we have :
$\overrightarrow{OM} = x(t)\overrightarrow{i} + y(t)\overrightarrow{j} + z(t)\overrightarrow{k}$
by definition :
$\frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\overrightarrow{i} + \frac{dy}{dt}\overrightarrow{j} + \frac{dz}{dt}\overrightarrow{k}$

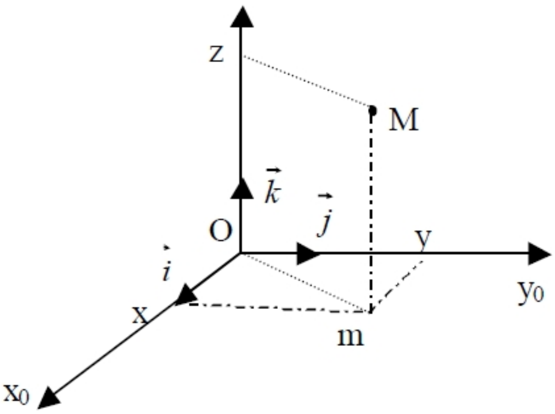
- **Properties**
  1. $\frac{d}{dt}(\overrightarrow{u} + \overrightarrow{v}) = \frac{d\overrightarrow{u}}{dt} + \frac{d\overrightarrow{V}}{dt}$

2. $\frac{d}{dt}(\overrightarrow{u} \wedge \overrightarrow{V}) = \frac{d\overrightarrow{u}}{dt} \wedge \overrightarrow{V} + \overrightarrow{u} \wedge \frac{d\overrightarrow{V}}{dt}$
3. $\frac{d}{dt}(\overrightarrow{u}.\overrightarrow{V}) = \frac{d\overrightarrow{u}}{dt}.\overrightarrow{V} + \frac{d\overrightarrow{V}}{dt}.\overrightarrow{u}$
4. $\frac{d}{dt}[\lambda(t)\overrightarrow{u}] = \frac{d\lambda(t)}{dt}\overrightarrow{u} + \lambda(t)\frac{d\overrightarrow{u}}{dt}$

1.2.11 coordinates system

Depending on the nature of a particle's path, its position will be identified by one of the following coordinate systems: Cartesian, cylindrical or spherical.

1. **Cartesian coordinate system**:
   In the reference frame $R_0(O, X_0, Y_0, Z_0)$ the particle's position M is given by its three Cartesian coordinates (x, y, z)



   where $-\infty\ < x, y, z < +\infty$

   - Vector position is written as :
     $\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = x\ \overrightarrow{i} + y\ \overrightarrow{j} + z\ \overrightarrow{k}$.

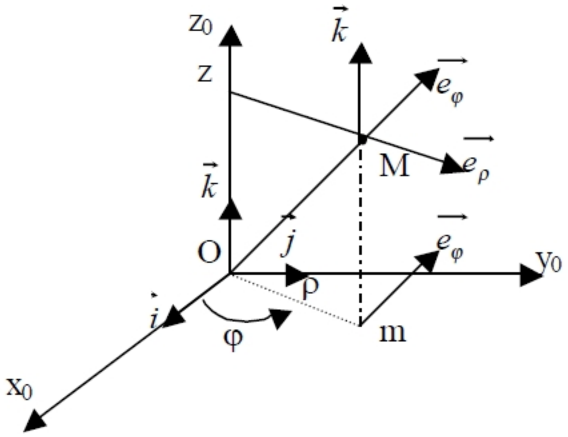
   - The elementary displacement is written as :
     $\overrightarrow{dl} = \overrightarrow{MM'} = dx\ \overrightarrow{i} + dy\ \overrightarrow{j} + dz\ \overrightarrow{k}$.

   - If there is movement, the coordinates x, y, z vary over time. The functions $x = f(t),\ y = g(t),\ z = h(t)$. Are called the time equations of motion. The motion of a point M is perfectly known if we know these equations of time.

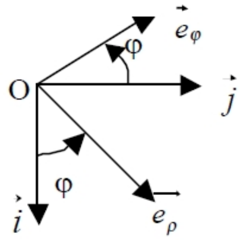
2 Cylindrical coordinate systems.

If the trajectory of the point M has axial symmetry of revolution it

is interesting to use cylindrical coordinates (ρ, φ, Z)
avec $0 < \rho < \infty,\ 0 < \varphi < 2\pi$ et $-\infty < z < +\infty$.



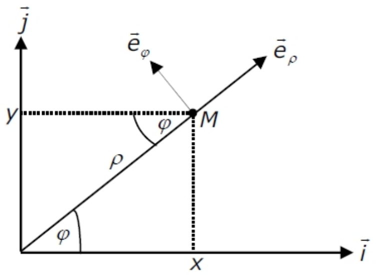
A new orthonormal direct base $(\overrightarrow{e}_\rho, \overrightarrow{e}_\varphi, \overrightarrow{k})$ is associated with this coordinate system such that :
$\overrightarrow{e}_\rho = cos\,\varphi\ \overrightarrow{i} + sin\varphi\ \overrightarrow{j}$
$\overrightarrow{e}_\varphi = -sin\varphi\ \overrightarrow{i} + cos\varphi\overrightarrow{j}$



with $\overrightarrow{e}_\varphi = \frac{d\,\overrightarrow{e}_\rho}{d\varphi}$ et $\overrightarrow{e}_\rho = -\frac{d\,\overrightarrow{e}_\varphi}{d\varphi}$

- Vector position written: $\overrightarrow{OM} = \rho\,\overrightarrow{e}_\varphi + z\,\overrightarrow{k}$
- The elementary displacement is written as :
  $\overrightarrow{MM'} = d\rho\overrightarrow{e}_\rho + \rho d\varphi\overrightarrow{e}_\varphi + z\,\overrightarrow{k}$

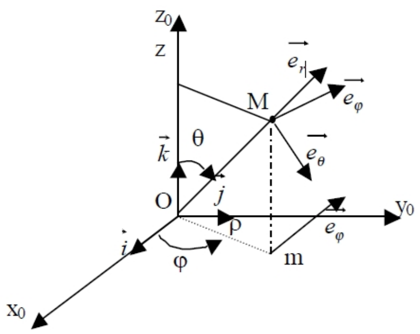
* **Special case** If the trajectory of M is plane, this point can be identified by its polar coordinates ρ and φ.



- Vector position is written as :
  $\overrightarrow{OM} = \rho\cos\varphi\,\overrightarrow{i} + \rho\sin\varphi\,\overrightarrow{j} = x\,\overrightarrow{i} + y\,\overrightarrow{j}$

3 Spherical coordinate system: When the problem has spherical symmetry about a point O which is taken as the origin of the space frame, it is convenient to use spherical coordinates (r, θ, φ) of the particle to be studied such that : :

$0 < r < \infty\ ,\ 0 < \varphi < 2\,\pi,\ \ 0 < \theta < \pi$

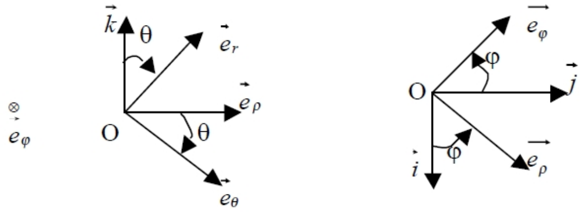


A new base is then introduced : $(\overrightarrow{e}_r, \overrightarrow{e}_\theta, \overrightarrow{e}_\varphi)$ where

$\overrightarrow{e}_r = \sin\theta\ \cos\varphi\ \overrightarrow{i} + \sin\theta\ \sin\varphi\ \overrightarrow{j} + \cos\theta\ \overrightarrow{k}$

$\overrightarrow{e}_\theta = \cos\theta\ \cos\varphi\ \overrightarrow{i} + \cos\theta\ \sin\varphi\ \overrightarrow{j} - \sin\theta\ \overrightarrow{k}$

$\overrightarrow{e}_\varphi = -\sin\varphi\ \overrightarrow{i} + \cos\varphi\ \overrightarrow{j}$



- vector position is written:
  $\overrightarrow{OM} = r\,\overrightarrow{e}_r$
- Elementary deplacement is written :
  $\overrightarrow{MM'} = dr\,\overrightarrow{e}_r + rd\theta\,\overrightarrow{e}_\theta + r\sin\theta\,d\varphi\,\overrightarrow{e}_\varphi$

1.2.11.1 Mathematical operators

1.2.11.1.1 Gradient operator The gradient of a scalar function f is a vector function noted $\overrightarrow{grad}f$ whose expression depends on the coordinate system in which the function f is expressed.

- In Cartesian coordinates:
  $f(x,y,z) \rightarrow \overrightarrow{grad}f = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k}$
- In cylindrical coordinates:
  $f(\rho,\varphi,z) \rightarrow \overrightarrow{grad}f = \frac{\partial f}{\partial \rho}\overrightarrow{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\overrightarrow{e}_\varphi + \frac{\partial f}{\partial z}\overrightarrow{e}_z$
- In spherical coordinates:
  $f(r,\theta,\varphi) \rightarrow \overrightarrow{grad}f = \frac{\partial f}{\partial r}\overrightarrow{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\overrightarrow{e}\varphi$
- The gradient of f can be expressed as $\overrightarrow{grad}f = \overrightarrow{\nabla}f$
  Where ∇ is the vector **nabla** whose components are defined by:

$$\overrightarrow{\nabla} = \begin{cases} \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial z}/(\overrightarrow{i},\overrightarrow{j},\overrightarrow{k}). \end{cases}$$

$$\overrightarrow{\nabla} = \begin{cases} \frac{\partial}{\partial \rho}, \\ \frac{1}{\rho}\frac{\partial}{\partial \varphi}, \\ \frac{\partial}{\partial z}/(\overrightarrow{e}_\rho,\overrightarrow{e}_\varphi,\overrightarrow{e}_z). \end{cases}$$

$$\overrightarrow{\nabla} = \begin{cases} \frac{\partial}{\partial r}, \\ \frac{1}{r}\frac{\partial}{\partial \theta}, \\ \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \varphi}/(\overrightarrow{e}_r,\overrightarrow{e}_\theta,\overrightarrow{e}_\varphi), \end{cases}$$

1.2.11.1.2 Divergence operator

- The divergence of a vector function $\overrightarrow{f}$ is the scalar defined by :
  $div\overrightarrow{f} = \overrightarrow{\nabla}.\overrightarrow{f}$

1.2.11.1.3 Rotational operator The rotationnel of a vector function $\overrightarrow{f}$ is the product defined by :
$\overrightarrow{rot}\overrightarrow{f} = \overrightarrow{\nabla}\Lambda\overrightarrow{f}$

- **Remarks**
  - To calculate $div\overrightarrow{f}$ and $\overrightarrow{rot}\overrightarrow{f}$, it is necessary to express $\overrightarrow{f}$ and $\overrightarrow{\nabla}$ in the same basis.
  - For any scalar function f, we have $\overrightarrow{rot}(\overrightarrow{grad}f) = \overrightarrow{0}$
  - Si $\overrightarrow{rot}\overrightarrow{f} = \overrightarrow{0}$, there exists a scalar function g such as $\overrightarrow{f} = \overrightarrow{grad}\,g$.We say that $\overrightarrow{f}$ derives from a potential g

1.3 Corrected exercises

- **Task 01:**
  Let two vector $\overrightarrow{A}$ and $\overrightarrow{B}$ located in the (Oxyz), defined as:
  $\overrightarrow{A} = 3\,\overrightarrow{i} + 4\,\overrightarrow{j} - 5\overrightarrow{k}$ and $\overrightarrow{B} = -\,\overrightarrow{i} + \,\overrightarrow{j} + 2\,\overrightarrow{k}$

  1. Calculate their magnitudes, then represent them.
  2. Calculate: $\overrightarrow{A} + \overrightarrow{B}$ and $\overrightarrow{A} - \overrightarrow{B}$
  3. Calculate the scalar products $\overrightarrow{A}.\overrightarrow{B}$ and $\overrightarrow{B} + \overrightarrow{A}$ .What do you notice ? .Calculate the angle $\theta = (\overrightarrow{A}, \overrightarrow{B})$.
  4. Calculate the vector products $(\overrightarrow{A} \wedge \overrightarrow{B})$ and $(\overrightarrow{B} \wedge \overrightarrow{A})$ .What do you notice?
  5. Consider the mixed product $\overrightarrow{A}.(\overrightarrow{A}\wedge\overrightarrow{B})$ What does this scalar present, with $\overrightarrow{D} = -2\,\overrightarrow{i} - 5\,\overrightarrow{j} + 5\overrightarrow{k}$

- **Solution:**
- The magnitudes of $\overrightarrow{A}$ and $\overrightarrow{B}$ $|\overrightarrow{A}| = \sqrt{3^2 + 4^2 + (-5)^2} = 5\sqrt{2}$
  $|\overrightarrow{B}| = \sqrt{(-1)^2 + 1^2 + 2^2} = \sqrt{6}$
- $\overrightarrow{A} + \overrightarrow{B} = 2\,\overrightarrow{i} + 5\,\overrightarrow{j} - 3\overrightarrow{k}$

- $\overrightarrow{A} - \overrightarrow{B} = 4\overrightarrow{i} + 3\overrightarrow{j} - 7\overrightarrow{k}$
- $\overrightarrow{A}.\overrightarrow{B} = -9$
  $\overrightarrow{B}.\overrightarrow{A} = -9$
  $\overrightarrow{A}.\overrightarrow{B} = \overrightarrow{B}.\overrightarrow{A}$ scalar product is commutative.
  $\cos\theta = \frac{\overrightarrow{A}.\overrightarrow{B}}{|\overrightarrow{A}||\overrightarrow{B}|} = -0.5196 \ , \theta = 121^{\circ}$
- $|\overrightarrow{A} \wedge \overrightarrow{B}| = 13\overrightarrow{i} - \overrightarrow{j} + 7\overrightarrow{k}$
  $|\overrightarrow{B} \wedge \overrightarrow{A}| = -13\overrightarrow{i} + \overrightarrow{j} - 7\overrightarrow{k}$

  $|\overrightarrow{A} \wedge \overrightarrow{B}| = -|\overrightarrow{B} \wedge \overrightarrow{A}|$ the scalar product is not commutative.

- **Task 02:**
  Consider the vector $\overrightarrow{A} = 3\overrightarrow{i} + 2\overrightarrow{j} + \overrightarrow{k}$

  1. Calculate the angles α,β ,γ, formed respectively between the vector $\overrightarrow{A}$ and the axes $\overrightarrow{ox}$, $\overrightarrow{oy}$, $\overrightarrow{oz}$.
  2. Check that $\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$

- **Solution:**
- $\overrightarrow{A}.\overrightarrow{i} = |\overrightarrow{A}|.|\overrightarrow{i}|\cos\alpha = |\overrightarrow{A}|\cos\alpha \Rightarrow \cos\alpha = \frac{\overrightarrow{A}.\overrightarrow{i}}{|\overrightarrow{A}|}$
  $\cos\alpha = \frac{3}{\sqrt{14}}$
- $\overrightarrow{A}.\overrightarrow{j} = |\overrightarrow{A}|.|\overrightarrow{j}|\cos\beta = |\overrightarrow{A}|\cos\beta \Rightarrow \cos\beta = \frac{\overrightarrow{A}.\overrightarrow{j}}{|\overrightarrow{A}|}$
  $\cos\beta = \frac{2}{\sqrt{14}}$
- $\overrightarrow{A}.\overrightarrow{k} = |\overrightarrow{A}|.|\overrightarrow{k}|\cos\gamma = |\overrightarrow{A}|\cos\gamma \Rightarrow \cos\gamma = \frac{\overrightarrow{A}.\overrightarrow{k}}{|\overrightarrow{A}|}$
  $\cos\gamma = \frac{1}{\sqrt{14}}$
- $(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = (\frac{3}{\sqrt{14}})^2 + (\frac{2}{\sqrt{14}})^2 + (\frac{1}{\sqrt{14}})^2 = 1$

- **Task 03:**
  Consider the vector $\overrightarrow{A}$ defined by: $\overrightarrow{A} = (2xy + z^3)\overrightarrow{i} + (x^2 + 2y)\overrightarrow{j} + (3xz^2 - 2)\overrightarrow{k}$

  1. Calculate $div\overrightarrow{A}$.
  2. Show that $\overrightarrow{rot}\overrightarrow{A} = 0$
  3. Find a scalar function $\varphi(x, y, z)$, such as $\overrightarrow{A} = \overrightarrow{grad}(\varphi)$

- **Solution:**
- $div\overrightarrow{A} = \frac{\partial(2xy+z^3)}{\partial x} + \frac{\partial(x^2+2y)}{\partial y} + \frac{\partial(3xz^2-2)}{\partial z} = 2y + 2 + 6xy$
  $div\overrightarrow{A} = 2y + 2 + 6XZ$

- $\overrightarrow{rot}\overrightarrow{A} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2+2y & 3xz^2-2 \end{vmatrix} = \overrightarrow{i}(0-0) - \overrightarrow{j}(3z^2-3z^2) +$
  $\overrightarrow{k}(2x-2x) = \overrightarrow{0} \Rightarrow \overrightarrow{rot}\overrightarrow{A} = \overrightarrow{0}$
- $\overrightarrow{A} = \overrightarrow{grad}(\varphi)$
  $\Rightarrow \begin{cases} \frac{\partial}{\partial x} = 2xy + z^3....(1) \\ \frac{\partial}{\partial y} = x^2 + 2y....(2) \\ \frac{\partial}{\partial z} = 3xz^2 - 2....(3) \end{cases}$
  $(1) \Rightarrow \frac{\partial}{\partial x} = 2xy + z^3 \Rightarrow \varphi(x,y,z) = \int(2xy + z^3)dx$
  $\varphi(x,y,z) = x^2y + z^3x + f(y,z).....(4)$
  $(2)and(4) \Rightarrow \frac{\partial}{\partial y} = x^2 + \frac{\partial f(x,y)}{\partial y} = x^2 + 2y$

  $\frac{\partial f(y,z)}{\partial y} = 2y \Rightarrow f(y,z) = \int 2y.dy = y^2 + g(z).....(5)$
  we replace (5) in (4) : $\varphi(x,y,z) = x^2y + z^3x + y^2 + g(z).....(6)$
  From (3) and (6) we get:
  $\frac{\partial\varphi(x,y,z)}{\partial z} = 3z^2x + \frac{\partial g(z)}{\partial z} = 3xz^2 - 2 \Rightarrow \frac{\partial g(z)}{\partial z} = -2 \Rightarrow g(z) = -2z + cst.....$
  So $\varphi(x,y,z) = x^2y + z^3x + y^2 - 2z + cte$

- **Task 04:**
  $\overrightarrow{r}$ is a vector defined as : $\overrightarrow{r} = \cos(2\,x)\,\overrightarrow{i} + \sin(5\,x)\,\overrightarrow{j} + e^{-\alpha\,x}\overrightarrow{k}$ (α is a real constant).

  1. Calculate the derivative vectors $\frac{d\,\overrightarrow{r}}{d\,x}, \frac{d\,\overrightarrow{r}}{d\,x^2}$. and evaluate their magnitude for $x = 0$, $\alpha = 1$
  2. Consider two vectors $\overrightarrow{A} = 3\,x\,\overrightarrow{i} + x^2\,\overrightarrow{j} - x^3\overrightarrow{k}$ and $\overrightarrow{B} = -x\,\overrightarrow{i} + 4x\,\overrightarrow{j} + x\,\overrightarrow{k}$
  3. Calculate the derivatives $\frac{d}{dx}(\overrightarrow{A}.\overrightarrow{B})$ and $\frac{d}{dx}(\overrightarrow{A} \wedge \overrightarrow{B})$ using two different methods:

     (a) By applying the vector derivation rules.

(b) By calculating the scalar product or the vector product. Then deriving the result.

- **Solution:**

- $\overrightarrow{r} = \cos(2\,x)\,\overrightarrow{i} + \sin(5\,x)\,\overrightarrow{j} + e^{-\alpha\,x}\overrightarrow{k}$
  $\frac{d\overrightarrow{r}}{dx} = -2\sin 2x\,\overrightarrow{i} + \sin 5x\,\overrightarrow{j} - \alpha e^{-\alpha x}\overrightarrow{k}$
  $|\frac{d\overrightarrow{r}}{dx}| = \sqrt{(-2\sin 2x)^2 + (\sin 5x)^2 + (-\alpha e^{-\alpha x})^2}$
  $x = 0\ \ \alpha = 1\ \ \Rightarrow |\frac{d\overrightarrow{r}}{dx}| = \sqrt{26}$
- $\frac{d^2\overrightarrow{r}}{d^2x} = -4\cos(2x)\,\overrightarrow{i} - 25\sin 5x\,\overrightarrow{j} + \alpha^2 e^{-\alpha x}\overrightarrow{j}$
  $|\frac{d^2\overrightarrow{r}}{d^2x}| = \sqrt{(-4\cos(2x))^2 + (-25\sin 5x)^2 + (\alpha^2 e^{-\alpha x})^2}$
  $if\ \ x = 0\ ,\alpha = 1\ \ \Rightarrow |\frac{d^2\overrightarrow{r}}{d^2x}| = \sqrt{17}$
- $\frac{d(\overrightarrow{A}.\overrightarrow{B})}{dx} = \overrightarrow{A}.\frac{d\overrightarrow{B}}{dx} + \frac{d\overrightarrow{A}}{dx}.\overrightarrow{B}$

  $\frac{d(\overrightarrow{A}.\overrightarrow{B})}{dx} = -6x + 12x^2 - 4x^3....(2)$

- $\overrightarrow{A}.\overrightarrow{B} = (3\,x\,\overrightarrow{i} + x^2\,\overrightarrow{j} - x^3\overrightarrow{k})(-x\,\overrightarrow{i} + 4x\,\overrightarrow{j} + x\,\overrightarrow{k})$
  $\overrightarrow{A}.\overrightarrow{B} = -2x^3 + 4x^3 - x^4$
  $\frac{d(\overrightarrow{A}.\overrightarrow{B})}{dx} = -6x + 12x^2 - 4x^3....(2)$

  $\frac{d(\overrightarrow{A}\wedge\overrightarrow{B})}{dx} = \frac{d\overrightarrow{A}}{dx}\wedge\overrightarrow{B} + \overrightarrow{A}\wedge\frac{d\overrightarrow{B}}{dx}$

  $$\frac{d(\overrightarrow{A}\wedge\overrightarrow{B})}{dx} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3 & 2x & -3x \\ -x & 4x & 3 \end{vmatrix} + \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 3x & x^2 & -x^2 \\ -1 & 4 & 1 \end{vmatrix}$$
  $= (3x^2 + 16x^3)\,\overrightarrow{i} - (6x - 4x^3)\,\overrightarrow{j} + (24x + 3x^2)\,\overrightarrow{k}....(3)$

  $\overrightarrow{A}\wedge\overrightarrow{B} = (x^3 + 4x^4)\,\overrightarrow{i} - (3x^2 - x^4)\,\overrightarrow{j} + (12x^2 + x^3)\,\overrightarrow{k}$
  $\frac{d(\overrightarrow{A}\wedge\overrightarrow{B})}{dx} = (3x^2 + 16x^3)\,\overrightarrow{i} - (6x - 4x^3)\,\overrightarrow{j} + (12x^2 + x^3)\,\overrightarrow{k}$

2

Kinematics of material point

2.1 Kinematics of material point

2.1.1 Definition

- Kinematics is the study of the movement of bodies independently of the causes that produce this movement.
- We will only consider bodies of small dimensions so that they can always be assimilated to a point called "the mobile"..
- The physical quantities of kinematics are time, position, speed and acceleration.

2.1.2 Vector velocity

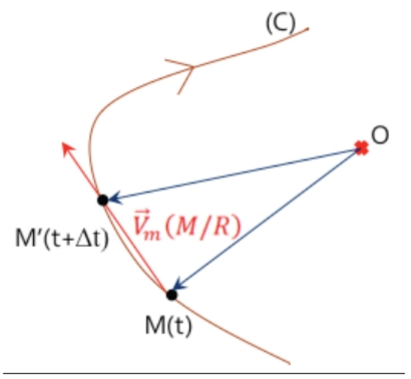
1. **Average velocity**

   Let a material point describe a trajectory (C) in a reference frame .The material point occupies the position M at time and the position M' at time $t' = t + \Delta t$

   the average velocity of material point between t and t' is given by:

   $\overrightarrow{V}_{moy}(M/R) = \frac{\overrightarrow{MM'}}{t'-t} = \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{\Delta t}$

   - The speed vector is therefore a vector which has the same direction and the same sense as $\overrightarrow{MM'}$ $(t' > t)$



2. **Instantaneous velocity**

   The instantaneous velocity vector of M with respect to the reference

frame R at an instant t is obtained by taking the limit $\Delta t \rightarrow 0$ in the definition of average velocity
$\overrightarrow{V} = \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt}$

2.1.3 Acceleration vector

Let $\overrightarrow{V}$ the velocity of the material point at time t, and $\overrightarrow{V'}$ its velocity at time t' .
so we define :

1. **Average acceleration**
   The average acceleration between the instances t and t' is defined as :
   $\overrightarrow{a}_{moy} = \frac{\overrightarrow{V'} - \overrightarrow{V}}{t' - t} = \frac{\overrightarrow{V}' - \overrightarrow{V}}{\triangle t}$
2. **Instantaneous acceleration**
   The instantaneous acceleration vector of M with respect to the reference frame R at time t is obtained by taking the limit $\Delta t \rightarrow 0$ in the definition of average acceleration
   $\overrightarrow{a} = \lim_{\Delta t \rightarrow 0} = \frac{d\overrightarrow{V}}{dt} = \frac{\overrightarrow{V}' - \overrightarrow{V}}{\triangle t} = \frac{d^2\overrightarrow{OM}}{dt^2}$

2.1.4 Velocity vector in different coordinate systems

Note that the Cartesian coordinate basis vectors are fixed,

- **Cartesian coordinates:**
  The velocity vector of point M is obtained by deriving its position vector with respect to time : :
  $\overrightarrow{V} = \frac{dx}{dt}\overrightarrow{i} + \frac{dy}{dt}\overrightarrow{j} + \frac{dz}{dt}\overrightarrow{k}$
  The following notation is also used
  $\overrightarrow{V} = \dot{x}\,\overrightarrow{i} + \dot{y}\,\overrightarrow{j} + \dot{z}\,\overrightarrow{k}$
  where the dot on the variable denote the derivative with respect to time.

- **Cylindrical coordinates:**
  To obtain the expression for the velocity vector in cylindrical coordinates, we derive the vector position :
  $\overrightarrow{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\,\overrightarrow{e}_\rho + z\,\overrightarrow{k})}{dt}$

$\overrightarrow{V}=\frac{d\rho}{dt}\overrightarrow{e}_\rho+\rho\frac{d\,\overrightarrow{e}_\rho}{dt}+\frac{dz}{dt}\overrightarrow{k}$

Knowing that $\overrightarrow{k}$ a fixed vector, its derivative is zero $\frac{d\,\overrightarrow{k}}{dt}=\overrightarrow{0}$

The vector being mobile $\overrightarrow{e_\rho}$ its derivative is generally not zero. In fact implicitly on t through its dependence on angle φ .So

$\frac{d\,\overrightarrow{e_\rho}}{dt}=\frac{d\overrightarrow{e_\rho}}{d\varphi}\frac{d\varphi}{dt}$

Using the expression for the vector $\overrightarrow{e_\rho}$ in the base ($\overrightarrow{i}$, $\overrightarrow{j}$, $\overrightarrow{k}$) we obtain:

$\frac{d\overrightarrow{e_\rho}}{d\varphi}=\frac{d(\cos\varphi\,\overrightarrow{i}+\sin\varphi\,\overrightarrow{j})}{d\varphi}=-\sin\varphi\,\overrightarrow{i}+\cos\varphi\,\overrightarrow{j}=\overrightarrow{e_\varphi}$

The derivative with respect to time is then given by:

$\frac{d\overrightarrow{e_\rho}}{dt}=\frac{d\varphi}{dt}\overrightarrow{e_\varphi}$

For the velocity vector, we obtain :

$\overrightarrow{V}=\dot{\rho}\overrightarrow{e_\rho}+\rho\,\dot{\varphi}\overrightarrow{e_\varphi}+\dot{z}\,\overrightarrow{k}$

- **Spherical coordinates** :

  The velocity vector is obtained by deriving the position vector :

  $\overrightarrow{V}=\frac{d\,\overrightarrow{OM}}{dt}=\frac{dr}{dt}\overrightarrow{e_r}+r\,\frac{d\overrightarrow{e_r}}{dt}$

  Avec $\frac{d\overrightarrow{e_r}}{dt}=\frac{\partial\overrightarrow{e_r}}{\partial\,\theta}\frac{d\theta}{dt}+\frac{\partial\overrightarrow{e_r}}{\partial\varphi}.\frac{d\varphi}{dt}$ Ainsi que

  $\frac{\partial\,\overrightarrow{e_r}}{\partial\theta}=\overrightarrow{e_\theta}.$

  $\frac{\partial\overrightarrow{e_r}}{\partial\,\varphi}=\sin\theta\,\overrightarrow{e_\varphi}$

  le vecteur vitesse s'écrit:

  $\overrightarrow{V}=\dot{r}\overrightarrow{e_r}+r\dot{\theta}\overrightarrow{e_\theta}+r\dot{\varphi}\sin\theta\overrightarrow{e_\varphi}$

2.1.5 Acceleration vector in different coordinate systems

- **Cartesian coordinates:**

  Using the velocity vector expression in cartesian coordinates, we have :

  $\overrightarrow{a}=\frac{d(\dot{x}\,\overrightarrow{i}+\dot{y}\,\overrightarrow{j}+\dot{z}\,\overrightarrow{k})}{dt}$

  Since the Cartesian coordinate basis vectors are fixed, we only need to derive the components of the velocity vector, which gives donne:

  $\overrightarrow{a}=\ddot{x}\,\overrightarrow{i}+\ddot{y}\,\overrightarrow{j}+\ddot{z}\,\overrightarrow{k}$

  where the two points on a variable mean the second derivative of the variable with respect to to time.

- **Cylindrical coordinates**:

  We use the expression for the velocity vector in cylindrical coordinates

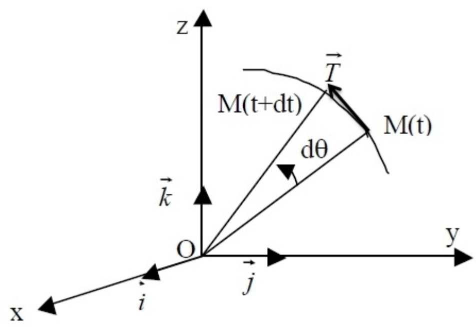
:
$\overrightarrow{a} = \frac{d(\dot{\rho}\overrightarrow{e_\rho}+\rho\dot{\varphi}\overrightarrow{e_\varphi}+\dot{z}\overrightarrow{k})}{dt}$
$\overrightarrow{a}(= \frac{d\dot{\rho}}{dt}\overrightarrow{e_\rho} + \dot{\rho}\frac{d\overrightarrow{e_\rho}}{dt} + \frac{d\rho}{dt}\dot{\varphi}\overrightarrow{e_\varphi} + \rho\dot{\varphi}\frac{d\overrightarrow{e_\varphi}}{dt} + \frac{d\dot{z}}{dt}\overrightarrow{k}$
Avec $\frac{d\overrightarrow{e_\rho}}{dt} = \dot{\varphi}\overrightarrow{e_\varphi}$ et
$\frac{d\overrightarrow{e_\varphi}}{dt} = \frac{d\overrightarrow{e_\varphi}}{d\varphi}.\frac{d\varphi}{dt} = \frac{d(-\sin\varphi\overrightarrow{i}+\cos\varphi\overrightarrow{j})}{d\varphi} = -\dot{\varphi}\overrightarrow{e_\rho}$
Substituting the above expression for the acceleration we obtain : $\overrightarrow{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\overrightarrow{e_\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\overrightarrow{e_\varphi} + \ddot{z}\overrightarrow{k}$

- **Spherical coordinates**:
  We use the velocity vector's expression in spherical coordinates :
  $\overrightarrow{a} = \frac{d(\dot{r}\overrightarrow{e_r}+r\dot{\theta}\overrightarrow{e_\theta}+r\dot{\varphi}\sin\theta\overrightarrow{e_\varphi})}{dt}$
  $\overrightarrow{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin\theta^2)\overrightarrow{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\,\cos\theta)\overrightarrow{e_\theta} + (2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta)\overrightarrow{e_\varphi}$

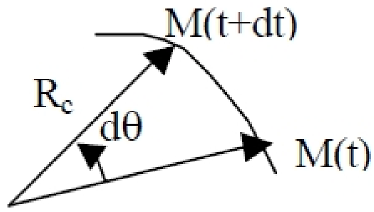
2.1.6 Components of velocity and acceleration vectors in curvilinear coordinates

Let (C) a trajectory described by a material point M relative to the frame $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ and P an arbitrary point considered as an origin on(C). The length of the arc (PM) is called the curvilinear abscissa s $(PM = s)$.

Let M' the position of M at time $(t+\Delta t)$
$\overrightarrow{OM}(t+\Delta t) - \overrightarrow{OM}(t) = \overrightarrow{OM'} - \overrightarrow{OM} = \overrightarrow{MM'}$
The speed vector is :
$\overrightarrow{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OM}}{ds}\frac{ds}{dt}$
with $\frac{d\overrightarrow{OM}}{ds} = \lim_{\Delta s \to 0} \frac{\overrightarrow{OM'} - \overrightarrow{OM}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\overrightarrow{MM'}}{\Delta s}$
If $\Delta t \to 0$, $\Delta s \to 0$ et M' very close to M. So $\|\overrightarrow{MM'}\| = \Delta s$ et $\overrightarrow{MM'}$ is tangent to the trajectory at point M and therefore $\lim_{\Delta s \to 0} \frac{\overrightarrow{MM'}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\overrightarrow{MM'}}{\|MM'\|} = \overrightarrow{T}$ ($\overrightarrow{T}$ is therefore a unit vector tangent to the trajectory at the point M).So,
The instantaneous velocity vector can therefore be written as
$\overrightarrow{V} = \frac{ds}{dt}\overrightarrow{T}$



- **Acceleration vector:** The derivative with respect to time of the velocity vector gives the acceleration vector, which is written as follows : $\overrightarrow{a} = \frac{d^2s}{dt^2}\overrightarrow{T} + \frac{ds}{dt}\frac{d\overrightarrow{T}}{dt}$
  Or $\frac{d\overrightarrow{T}}{dt} = \frac{d\overrightarrow{T}}{d\theta}\frac{d\theta}{dt}$ and
  $\frac{d\overrightarrow{T}}{d\theta} = \overrightarrow{N}$ is the normal vector directed towards the centre of curvature of the trajectory of point M.
  also we have, $\Delta s = R_s\theta$ où $ds = R_c d\theta$.
  So $\frac{d\theta}{dt} = \frac{1}{R_c}\frac{ds}{dt} = \frac{v}{R_c}$
  Therefore, $\frac{d\overrightarrow{T}}{dt} = \frac{v}{R_c}\overrightarrow{N}$
  The instantaneous acceleration vector of point M is written as :
  $\overrightarrow{a} = \frac{dv}{dt}\overrightarrow{T} + \frac{v^2}{R_c}\overrightarrow{N} = a_T\overrightarrow{T} + a_N\overrightarrow{N}$



  - The direction defined by the vector $\overrightarrow{N}$ is the principal normal at M to the trajectory..
  - The unit vector $\overrightarrow{B} = \overrightarrow{T} \wedge \overrightarrow{N}$ is called the binormal vector.
  - The reference frame $(M, \overrightarrow{T}, \overrightarrow{N}, \overrightarrow{B})$ is the reference of Frenet-Serret.

- **Calculation of the curvature's radius**
  $\overrightarrow{V} \wedge \overrightarrow{a} = \dot{s}\overrightarrow{T} \wedge [\ddot{s}\overrightarrow{T} + \frac{\dot{s}^2}{R_c}\overrightarrow{N}] = \dot{s}\overrightarrow{T} \wedge \ddot{s}\overrightarrow{T} + \dot{s}\overrightarrow{T} \wedge \frac{\dot{s}^2}{R_c}\overrightarrow{N} = \frac{\dot{s}^3}{R_c}\overrightarrow{B}$
  $\Rightarrow \|\overrightarrow{V} \wedge \overrightarrow{a}\| = \frac{\dot{s}^3}{R_c}\|B\| = \frac{\dot{s}^3}{R_c} = \frac{\|\overrightarrow{V}\|^3}{R_c}$
  So the radius of curvature is calculated by:
  $R_c = \frac{\|\overrightarrow{V}\|^3}{\|\overrightarrow{V} \wedge \overrightarrow{a}\|}$

- **Remarks**

  - $\overrightarrow{a}\overrightarrow{V} > 0 \Rightarrow \overrightarrow{a_T}$ and $\overrightarrow{V}$ have the same direction. The motion is therefore accelerated

- $\vec{a}\vec{V} < 0 \Rightarrow \vec{a_T}$ et $\vec{V}$ have two opposite directions. The movement is decelerated.
- $\vec{a}\vec{V} = 0 \Rightarrow \ddot{s} = 0$, so the movement is uniform.

2.1.7 Examples of simple movements

2.1.7.1 Rectilinear motion

This is a movement where the trajectory is a straight line

- **Example**

  Let a point M move along the Ox axis of a direct orthonormal reference frame $R(O, \vec{i}, \vec{j}, \vec{k})$
  such as $\overrightarrow{OM} = x(t)\vec{i}$.

  - $\vec{V} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(x\vec{i})}{dt} = \dot{x}\vec{i} \Longrightarrow \|\vec{V}\| = \dot{x}$.
  - $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d(\dot{x}\vec{i})}{dt} = \ddot{x}\vec{i} \Rightarrow \|\vec{a}\| = \ddot{x}$
  - $\vec{T} = \frac{\vec{V}}{\|\vec{V}\|} = \vec{i} \Longrightarrow s(t) = x(t)$
  - $\vec{a_t} = (\frac{d\|\vec{V}\|}{dt})\vec{T} = \ddot{x}\vec{T} \Longrightarrow \vec{a} = \vec{a}_T \Rightarrow \vec{a_N} = \vec{0}$
  - $\vec{V} \wedge \vec{a} = \dot{x}\vec{i} \wedge \ddot{x}\vec{i} = \vec{0} \Rightarrow \|\vec{V} \wedge \vec{a}\| = 0$
  - $R_c = \frac{\|\vec{V}\|^3}{\|\vec{V} \wedge \vec{a}\|} = \frac{\dot{x}}{0} = \infty$

    * **Remarks**

      · If $\|\vec{V}\| = V_0 = cte \Rightarrow \vec{a} = \vec{0} \Rightarrow$ the motion is uniformly rectilinear.
      $\dot{x} = V_0 \Rightarrow x(t) = V_0 t + x_0$ with $x(t=0) = x_0$

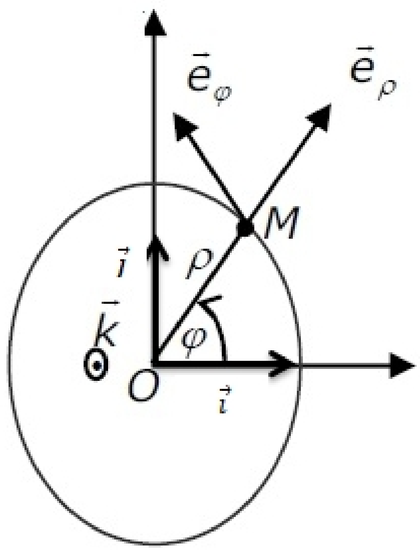
      · If$\|\vec{a}\| = a_0 = cte$ the motion is uniformly rectilinear accelerated if $(\vec{a}\vec{V}) > 0$ or decelerated $(\vec{a}\vec{V} < 0)$.
      $\ddot{x} = a_0 \Rightarrow x(t) = \frac{1}{2}a_0 t^2 + V_0 t + x_0$
      $v(M) = a_0 t + V_0$ avec $(x(t=0) = x_0)$ et $(V(t=0) = V_0)$.

2.1.7.2 Rectilinear sinusoidal motion

Rectilinear sinusoidal motion if the position vector in the reference frame $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ can be written as:
$\overrightarrow{OM} = x(t)\overrightarrow{i} = X_m \cos(\omega t + \phi)\overrightarrow{i}$
With
X_m : the amplitude of movement (m)
ω : pulse $(rad.s^{-1})$
ϕ : phase at origin(rad).
So
$\overrightarrow{V}(M/R) = \frac{d\overrightarrow{OM}(M/R)}{dt} = -X_m\omega \sin(\omega t + \phi)\overrightarrow{i}$
et $\overrightarrow{\gamma} = \frac{d\overrightarrow{V}(M/R)}{dt} = -X_m\omega^2 \cos(\omega t + \phi)\overrightarrow{i}$
We can see that : $\overrightarrow{\gamma}(M/R) = -\omega^2\overrightarrow{OM}$
and as $\overrightarrow{\gamma}(M/R) = (\frac{d^2\overrightarrow{OM}(M/R)}{dt^2})$
Rectilinear sinusoidal motion is governed by the differential equation $\frac{d^2\overrightarrow{OM}}{dt^2} + \omega^2\overrightarrow{OM} = 0$
Projected according to $\overrightarrow{i}$, this differential equation is written as:
$\ddot{x} + \omega^2 x = 0$

2.1.7.3 Circular movement

This is a movement such that the trajectory is a circle of constant radius.
polar coordinates are well adapted to the study of the motion of a point describing a circular trajectory.
In the case of circular motion, the constant radius $(\rho = R)$



In this case the position vector can be written as :
$\overrightarrow{OM} = R\overrightarrow{e_\rho}$.

The velocity vector:
$\overrightarrow{V}(M) = \frac{d\overrightarrow{OM}}{dt} = \frac{(dR\overrightarrow{e_\rho})}{dt} = \frac{Rd(\overrightarrow{e_\rho})}{dt} = R\dot{\varphi}\overrightarrow{e_\varphi} \Rightarrow \|\overrightarrow{V}(M)\| = R\dot{\varphi}$

The acceleration vector :
$\overrightarrow{\gamma}(M) = \frac{d\overrightarrow{V}}{dt} = \frac{d(R\dot{\varphi}\overrightarrow{e_\varphi})}{dt} = R\ddot{\varphi}\overrightarrow{e_\varphi} + R\dot{\varphi}\frac{d(\overrightarrow{e}_\varphi)}{dt} = R\ddot{\varphi}\overrightarrow{e_\varphi} - R\dot{\varphi}^2\overrightarrow{e_\rho} \Rightarrow \|\overrightarrow{\gamma}(R)\| = R\sqrt{\ddot{\varphi} + \dot{\varphi}^4}$

- $\overrightarrow{\gamma_T} = (\frac{d\|\overrightarrow{V}(M)\|}{dt})\overrightarrow{T} = \frac{d(R\dot{\varphi})}{dt}\overrightarrow{T} = R\ddot{\varphi}\overrightarrow{T}$
- $\overrightarrow{\gamma_N} = \overrightarrow{\gamma}(M) - \overrightarrow{\gamma_T} = -R\dot{\varphi}^2\overrightarrow{e_\rho} = R\dot{\varphi}^2\overrightarrow{N}$
  with $\overrightarrow{N} = \overrightarrow{B} \wedge \overrightarrow{T} = \overrightarrow{e_z} \wedge \overrightarrow{e_\varphi} = -\overrightarrow{e_\rho}$

- **Remarques**

- $\dot{\varphi}$: Angular velocity
- $\overrightarrow{\omega} = \dot{\varphi}\overrightarrow{k}$: Angular velocity vector.
- $\overrightarrow{V}(M) = R\dot{\varphi}\overrightarrow{e_\varphi} = R\dot{\varphi}\overrightarrow{k} \wedge \overrightarrow{e_\rho} = \overrightarrow{\omega} \wedge \overrightarrow{OM}$.
- $\ddot{\omega}$: angular acceleration.
- Si $\dot{\varphi} = \omega_0 = cte \Rightarrow$ the movement is uniformly circular $(\varphi(t) = \omega_0 t + \varphi_0)$
- Si $\ddot{\varphi} = \gamma_0 = cte \Rightarrow$ the motion is circular and uniformly varied.

2.2 Change of reference frame

Let's study the motion of a particle M relative to a fixed reference frame R, called absolute reference.

It is sometimes useful to introduce a second reference frame R', known as the relative frame, compared with in which the movement of M is simple to study .

- $R(O, X, Y, Z)$ an absolute reference (fixed reference point).
- $R'(O', X', Y', Z')$ a relative reference (a reference that moves relative to R).

  R'can be animated by a movement of translation or rotation relative to R.

  The rotation of R' with respect to R occurs with an angular velocity $\omega(R'/R)$ such that :

  In the R reference ,

  $\frac{d\overrightarrow{i'}}{dt}|_R = \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{i'}$

  $\frac{d\overrightarrow{j'}}{dt}|_R = \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{j'}$

  $\frac{d\overrightarrow{k'}}{dt}|_R = \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{k'}$

  In the R' reference ,

  $\frac{d\overrightarrow{i'}}{dt}|_{R'} = \frac{d\overrightarrow{j'}}{dt}|_{R'} = \frac{d\overrightarrow{k'}}{dt}|_{R'} = \overrightarrow{0}$

  Derivation in a moving reference :

  Lets $\overrightarrow{A}$ an unusual vector. In the R frame, this vector is written as :

  $\overrightarrow{A} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}.$

  In the frame R' the vector $\overrightarrow{A}$ is written as,

  $\overrightarrow{A} = x'\,\overrightarrow{i'} + y'\,\overrightarrow{j'} + z'\,\overrightarrow{k'}.$

  $\frac{d\overrightarrow{A}}{dt}|_R = \dot{x}\,\overrightarrow{i} + \dot{y}\,\overrightarrow{j} + \dot{z}\,\overrightarrow{k} = x'\,\overrightarrow{i'} + x'\frac{d\overrightarrow{i'}}{dt}|_R + y'\,\overrightarrow{j'} + y'\frac{d\overrightarrow{j'}}{dt}|_R + z'\,\overrightarrow{k'} + z'\frac{dk'}{dt}|_R$

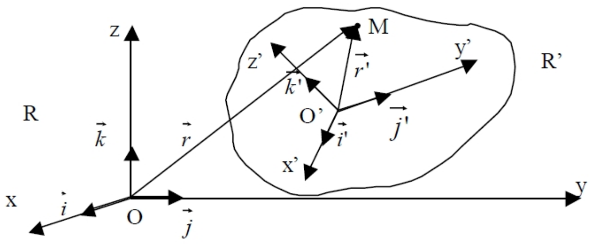
  which can also be written as,

  $\frac{d\overrightarrow{A}}{dt}|_R = x'\,\overrightarrow{i'} + y'\,\overrightarrow{j'} + z'\,\overrightarrow{k'} + x'\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{i'} + y'\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{j'} + z'\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{k'}.$

  $\frac{d\overrightarrow{A}}{dt}|_R = \frac{d\overrightarrow{A}}{dt}|_{R'} + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{A}$

– **Velocity composition**

Let $R(O, XYZ)$ an absolute reference and $R'(O', x', Y', Z')$ relative reference.



the position vectors of the particle M in the reference frames R and R' are, respectively :
$\overrightarrow{OM} = \overrightarrow{r}$ et $\overrightarrow{O'M} = \overrightarrow{r'}$.
We can write,
$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$.
So the absolute velocity of point M is,
$\overrightarrow{V_a}(M) = \overrightarrow{V}(M/R) = \frac{d\overrightarrow{OM}}{dt}|_R = \frac{d\overrightarrow{OO'}}{dt}|_R + \frac{d\overrightarrow{O'M}}{dt}|_R = \frac{d\overrightarrow{OO'}}{dt}|_R + \frac{d\overrightarrow{O'M}}{dt}|_{R'} + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M}...$
où $\overrightarrow{V}(M/R') = \overrightarrow{V}_r(M) = \frac{d\overrightarrow{O'M}}{dt}|_{R'}$ denotes the relative velocity point M. And,
$\overrightarrow{V_e}(M) = \frac{d\overrightarrow{OO'}}{dt}|_R + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M}$.
is the training velocity of M. Training speed of M is the absolute speed of the (imaginary) point which coincides with M at time t and assumed to be fixed in t R'.
We can also note the training speed of M as follows,
$\overrightarrow{V_e}(M) = \frac{d\overrightarrow{OM}}{dt}|_R(\text{M}\, fixed\ in \text{R'})$.
So we have,
$\overrightarrow{V_a}(M) = \overrightarrow{V_r}(M) + \overrightarrow{V_e}(M)$.

∗ **Acceleration composition**

The absolute acceleration of point M is,

$\overrightarrow{\gamma_a}(M) = \overrightarrow{\gamma}(M/R) = \frac{d^2\overrightarrow{OM}}{dt^2}|_R = \frac{d\overrightarrow{V_a}}{dt}|_R$
$\overrightarrow{\gamma_a}(M) = \frac{d(\overrightarrow{V_r}(M)+\overrightarrow{V_e}(M))}{dt}|_R = \frac{d\overrightarrow{V_r}(M)}{dt}|_R + \frac{d}{dt}[\frac{d\overrightarrow{OO'}}{dt}|_R + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M}|_R]$
$\frac{d\overrightarrow{V_r}(M)}{dt}|_R = \frac{d\overrightarrow{V_r}(M)}{dt}|_{R'} + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{V_r} = \overrightarrow{\gamma}_r(M) + \overrightarrow{\omega}(R'/R) \wedge \overrightarrow{V}_r.$
$\frac{d}{dt}(\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M})|_R = \frac{d\overrightarrow{\omega}(R'/R)}{dt} \wedge \overrightarrow{O'M} + \overrightarrow{\omega}(R'/R) \wedge \frac{d\overrightarrow{O'M}}{dt}|_R$
The absolute acceleration can therefore be written as:
$\overrightarrow{\gamma_a}(M) = \overrightarrow{\gamma_r}(M) + 2\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{V_r}(M) + \frac{d^2\overrightarrow{OO'}}{dt^2}|_R + \frac{d\overrightarrow{\omega}(R'/R)}{dt} \wedge \overrightarrow{O'M} + \overrightarrow{\omega}(R'/R) \wedge (\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M}).$
Of which
$\frac{d^2\overrightarrow{OO'}}{dt^2}|_R + \frac{d\overrightarrow{\omega}(R'/R)}{dt} \wedge \overrightarrow{OM'} + \overrightarrow{\omega}(R'/R) \wedge (\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{O'M}) = \overrightarrow{\gamma_e}(M)$
denotes the training acceleration, and
$2\overrightarrow{\omega}(R'/R) \wedge \overrightarrow{V_r}(M) = \overrightarrow{\gamma_C}(M).$
is the Coriolis or complementary acceleration. We then write.
$\overrightarrow{\gamma_a}(M) = \overrightarrow{\gamma_r}(M) + \overrightarrow{\gamma_e}(M) + \overrightarrow{\gamma_c}(M).$

· **Special cases**

When the reference R' is in translation with respect to R,
$\overrightarrow{\omega}(R'/R) = \overrightarrow{o} \Rightarrow \overrightarrow{\gamma_c}(M) = \overrightarrow{0}$
Therefore :

$$\overrightarrow{\gamma_a}(M) = \overrightarrow{\gamma_r}(M) + \overrightarrow{\gamma_e}(M)$$

2.3 Corrected exercises

- **Task 01:**
Let $R(0, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ be a direct orthonormal reference. Consider a material point M which describes an orthonormal motion in the plane $(O, \overrightarrow{i}, \overrightarrow{j})$ along the path shown in Figure 1. The equation of this trajectory is given in polar coordinates by:
$\rho = \frac{1}{2}\rho_0(1 + \cos\varphi).$

Where ρ_0 is a given length, $0 \leq \varphi \leq \pi$ and $\dot{\varphi} > 0$.

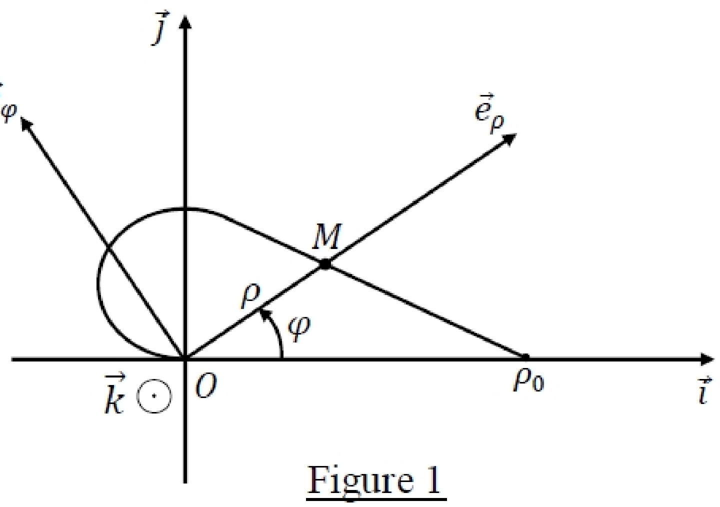
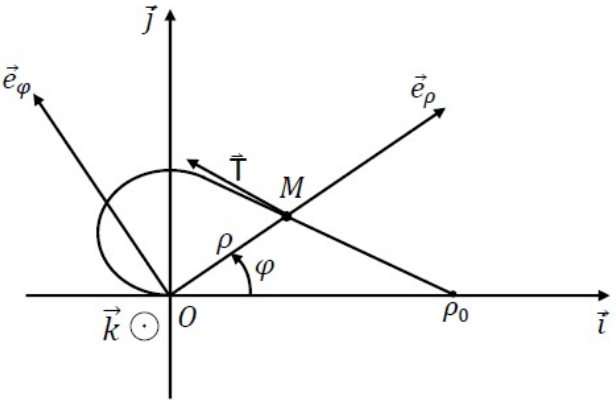


Figure 1

1. Show that the velocity of M in R can be written in the basis $(\overrightarrow{e}_\rho, \overrightarrow{e}_\varphi)$ in the form: :
   $\overrightarrow{V}(M/R) = \rho_0\dot{\varphi}\cos(\frac{\varphi}{2})[-\sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi]$
2. Determine $\|\overrightarrow{V}(M/R)\|$ the magnitude of the velocity. $\overrightarrow{V}(M/R)$.
3. Deduce $\overrightarrow{T}$ the unit vector tangent to the trajectory.
4. Show that the angle $(\widehat{\overrightarrow{e}_\varphi, \overrightarrow{T}}) = \frac{\varphi}{2}$.
5. Represent graphically the vector $\overrightarrow{T}$.
6. Determine γ_T and γ_N the tangential and normal acceleration vectors respectively..
7. Deduce ρ_c the radius of curvature of the trajectory and the unit vector $\overrightarrow{N}$.

   Solution:

   Let $R(0, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ be a direct orthonormal reference. Consider a material point M which describes an orthonormal motion in the plane $(O, \overrightarrow{i}, \overrightarrow{j})$ along the path shown in Figure 1. The equation of this trajectory is given in polar coordinates by:
   $\rho = \frac{1}{2}\rho_0(1+\cos\varphi)$. 1- We have: $\overrightarrow{OM} = \rho\overrightarrow{e}_\rho = \rho = \frac{1}{2}\rho_0(1+\cos\varphi)\overrightarrow{e}_\rho$
   $\overrightarrow{V}(M/R) = -\frac{1}{2}\rho_0\dot{\varphi}\sin\varphi\overrightarrow{e}_\rho + \frac{1}{2}\rho_0\dot{\varphi}(1+\cos\varphi)\overrightarrow{e}_\varphi$
   Similarly, we have



$\sin\varphi = \sin(\frac{\varphi}{2}+\frac{\varphi}{2}) = 2\sin(\frac{\varphi}{2})\cos(\frac{\varphi}{2})$.

$\cos\varphi = \cos(\frac{\varphi}{2}+\frac{\varphi}{2}) = 2\cos^2(\frac{\varphi}{2}) - 1 \Rightarrow 1+\cos\varphi = 2\cos^2(\frac{\varphi}{2}) \Rightarrow$
$\overrightarrow{V}(M/R) = -\rho_0\dot{\varphi}\sin(\frac{\varphi}{2})\cos(\frac{\varphi}{2})\overrightarrow{e}_\rho + \rho_0\dot{\varphi}\cos^2(\frac{\varphi}{2})\overrightarrow{e}_\varphi$

Hence: $\overrightarrow{V}(M/R) = \rho_0\dot{\varphi}\cos(\frac{\varphi}{2})[-\sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi]$

2-$\|\overrightarrow{V}(M/R)\| = \|\rho_0\dot{\varphi}\cos(\frac{\varphi}{2})[-\sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi]\|$
$=\|\rho_0\dot{\varphi}\cos(\frac{\varphi}{2})\|\ \|[-\sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi]\| = \rho_0\dot{\varphi}\cos(\frac{\varphi}{2})$

3-We know that $\overrightarrow{T} = \frac{\overrightarrow{V}(M/R)}{\|\overrightarrow{V}(M/R)\|}$

So: $\overrightarrow{T} = -sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi$

4-We have:$\overrightarrow{T}\overrightarrow{e}_\varphi = \cos(\frac{\varphi}{2})$.

So $(\widehat{\overrightarrow{e}_\varphi, \overrightarrow{T}}) = \frac{\varphi}{2}$.

5-See figure1.

6- $\overrightarrow{\gamma}_T = \gamma_T\overrightarrow{T} = \frac{d\|\overrightarrow{V}(M/R)\|}{dt}\overrightarrow{T} = \frac{d(\rho_0\dot{\varphi}\cos(\frac{\varphi}{2}))}{dt}\overrightarrow{T} = [\rho_0\ddot{\varphi}\cos(\frac{\varphi}{2}) - \frac{1}{2}\rho_0\dot{\varphi}^2\sin(\frac{\varphi}{2})]\overrightarrow{T}$

$\gamma_N = \gamma_N\overrightarrow{N} = \|\overrightarrow{V}(M/R)\|\frac{d\overrightarrow{T}}{dt} = \rho_0\dot{\varphi}\cos(\frac{\dot{\varphi}}{2})\frac{d}{dt}[-sin(\frac{\varphi}{2})\overrightarrow{e}_\rho + \cos(\frac{\varphi}{2})\overrightarrow{e}_\varphi] =$
$\rho_0\dot{\varphi}\cos(\frac{\varphi}{2})[-\frac{\dot{\varphi}}{2}\cos(\frac{\varphi}{2})\overrightarrow{e}_\rho - \dot{\varphi}\sin(\frac{\varphi}{2})\overrightarrow{e}_\varphi - \frac{\dot{\varphi}}{2}\sin(\frac{\varphi}{2})\overrightarrow{e}_\varphi - \dot{\varphi}\cos(\frac{\varphi}{2})\overrightarrow{e}_\rho]$
$= \rho_0\dot{\varphi}\cos(\frac{\varphi}{2})[-\frac{3}{2}\dot{\varphi}\cos(\frac{\varphi}{2})\overrightarrow{e}_\rho - \frac{3}{2}\dot{\varphi}\sin(\frac{\varphi}{2}\overrightarrow{e}_\varphi])$
$= \frac{3}{2}\rho_0\dot{\varphi}^2\cos(\frac{\varphi}{2})[-\cos(\frac{\varphi}{2})\overrightarrow{e}_\rho - \sin(\frac{\varphi}{2})\overrightarrow{e}_\varphi]$

7- We know that :$\gamma_N = \frac{\|\overrightarrow{V}(M/R)\|^2}{\rho_c} = \frac{\rho_2^2\dot{\varphi}^2\cos^2(\frac{\varphi}{2})}{\rho_c}$

and like: $\gamma_N = \|\gamma_N\| = \frac{3}{2}\rho_0\dot{\varphi}^2\cos(\frac{\varphi}{2})$

So:

$\frac{\rho_0^2\dot{\varphi}^2\cos^2(\frac{\varphi}{2})}{\rho_c} = \frac{3}{2}\rho_0\dot{\varphi}^2\cos(\frac{\varphi}{2})$

$\Rightarrow \rho_c = \frac{2\,\rho_0\cos(\frac{\varphi}{2})}{3}$.

And as :$\gamma_N = \gamma_N\overrightarrow{N}$ then $\overrightarrow{N} = [-\cos(\frac{\varphi}{2})\overrightarrow{e}\rho - \sin(\frac{\varphi}{2})\overrightarrow{e}_\varphi]$.

- **Task 02:**

  The velocity diagram for a moving body A animated by rectilinear motion on an axis Ox is given Figure 2, knowing that at time $t = 0s$, $x = 0m$, $v = 0m/s$

  1. Plot the acceleration versus time diagram.
  2. Plot the space diagram for the time interval [0, 7s].
  3. Determine the position of the mobile at time $t = 10s$.
  4. What is the distance travelled between the two instants $t = 0$ and $t = 10s$
  5. Specify the nature of the movement in each phase
  6. Determine the equations of velocity $v(t)$ and motion $x(t)$ for each phase

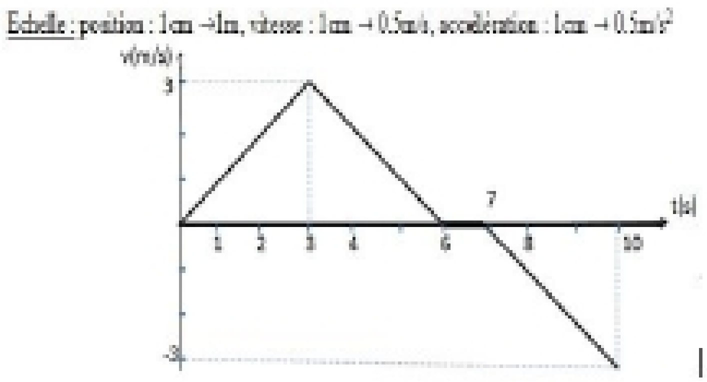


Figure 2

  7. Plot the vectors: position, velocity and acceleration at time $t = 8s$ on the trajectory.

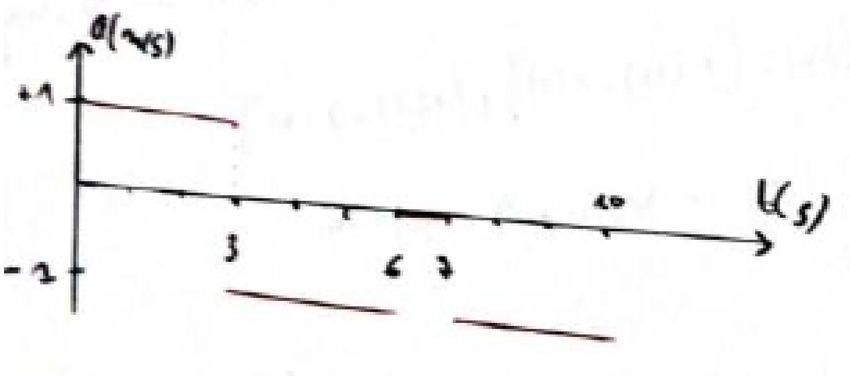
Solution:

1. The acceleration versus time diagram. Acceleration is measured by the slope of the curve $(\tan)$

   $a = \frac{v_2 - v_1}{t_2 - t_1}$

   - $t \in [0, 3]s; a_1 = \frac{3-0}{3-0} = 1m/s^2$

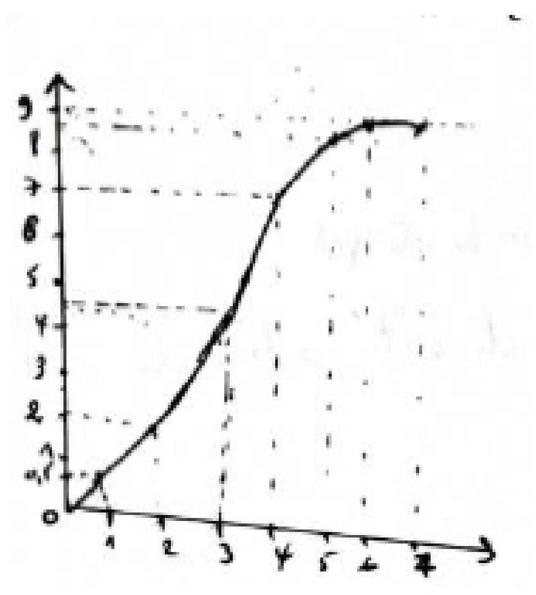
- $t \in [3, 6]s$ $a_2 = -1m/s^2$
- $t \in [6, 7]s$ $a_3 = 0m/s^2$
- $t \in [7, 10]s$ $a_4 = -1m/s^2$



2. The space diagram for the time interval [0, 7s].

t(s)	0	1	2	3	4	5	6	7
x(m)	0	0,5	2	4,5	7	8,5	9	9

3. Position of the mobile at time $t = 10s$ $X(10) = [x(10) - x(0)] + [x(7) - x(0)]$
   $x(10) = -4.5 + 9 = 4.5m$
4. Distance travelled between the two instants $t = 0$ and $t = 10s$ $d = X(10) = |[x(10) - x(0)]| + |[x(7) - x(0)]|$
5. The nature of the movement in each phase
   - $[0, 3s] : v.a > 0$ uniform movement
   - $[3, 6s] : a.v < 0$ decelerate movement
   - $[6, 7s] : a.v = 0$ at rest
   - $[7, 10s] : a.v > 0$ accelerate movement
6. the equations of velocity $v(t)$ and motion $x(t)$ for each phase



- $[0, 3s]$
  $a_1 = 1m/s^2$
  $a_1 = \frac{dv_1(t)}{dt} \Rightarrow dv_1 = a_1 dt \Rightarrow \int_0^{v_1(t)} dv(t) = \int_{t_0=0}^{t} dt$
  $v_1(t) = t$
  $v_1(t) = \frac{dx(t)}{dt} \Rightarrow$
  $\int_{x(0)=0}^{x_1(t)} dx(t) = \int_{t}^{t=0} v_1(t)dt$
  $\Rightarrow x_1(t) = \frac{1}{2}t^2$
- $[3, 6s]$ $a_2 = -1m/s^2$
  $a_2 = \frac{dv_2(t)}{dt} \Rightarrow dv_2 = a_2 dt \Rightarrow \int_{v(3)}^{v_2(t)} dv(t) = \int_{t=3}^{t} a_2 dt$
  $v_2(t) - 3 = -1(t - 3)$
  $\Rightarrow v_3(t) = -t + 6$

  $v_2(t) = \frac{dx(t)}{dt} \Rightarrow$
  $\int_{x(3)=4.5}^{x_2(t)} dx(t) = \int_{t=3}^{t} v_2(t)dt$
  $x_2(t) - 4.5 = \int_{t=3}^{t} (-t + 6)dt \Rightarrow x_2(t) = -\frac{t^2}{2} + 6t - 9$
- $[6, 7s]$ $a_3 = 0m/s^2$, $v = 0$, $x_3(t) = 9\,m$
- $[7, 10s]$ $a_4 = -1m/s^2$

$a_4 = \frac{dv_4(t)}{dt} \Rightarrow dv_4 = a_4 dt \Rightarrow \int_{v(7)=0}^{v_4(t)} dv(t) = \int_{t=7}^{t} a_4 dt$
$v_4(t) = -t + 7$

$v_4(t) = \frac{dx(t)}{dt} \Rightarrow$
$\int_{x(7)=9}^{x_4(t)} dx(t) = \int_{t=7}^{t} (-t+7)dt$
$x_4(t) = -\frac{t^2}{2} + 7t - 15.5$

7. Piloting of vectors: position, velocity and acceleration

- **Task 03:**

The coordinates of a material point M in a direct orthonormal reference frame $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ are given as a function of time by :
$x(t) = t - 1$, $y(t) = -t^2 + 1$ et $Z(t) = 0$.

1. Determine the equation of the trajectory of M.
2. Determine the velocity and acceleration vectors of M.
3. Calculate the tangential and normal accelerations of M.
4. Deduce the radius of curvature R_c of the trajectory as a function of time

Solution:

1. $x(t) + 1 = t \Rightarrow y = -x^2 - 2x \Rightarrow$ the trajectory is a parabola.
2. $\overrightarrow{V}(M/R) = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\overrightarrow{i} + \frac{dy}{dt}\overrightarrow{j} + \frac{dZ}{dt}\overrightarrow{k} = \overrightarrow{i} - 2t\overrightarrow{j}$
$\overrightarrow{\gamma}(M/R) = \frac{d\overrightarrow{V}(M/R)}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{d^2t} + \frac{d^2y}{d^2t} + \frac{d^2Z}{d^2t} = -2\overrightarrow{j}$
3.Tangential acceleration: $\gamma_T = \frac{d\|\overrightarrow{V}(M/R)\|}{dt}\overrightarrow{T}$
$\overrightarrow{T} = \frac{\overrightarrow{V}(M/R)}{\|\overrightarrow{V}(M/R)\|} = \frac{1}{\sqrt{1+4t^2}}\overrightarrow{i} - \frac{2t}{\sqrt{1+4t^2}}\overrightarrow{j}$

$\frac{d\|\overrightarrow{V}(M/R)\|}{dt} = \frac{d(\sqrt{1+4t^2})}{dt} = \frac{4t}{\sqrt{1+4t^2}}$
$\overrightarrow{\gamma}_T = \frac{4t}{1+4t^2}\overrightarrow{i} - \frac{8t^2}{1+4t^2}\overrightarrow{j}$.
Normal acceleration: $\overrightarrow{\gamma}_N = \overrightarrow{\gamma}(M/R) - \overrightarrow{\gamma}_T \Rightarrow \overrightarrow{\gamma}_N = -\frac{4t}{1+4t^2}\overrightarrow{i} - \frac{2}{1+4t^2}\overrightarrow{j}$
4.Radius of curvature: $R_c = \frac{\|\overrightarrow{V}(M/R)\|^3}{\|\overrightarrow{V}(M/R)\wedge\overrightarrow{\gamma}(M/R)\|} = \frac{(1+4t^2)^{\frac{3}{2}}}{2}$
$\Rightarrow R_c = \frac{(1+4t^2)^{\frac{3}{2}}}{2}$.
or else $\overrightarrow{\gamma}_N = \frac{\|\overrightarrow{V}(M/R)\|^2}{R_c}\overrightarrow{N} \Rightarrow \|\overrightarrow{\gamma}_N\| = \frac{\|\overrightarrow{V}(M/R)\|^2}{R_c}\|\overrightarrow{N}\|$, $\|\overrightarrow{N}\| = 1 \Rightarrow$
$R_c = \frac{\|\overrightarrow{V}(M/R)\|^2}{\|\overrightarrow{\gamma}_N\|} = \frac{(1+4t^2)}{\sqrt{(\frac{4t}{1+4t^2})^2+(\frac{2}{1+4t^2})^2}} = \frac{(1+4t^2)^{\frac{3}{2}}}{2}$

- **Task 04** :

Consider a moving point M whose cylindrical coordinates at each instant are $\rho(t) = a_0t^2 + \rho_0$, $\varphi(t) = \omega t - \varphi_0$ and $Z(t) = -Vt$, with $\rho_0 = 1m$, $a_0 = 1m.s^{-2}$, $\omega = 3rad.s^{-1}$, $\varphi_0 = 2\,rad$ and $V = 2m.s^{-1}$.

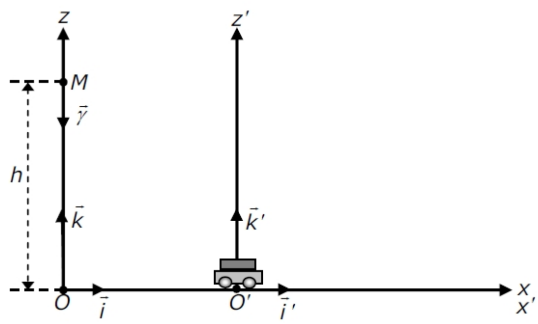
1. Give in the cylindrical base $(\overrightarrow{e}_\rho, \overrightarrow{e}_\varphi, \overrightarrow{e}_Z)$, the velocity and acceleration vectors of M.
2. Calculate the magnitude of the velocity vector of M at time $t = 1s$.
3. Calculate the magnitude of the acceleration vector of M at the initial instant $(t = 0s)$.

Solution:
1. $\overrightarrow{V}(M/R) = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\overrightarrow{e}_\rho + Z\overrightarrow{k})}{dt} = \frac{d\rho}{dt}\overrightarrow{e}_\rho + \rho\frac{d\overrightarrow{e}_\rho}{dt} + \frac{dZ}{dt}\overrightarrow{k}$
$\Rightarrow \overrightarrow{V}(M/R) = 2a_at\overrightarrow{e}_\rho + (a_0t^2 + \rho_0)\omega\overrightarrow{e}_\varphi - V\overrightarrow{k}$
$\Rightarrow \|\overrightarrow{V}(M/R)\| = \sqrt{(2a_0t)^2 + ((a_0t^2 + \rho_0)\omega)^2 + V^2}$
$\overrightarrow{\gamma}(M/R) = \frac{d\overrightarrow{V}(M/R)}{dt} = 2a_0\overrightarrow{e}_\rho + 2a_0\omega t\overrightarrow{e}_\varphi - (a_0t^2 + \rho_0)\omega^2\overrightarrow{e}_\rho$.
$\Rightarrow \gamma(M/R) = [2a_0 - (a_0t^2 + \rho_0\omega^2)]^2 + (4a_0\omega t)^2$
2. The magnitude of the velocity vector of M at time $t = 1s$.
$\|\overrightarrow{V}(M/R)\| = \sqrt{(2a_0)^2 + ((a_0 + \rho_0)\omega)^2 + V^2} = 6.63m/s$.
3.The magnitude of the acceleration vector of M at the initial instant $(t = 0s)$
$\|\overrightarrow{\gamma}(M/R)\| = \sqrt{(2a_0 - \rho\omega^2)^2} = 7ms^{-2}$

- **Task 05** :

A ball h is dropped from a building of height M with no initial velocity.In a reference frame $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ linked to this building it falls vertically according to a uniformly accelerated movement of acceleration $\overrightarrow{\gamma} = -g\overrightarrow{k}$ (See the figure below).

1. Determine the position vector $\overrightarrow{O'M}$ of the ball in a reference frame ntiel $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$ related to a car moving in a uniform rectilinear motion with speed $\overrightarrow{u} = u\overrightarrow{i}$ and passing through the vertical fall at the moment of release.Derive the equation of the trajectory of the ball in $R'(O', \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}.)$
2. Determine the position vector $\overrightarrow{O'M}$ of the ball in the same frame of reference $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$ if we assume that, at the moment of release and starting from the vertical of the fall a uniformly accelerated rectilinear movement of acceleration $\gamma = a\overrightarrow{i}$. Deduce the equation for the trajectory of the ball in $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$.



Solution:

The position vector of the ball in the reference frame $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ is :

$\overrightarrow{OM} = z\overrightarrow{k}$ $(x = y = 0)$.

The ball is in free fall with no initial velocity in $R(O, \overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k})$ with acceleration $\overrightarrow{\gamma} = -g\overrightarrow{k}$

$\Rightarrow \gamma_z = \frac{d^2z}{dt^2} = -g \Rightarrow z = -\frac{1}{2}gt^2 + V_0t + z_0$

$At = 0, V_0 = 0, z_0 = h \Rightarrow z = -\frac{1}{2}gt^2 + h \Rightarrow \overrightarrow{OM} = (-\frac{1}{2}gt^2 + h)\overrightarrow{k}$

The position vector of the ball in the reference frame $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$

is:
$\overrightarrow{O'M} = x'\overrightarrow{i'} + y'\overrightarrow{j'} + z'\overrightarrow{k'}$
$\overrightarrow{i} = \overrightarrow{i'}, \overrightarrow{k} = \overrightarrow{k'}$ et $y' = 0$ (M moves in the plane (xoz))$\Rightarrow \overrightarrow{O'M} = x'\overrightarrow{i} + z'\overrightarrow{k}$
$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} \Rightarrow \overrightarrow{O'M} = \overrightarrow{OM} - \overrightarrow{O'O}$.

1. 1^{st} **case:** $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$ is in uniform rectilinear motion of speed $\overrightarrow{u} = u\overrightarrow{i}$ and passing through the vertical fall at the moment of release:
   The point O' is fixed in $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'}) \Rightarrow \overrightarrow{V}(M/R) = \frac{d\overrightarrow{OO'}}{dt} = u\overrightarrow{i} = \frac{dx_{O'}}{dt}\overrightarrow{i} \Rightarrow x_{O'} = ut + c$
   A $t = 0 \Rightarrow O = O' \Rightarrow x_{O'} = 0 \Rightarrow c = 0 \Rightarrow x_{O'} = ut \Rightarrow \overrightarrow{OO'} = ut\overrightarrow{i} \Rightarrow \overrightarrow{O'M} = x'\overrightarrow{i'} + z'\overrightarrow{k'} = \overrightarrow{OM} - \overrightarrow{OO'} = (-\frac{1}{2}gt^2 + h)\overrightarrow{k'} - ut\overrightarrow{i'} \Rightarrow$
   $x' = -ut$ et $z' = -\frac{1}{2}gt^2 + h \Rightarrow z' = -\frac{1}{2}g\frac{x'^2}{u^2+h} \Rightarrow$ la the trajectory of the ball in $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$.
2. 2^{ed} **case:** $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$ is in uniformly rectilinear motion acceleration $\overrightarrow{\gamma} = a\overrightarrow{i}$ and passing through the vertical fall at the moment of release:
   The point O' is fixed in $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'}) \Rightarrow \overrightarrow{\gamma}(O'/R) = \frac{d^2\overrightarrow{OO'}}{dt^2} = a\overrightarrow{i} = \frac{d^2x_{O'}}{dt^2}\overrightarrow{i} \Rightarrow x_0 = \frac{1}{2}at^2 + c_1t + c_2$.
   At $t = 0 \Rightarrow O = O' \Rightarrow x_{O'} = 0, \dot{x}_{O'} = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow x_{O'} = \frac{1}{2}at^2 \Rightarrow \overrightarrow{OO'} = \frac{1}{2}at^2\overrightarrow{i}$
   $\Rightarrow x' = -\frac{1}{2}at^2$ et $z' = -\frac{1}{2}gt^2 + h \Rightarrow z' = \frac{g}{a}x' + h$.
   $\Rightarrow$ the trajectory of the ball in $R'(O', \overrightarrow{i'}, \overrightarrow{j'}, \overrightarrow{k'})$ is a straight line with equation $z' = \frac{g}{a}x' + h$

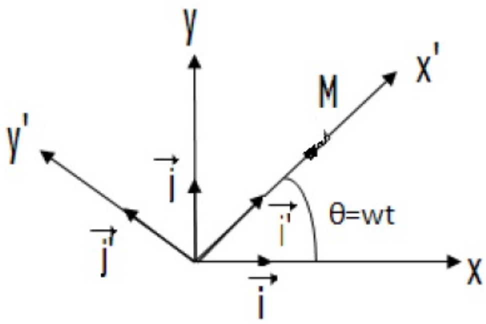
- **Task 06**:
  In the xOy a straight line Ox' rotates about Oz with a constant angular velocity A moving body M moves along the straight line Ox' according to the law : $r = a\sin\theta$ avec $\theta = \omega t$ et $a = cte$.

1. Determine at time t as a function of a and ω the relative speed and the driving speed of M by their projections in the moving reference

frame $x'Oy'$ .Deduce the absolute velocity expressed in this same projection basis, and show that the modulus of is constant.

2. Determine at time t as a function of a and ω the relative acceleration, training acceleration of M by their projections in the moving reference frame $x'Oy'$ and complementary acceleration. Deduce the absolute acceleration expressed in this same projection basis, and show that the modulus of is constant.

Solution:



1. $\overrightarrow{V}_r = \frac{d\overrightarrow{OM}}{dt}|_{R'} = a\omega\cos\omega t\,\overrightarrow{i'}$
   $\overrightarrow{V}_e = \omega\overrightarrow{k'}\wedge\overrightarrow{OM} = a\omega\sin\omega t\,\overrightarrow{j'}.$
   $\overrightarrow{V}_a = \overrightarrow{V}_r + \overrightarrow{V}_e = a\omega\cos\omega t\,\overrightarrow{i'} + a\omega\sin\omega t\,\overrightarrow{j'}$
   The unit vectors are written as follows
   $\overrightarrow{i'} = \cos\omega t\,\overrightarrow{i} + \sin\omega t\,\overrightarrow{j}$ et
   $\overrightarrow{j'} = -\sin\omega t\,\overrightarrow{i} + \cos\omega t\,\overrightarrow{j}$
   If we replace these expressions in the absolute speed:
   $\overrightarrow{V}_a = a\omega\cos\omega t(\cos\omega t\,\overrightarrow{i} + \sin\omega t\,\overrightarrow{j}) + \omega\sin\omega t(-\sin\omega t\,\overrightarrow{i} + \cos\omega t\,\overrightarrow{j})$
   $\overrightarrow{V}_a = a\omega\omega(\cos\omega t\cos\omega t - \sin\omega t\sin\omega t)\,\overrightarrow{i} + (\cos\omega t\sin\omega t + \sin\omega t\cos\omega t)\,\overrightarrow{j}$
2. $\overrightarrow{\gamma}_r = \frac{d^2\overrightarrow{OM}}{dt^2}|_{R'} = -a\omega^2\sin\omega t\,\overrightarrow{i'}.$
   $\overrightarrow{\gamma}_e = -a\omega^2\sin\omega t\,\overrightarrow{i'}$
   $\overrightarrow{\gamma}_c = 2a\omega^2\cos\omega t\,\overrightarrow{j'}$
   $\overrightarrow{\gamma}_r + \overrightarrow{\gamma}_e + \overrightarrow{\gamma}_c = -2a\omega^2\sin\omega t\,\overrightarrow{i'} + 2a\omega^2\cos\omega t\,\overrightarrow{j'}$

3

Point material dynamics

3.1 Point material dynamics

3.1.1 Definition

- Dynamics is the study of the movement of bodies as a function of the forces exerted on them.
- All the knowledge of so-called classical mechanics: dynamics, statics, hydrodynamics, resistance of materials, etc... are based on three principles (or laws) set out and published by Isaac Newton in 1687.
- A principle is a statement that is accepted without demonstration.

3.1.2 Newton's laws

3.1.2.1 First Principle (Principle of inertia)

- **Definition of inertia** : is the tendency of an object to resist changes in its state of motion, i.e. to resist a change in speed or orientation.
- **Galilean reference frame** : A galilean or inertial frame of reference is a frame of reference in which the trajectory of a particle undergoing no external action is straight.
- **Principle statement** : In a Galilean reference frame, an isolated material point (which is not subject to any force) is either at rest or in uniform rectilinear motion.

3.1.2.2 The second principle (Fundamental Principle of Dynamics)

- **Principle statement**: In a Galilean reference frame (R), the resultant of the external forces applied to a material point is proportional to its acceleration vector, the proportionality coefficient being its mass **m**. $\sum \overrightarrow{F}_{ext} = m\overrightarrow{\gamma}(M)$
- The acceleration of a particle is directly proportional to the force applied to it and inversely proportional to its mass.
- Force units in the system IS: $N(Newton) = \frac{K_g m}{s^2}$

3.1.2.3 Third principle (action and reaction)

- **Third principle (action and reaction)** When a body A acts on a body B with a force $\overrightarrow{F}_{A\to B}$ then body B acts on body A in a reciprocal way. ,with a force $\overrightarrow{F}_{B\to A}$, which has the same direction and the same magnitude,but in the opposite sense:$\overrightarrow{F}_{A\to B} = -\overrightarrow{F}_{B\to A}$

- For every action there is an equal and opposite reaction.

3.1.3 Base Units and Physical Standards

The concepts of length, time, and mass are fundamental to every branch of physics. The units of length, time, and mass are known as the base units of physics, and are defined by a set of physical standards that are augmented by descriptions of the procedures for employing them. Base units are not mere matters of practical convenience: because they embody the underlying concepts, they are foundations of physical science. This section presents a brief description of the base units and the systems of units derived from them that are universally used in physics. The SI base units of length, mass, and time are the meter, kilogram, and second. A related system, the CGS system (for centimeter, gram, second), differs from SI only in scaling factors.CGS units appear in older databases and are sometimes used in chemical and biological research.Yet another system of units, the English system, is used for non-scientific measurements in Britain and North America, although Britain also uses the SI system. English units are related to SI units by legally agreed scaling factors; for example, the inch is legally defined as $2.54cm$. The table lists some principal units in the SI, CGS, and English systems.

3.1.4 The Algebra of Dimensions

Equations in physics are not meaningful unless they are dimensionally consistent. In this context, the term dimension refers to the type of physical quantity (ultimately expressed in units of mass, length, and time), in contrast to usage in mathematics, where dimension refers to the number of coordi-

	SI	CGS	English
Length	1 meter (m)	1 centimeter (cm)	1 inch (in)
Mass	1 kilogram (kg)	1 gram (g)	1 slug
Time	1 second (s)	1 second (s)	1 second (s)
Acceleration	1 m/s^2	1 cm/s^2	1 ft/s^2
Force	1 newton (N) = 1 kg·m/s^2	1 dyne = 1 g·cm/s^2	1 pound (lb) = 1 slug·ft/s^2

Here are some useful relations between these units systems.

1 m = 100 cm	1 m ≈ 39.4 in
1 ft = 12 in	1 mile = 5280 ft
1 kg = 1000 g	1 slug ≈ 14.6 kg
1 N = 10^5 dyne	1 N ≈ 0.224 lb

nates needed to specify a point. A useful check on a calculation is to see whether the units agree on both sides of the final result. If they don't, there is evidently an error somewhere.in mechanics the physical quantities such as velocity and force are measured in units constructed from the base units of mass, length, and time.Regardless of the system of units we use, in Newtonian mechanics every quantity depends on mass, length, and time in a unique way. For example, the units of velocity are m/s in the SI system and cm/s in the CGS system, but both have dimensions length/time. In analyzing the consistency of units in an equation, the dimension of mass is abbreviated M, the dimension of length is L, and the dimension of time is T. James Clerk Maxwell, who developed the theory of electromagnetism, was the first to use the convenient notation of square brackets to stand for the dimensions of a quantity.

[mass] = M , [length] = L [time] = T.

The dimensions of quantities in mechanics can always be expressed in terms of powers of M, L, and T. For instance, [velocity] = LT^{-1}, [force] = MLT^{-2}. Units must agree on both sides of an equation and this is only possible if the underlying dimensions also agree.Note that M, L, and T are independent quantities; we cannot express mass in terms of time, or length in terms of mass. Consequently, for an equation to be valid, the powers of M, L, and T must separately agree no matter what system of units we choose to employ.

S.No.	Physical Quantity	Relation with other physical quantities	Dimensional formula	SI-Unit
1	Area	Length x breadth	$[L] \times [L] = [M^0 L^2 T^0]$	m^2
2	Volume	Length x breadth x height	$[L] \times [L] \times [L] = [M^0 L^3 T^0]$	m^3
3	Density	$\frac{\text{mass}}{\text{volume}}$	$\frac{[M]}{[L^3]} = [M L^{-3} T^0]$	$Kg\ m^{-3}$
4	Speed or velocity	$\frac{\text{distance}}{\text{time}}$	$\frac{[L]}{[T]} = [M^0 L T^{-1}]$	$m\ s^{-1}$
5	Acceleration	$\frac{\text{velocity}}{\text{time}}$	$\frac{[L T^{-1}]}{[T]} = [M^0 L T^{-2}]$	$m\ s^{-2}$
6	Momentum	Mass x velocity	$[M] \times [L T^{-1}] = [M L T^{-1}]$	$Kg\ m\ s^{-1}$
7	Force	Mass x acceleration	$[M] \times [L T^{-2}] = [M L T^{-2}]$	N (Newton)
8	Pressure	$\frac{\text{force}}{\text{area}}$	$\frac{[M L T^{-2}]}{[L^2]} = [M L^{-1} T^{-2}]$	$N\ m^{-2}$ or Pa (Pascal)
9	Work	Force x distance	$[M L T^{-2}] X [L] = [M L^2 T^{-2}]$	J (Joule)
10	Energy	Work	$[M L^2 T^{-2}]$	J
11	Power	$\frac{\text{work}}{\text{time}}$	$\frac{[M L^2 T^{-2}]}{[T]} = [M L^{-3} T^{-3}]$	W (Watt)
12	Gravitational constant (G)	$\frac{\text{force x distance}^2}{\text{mass}^2}$	$[M^{-1} L^3 T^{-2}]$	$N\ m^2\ kg^{-2}$
13	Impulse	Force x time	$[M L T^{-2}] \times [T] = [M L T^{-1}]$	N s
14	Surface tension	$\frac{\text{force}}{\text{length}}$	$\frac{[M L T^{-2}]}{[L]} = [M L^0 T^{-2}]$	$N\ m^{-1}$
15	Coefficient of viscosity	$\frac{\text{force}}{\text{area x velocity gradient}}$	$[M L^{-1} T^{-1}]$	daP (decapoise)
16	Angle	$\frac{\text{arc}}{\text{radius}}$	Dimensionless	rad
17	Moment of inertia	Mass x distance^2	$[M L^2 T^0]$	$Kg\ m^2$
18	Angular momentum	Moment of inertia x angular velocity	$[M L^2] \times [T^{-1}] = [M L^2 T^{-1}]$	$Kg\ m^2\ s^{-1}$
19	Torque or couple	Force x perpendicular distance	$[M L T^{-2}] \times [L] = [M L^2 T^{-2}]$	N m

3.1.5 Force classification

3.1.5.1 Real (or external) forces

There are two types of real force:

3.1.5.2 Action-at-a-Distance Forces

It is a force exerted without the help of a material support.

Exemples

- **Electrostatic force :**

  force exerted by one charge on another

  $\overrightarrow{F} = \frac{k\,q\,\acute{q}}{r^2}\overrightarrow{u_r}$

  $\overrightarrow{F}$:electrostatic force exerted on q' by q, with $K = 9.10^9(SI)$ Coulomb's constant

r: Distance between q' and q.
$\vec{u_r}$:Unit vector.

- **Electromagnetic force :**

- **Electromagnetic force :**

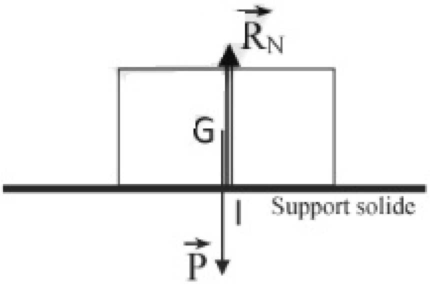
  is a force exerted on a point of charge q, of velocity $\vec{V}$ placed in an electrostatic field $\vec{E}$ and a magnetic field $\vec{B}$: $\vec{F} = q(\vec{E} + \vec{V} \wedge \vec{B})$

- **Gravitational force**

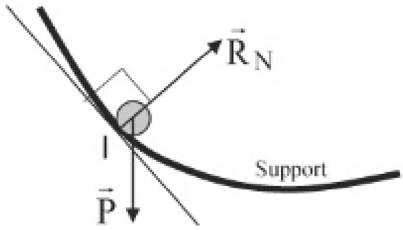
  It is an interaction force at a distance. It is exerted between two masses. It was formulated by Newton in 1650.
  A mass M at point O is interacting with another mass m at point P , such that the distance between O and P is equal to r, is written as :
  $\vec{F} = G\frac{mM}{r^2}\vec{u}_{op}$
  $G = 6.6710^{-11}(SI)$ gravitational constant.

  This force can be written as
  $\vec{F} = -m\vec{g}(P)$
  $\vec{g} = G\frac{M}{r^2}\vec{u}_{op}$
  This last expression is called the gravitational field created by m at any point P in space. It has the same size as a acceleration.
  The Earth's gravitational field at a point P in space outside the Earth. this field has the expression:
  $\vec{g} = G\frac{M}{(R_T+r)^2}\vec{u}_{op}$

  - M M Earth's mass: $M = 5.9810^{24}kg$
  - R_T radius of the Earth $= 6.3710^6 m$
  - r Represents the altitude of point P relative to the earth's surface. This field at the earth's surface will have a value of $g_0 = 9.83$



(Figure 1)



(Figure 2)

3.1.5.3 Contact force

Are types of forces in which the two interacting objects are physically contacting each other.

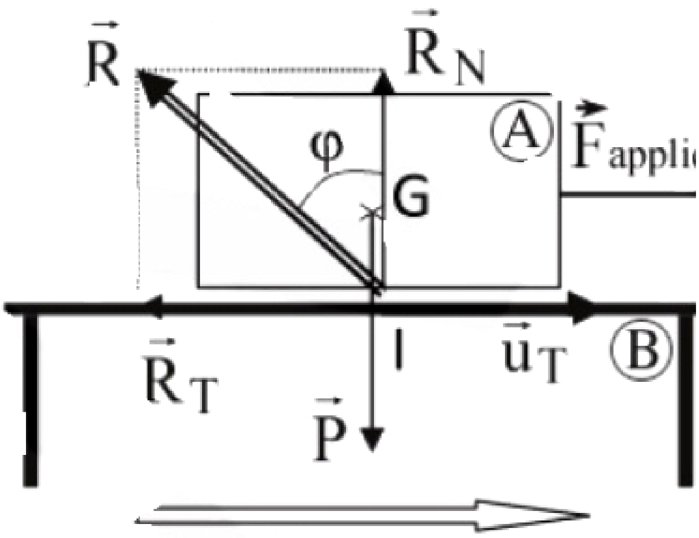
Exemple

1. **Normal Force:**

   A body of mass m is placed on a solid horizontal support, for example, a table (Figure 1).Its weight P = m g is applied to its centre of gravity G and normal to the surface of the table . By virtue of the principle of action and reaction, the table exerts an equal and opposite force to $\overrightarrow{p}$:
   $\overrightarrow{R}_N = -\overrightarrow{p}$
   its magnitude $\|\overrightarrow{R}_N\| = \|\overrightarrow{p}\| = mg$
   For a perfectly smooth curvilinear support, its reaction on the object is normal to the tangent to the support at the point of contact I (figure 2).

2. **Friction force**

   These forces occur between material bodies in motion relative to each other. There are two types of friction : solid friction and viscous friction.
   a-Solid friction:
   Two solid bodies A and B with a certain roughness are in contact. Body B (the support) exerts a force on body A $\overrightarrow{R}$ (called reaction)composed of a normal reaction $\overrightarrow{R}_N$(at the contact surface) and a tangential reaction called $\overrightarrow{R}_T$ friction force that opposes movement.



Experience shows that the magnitude of the normal reaction force is proportional to the magnitude of the normal reaction force $\overrightarrow{R}_N$ (Coulomb's Law) : $\|\overrightarrow{R}_N\| \propto \|\overrightarrow{R}_T\|$

The coefficient of friction is defined as follows :

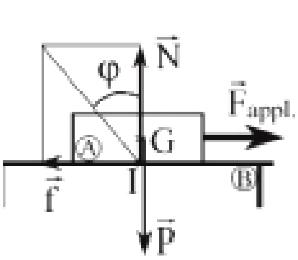
$\mu = tg\varphi = \frac{\|\overrightarrow{R}_T\|}{\|\overrightarrow{R}_N\|}$ with : $\varphi = (\overrightarrow{R}_N, \overrightarrow{R})$.

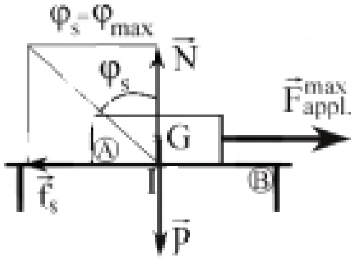
Remarks: μ has no dimension (without unit).

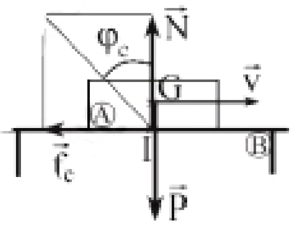
The total reaction of the solid support on the object is :$\overrightarrow{R} = \overrightarrow{R}_T + \overrightarrow{R}_N$

N.B. : In the following, we will refer to the friction force as $\overrightarrow{f}$ and the normal reaction as $\overrightarrow{N}$. There are two types of solid friction :

- **Static solid friction:** To make A slide over B, the force applied to A must have a certain value below which it remains stationary. The limiting friction force which balances the driving force and prevents A from moving is then at its maximum. **This is called the static friction force to which corresponds the coefficient of static friction noted** μ_s and we have :
$\|\overrightarrow{f}\|_{max} = \|\overrightarrow{f}_s\| = \mu_s\|\overrightarrow{N}\|$ with $\mu_s = tg\varphi_{max} = \frac{\|\overrightarrow{f}\|_{max}}{\|\overrightarrow{N}\|}$.
- Solid kinetic friction :
As soon as body A begins to slide on support B, the frictional force decreases until it reaches a value of $\|\overrightarrow{f}_c\| = \mu_c\|\overrightarrow{N}\|$ called the ki-







netic friction force.

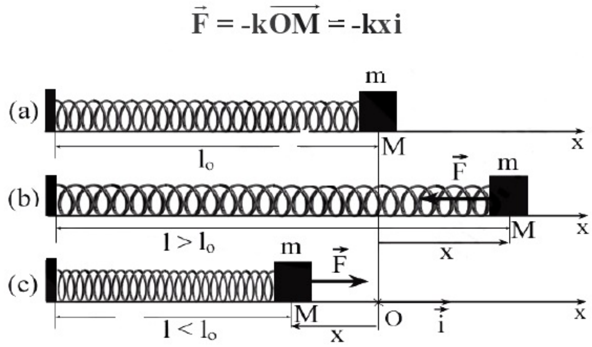
μ_c **being the coefficient of kinetic friction, which is therefore less than** μ_s :$\mu_c < \mu_s$

The kinetic friction force is opposite to the direction of motion and is tangent to the trajectory:
$\overrightarrow{f}_c = -\mu_c \|\overrightarrow{N}\| \frac{\overrightarrow{V}}{\|\overrightarrow{V}\|}$.

b-Viscous friction : When the movement of a solid body occurs in a liquid (e.g. water) or a gas (e.g. air), the friction is called viscous friction.In the case where the velocity $\overrightarrow{V}$ of the object is relatively low, the expression for the viscous friction force opposing the motion of the solid is :
$\overrightarrow{f} = -k\eta\overrightarrow{V}$.Where
k: is a constant that depends on the shape of the object (Unit : m)
η :is the coefficient of viscosity of the medium (unit : $N.m-2.s$ or $Pa.s$)
Example: for a spherical object in a medium of viscosity, the friction force experienced by the object in contact with the medium is given by is given by Stockes' law :
$\overrightarrow{f} = -6\pi R\eta\overrightarrow{V}$.
Note: The viscous friction force is often denoted $\overrightarrow{f} = -\mu\overrightarrow{V}$, avec μ coefficient of viscous friction (Unit S.I. : $kg.s^{-1}$).

3. **La force de rappel (ou élastique) d'un ressort (loi de Hooke)**
A spring is characterised by its initial length l_0 and its stiffness constant k k (unit :N/m) which depends on the nature of the material. When sub-

jected to elongation, it tends to return to its initial length provided that this elongation is not too great (to avoid irreversible deformation).The spring is placed on a horizontal plane without friction, with one end fixed,,and at the other is attached an object M of mass m .It is stretched by an elongation a, its length is $l = l_0 + a$, then release the M. It begins to oscillate about point O, the position of the mass at the initial instant (immobility).The force exerted by the spring on the mass is proportional to the instantaneous elongation
$X = l - l_0$, with : $\|\overrightarrow{F}\| = k|l - l_0|$
Figure (a) :Initial state. The spring has a length l_o.
Figure (b) : The spring has a length $l > l_o$.It is stretched and acts on the mass so as to regain its equilibrium length by exerting a restoring force $\overrightarrow{F}$ from right to left, opposite to the vector $\overrightarrow{i}$
$\overrightarrow{F} = -k\overrightarrow{OM} = -kX\overrightarrow{i}$ with $X > 0$
Figure (c) :The spring has a length $l < lo$.It is compressed and acts on the mass so as to regain its length at equilibrium by exerting a restoring force $\overrightarrow{F}$ from left to right, in the same direction as the vector $\overrightarrow{i}$: we also have $\overrightarrow{F} = -k\overrightarrow{OM} = -kX\overrightarrow{i}$ but with $X < 0$ (x, the abscissa of M, is an algebraic quantity).
Therefore, in all configurations, the spring tension, or return force is written as : $\overrightarrow{F} = -k\overrightarrow{OM} = -kX\overrightarrow{i}$



3.1.6 Quantity of movement and kinetic momentum

3.1.6.1 Definitions

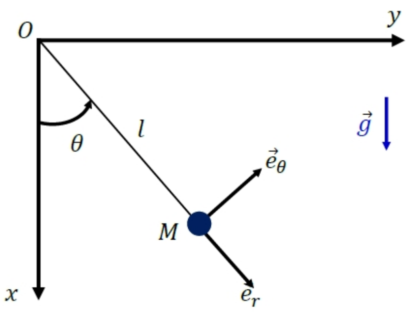
- Assume a material point M with mass m and velocity vector $\overrightarrow{V}(M)$ in a reference frame $R(O, xyz)$.
- The momentum of M in frame R is : $\overrightarrow{P}(M) = m\overrightarrow{V}(M)$
- The angular momentum of M relative to the fixed point O is:
  $\overrightarrow{\sigma}_0(M) = \overrightarrow{OM} \wedge \overrightarrow{P}(M) = \overrightarrow{OM} \wedge m\overrightarrow{V}(M)$
- The angular momentum of point M with respect to a straight line (D), passing through O and having unit vector $\overrightarrow{u}$, is given by the scalar,
  $M_D(\overrightarrow{P}) = \overrightarrow{\sigma}_0(M).\overrightarrow{u}$
- The dynamic moment of a particle M at a fixed point O , in a reference frame R, is by definition:
  $\overrightarrow{\delta}_0(M) = \overrightarrow{OM} \wedge m\overrightarrow{\gamma}$

3.1.6.2 The angular momentum theorem

$\frac{d\overrightarrow{\sigma}_0(M)}{dt}|_R = \frac{d(\overrightarrow{OM} \wedge m\overrightarrow{V}(M))}{dt}|_R = \overrightarrow{V}(M) \wedge m\overrightarrow{V}(M) + \overrightarrow{OM} \wedge m\overrightarrow{\overrightarrow{\gamma}}$
So: $\overrightarrow{\delta}_0(M)|_R = \frac{d\overrightarrow{\sigma}_0(M/R)}{dt}|_R = \overrightarrow{OM} \wedge \sum F_{ext}$

3.2 Corrected exercises

- **Exercice 01**
  Consider a simple pendulum consisting of a point object M M of mass m attached to an inextensible wire of length l l and negligible mass.It moves in the vertical plane (xOy) of the fixed reference frame $R(o, xyz)$. The pendulum is moved away from its equilibrium position by an angle θ $(\theta = 0)$ and release without initial speed.Frictional forces are assumed to be non-existent.
  The assembly is located in the gravity field g considered uniform.



1. Express the forces applied to point M in the base $(\overrightarrow{e}_r, \overrightarrow{e}_\theta, \overrightarrow{k})$.
2. Calculate $\overrightarrow{V}(M/R)$ and $\overrightarrow{\gamma}(M/R)$ respectively the vectors velocity and acceleration of M on R.
3. Applying the PFD in the Galilean reference frame R: a-Establish the differential equation of motion in the case of small oscillations. b- Solve this differential equation.
4. Establish the expression for the wire voltage T .
5. Find the differential equation of motion by applying the kinetic energy theorem..

Solution:

1- The forces applied at point M are :

- Its weight $\overrightarrow{P}$ with $\overrightarrow{P} = m\overrightarrow{g} = mg\cos\theta\overrightarrow{e}_r - mg\sin\theta\overrightarrow{e}_\theta$
- Wire tension $\overrightarrow{T}$ with :$\overrightarrow{T} = -T\overrightarrow{e}_r$

2- The PDF in this Galilean reference frame is as follows:
$\sum\overrightarrow{F}_{ext} = m\overrightarrow{\gamma}(M/R) \Rightarrow m\overrightarrow{\gamma}(M/R) = \overrightarrow{P} + \overrightarrow{T}$
$\Rightarrow -ml\dot{\theta}^2\overrightarrow{e}_r + ml\ddot{\theta}\overrightarrow{e}_\theta = (mg\cos\theta - T)\overrightarrow{e}_r - mg\sin\theta\overrightarrow{e}_\theta.$
3- a-The projection of PFD onto $\overrightarrow{e}_\theta$ gives:
$ml\ddot{\theta} = -mg\sin\theta$
$\Rightarrow ml\ddot{\theta} + mg\sin\theta = 0$

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

For small oscillations, $\sin\theta \cong \theta$. θ very small

Finally, we find :

$\ddot{\theta} + \frac{g}{l}\theta = 0$

b- $\omega_0 = \sqrt{\frac{g}{l}}$.

4-The PFD projection on $\overrightarrow{e}_r$ gives:

$-ml\dot{\theta}^2 = mgcos\theta - T$

$\Rightarrow mg\cos\theta + ml\dot{\theta}^2$.

5-The angular momentum theorem can be written as follows:

$\frac{d\overrightarrow{\sigma}_0(M/R)}{dt}|_R = \overrightarrow{OM} \wedge \overrightarrow{P} + \overrightarrow{OM} \wedge \overrightarrow{T}$

with

- $\overrightarrow{\sigma}_0(M/R) = \overrightarrow{OM} \wedge m\overrightarrow{V}(M/R) = l\overrightarrow{e}_r \wedge ml\dot{\theta}\overrightarrow{e}_\theta = ml^2\dot{\theta}\overrightarrow{k} \Rightarrow \frac{d\overrightarrow{\sigma}_0(M/R)}{dt}|_R = ml\ddot{\theta}\overrightarrow{k}$
- $\overrightarrow{OM} \wedge \overrightarrow{P} = l\overrightarrow{e}_r \wedge (mg\cos\theta\overrightarrow{e}_r - mg\sin\theta\overrightarrow{e}_\theta) = -mgl\sin\theta\overrightarrow{k}$
- $\overrightarrow{OM} \wedge \overrightarrow{T} = \overrightarrow{0}$.

This means that: $ml^2\ddot{\theta}\overrightarrow{k} = -mgl\sin\theta\overrightarrow{k}$

$\Rightarrow ml^2\ddot{\theta} = -mgl\sin\theta$.

We finally obtain:

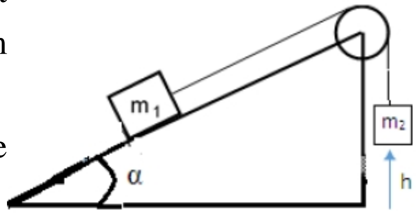
$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$.

- **Task 02**

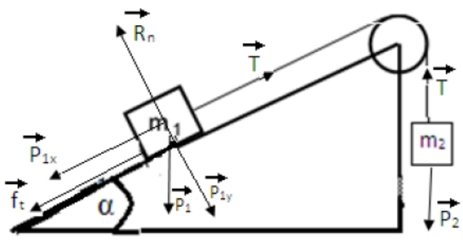
  Two masses $m1$ and $m2$ are linked by an inextensible wire which passes through a pulley of negligible mass and fixed axis.The mass $m1$ slides on a non-smooth inclined plane which makes an angle $\alpha = 30°$ to the horizontal (see figure).knowing that the coefficients of friction are respectively $\mu_s = 0.7$ μ_d On prendra $g = 9.8m/s^2$,$m_1 = 1kg$.

1. calculate the minimum mass of m_{2min} that maintains the system in equilibrium..

2. OWe now take the mass $m_2 = 1.5kg$. It is released, without initial velocity, from a height h for a time of 2 seconds..
   a- a- Calculate the accelerations of the two masses.
   b- Calculate the height h. Deduce the speeds of the two masses when the m2 mass hits the ground.



Solution:



We apply the fundamental principle of dynamics to the mass m_1
$\overrightarrow{P} + \overrightarrow{T} + \overrightarrow{R}_n + \overrightarrow{f}_t = \overrightarrow{0} \Rightarrow$
$T - m_1 g \sin\alpha - f_t = 0$
$R_n - m_1 g \cos\alpha = 0$
On the mass M_2 : $\overrightarrow{P}_2 + \overrightarrow{T} = \overrightarrow{0}$
$m_2 g - T \Rightarrow m_2 g = T$
With $P_1 \cos\alpha = R_n$ et $m_2 g = T$
and $f_1 = \mu_s R_n = \mu_s m_1 \cos\alpha g$
$\Rightarrow m_{2min} = m_1(\mu_s \cos\alpha + \sin\alpha) = 1.1kg$
2-
a- Since $m_2 > m_{2min}$ the movement is in the direction of m_2 we apply the fundamental principle of dynamics
Forces acting on the mass m_2 : $\overrightarrow{P}_2 + \overrightarrow{T} = m_2\overrightarrow{\gamma}$
$m_2 g - T = m_2\gamma$
On the mass m_1
$\overrightarrow{P} + \overrightarrow{T} + \overrightarrow{R}_n + \overrightarrow{f}_t = m_1\overrightarrow{\gamma} \Rightarrow$

$T - m_1 g \sin\alpha - f_t = m_1\gamma$

$R_n - m_1 g \cos\alpha = 0$

$P_1 \cos\alpha = R_n$

$f_t = \mu_s R_n = \mu_s m_1 \cos\alpha g$

So $\gamma = \frac{m_2 - m_1 \sin\alpha - \mu_d m_1 \cos\alpha}{m_2 + m_1} g = 2.91 m/s^2$

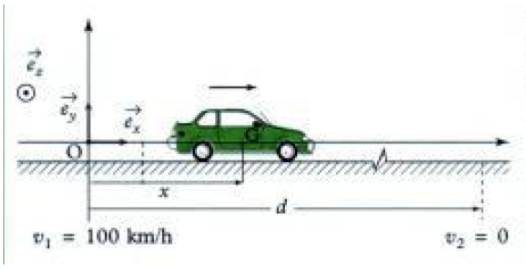
b-$h = \frac{1}{2}\gamma^2 t = 5.83m$

$V = \gamma t = 2.91 m/s$

- **Exercice 03**

  During a braking test, a car, assimilated to a material point G of mass $m = 1300kg$, is runs on a horizontal road and brakes while its speed $v_1 = 100km.h^{-1}$.

  The time required for the vehicle to come to a complete stop is $T = 7s$.It is assumed that the deceleration is constant and that the reference frame $R(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$ is Galilean



1. 1.Determine the braking force F_0 and the stopping distance d
2. 2. It is assumed that the braking force remains the same F_0 but several starting speeds are tested:

   $0 < v_1 < 130km.h^{-1}$.Draw the curve $d = f(v_1)$. What can we conclude about compliance with speed limits on the road?

Solution

1) To find the braking force, we will apply the PFD System studied: the car of mass m assimilated to point G

Galilean study reference system: $R(O, \vec{e}_x, \vec{e}_y, \vec{e}_z)$.

Forces applied to the system:

- weight $\overrightarrow{P} = -mg\overrightarrow{e}_y$
- Soil reaction $\overrightarrow{N} = N\overrightarrow{e}_y$
- braking force $\overrightarrow{F} = -F\overrightarrow{e}_x$

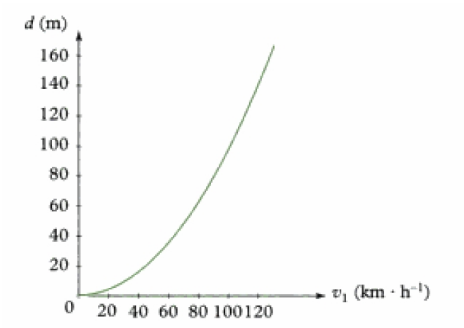
The PFD gives :$m\overrightarrow{a}(G) = \overrightarrow{P} + \overrightarrow{N} + \overrightarrow{F}$
Position vector:$\overrightarrow{OM} = x\overrightarrow{e}_x$;velocity vector:$\overrightarrow{v}(M/R) = \dot{x}\overrightarrow{e}_x$; acceleration vector:$\overrightarrow{a}(M/R) = \ddot{x}\overrightarrow{e}_x$.
The vector equation from the PFD is therefore written:
$m\ddot{x}\overrightarrow{e}_x = -mg\overrightarrow{e}_y + N\overrightarrow{e}_y - F\overrightarrow{e}_x$.

- Since we want to calculate F and we do not know N, we will project this relationship on $\overrightarrow{e}_x$:

  $m\ddot{x} = -F$

- As $\ddot{x}$ is a constant, so is F ; we can therefore integrate the previous differential equation once: $\ddot{x} = -\frac{F}{m}$,then $\dot{x} = -\frac{F}{m}t + cte$.
- $t = 0, \dot{x}(t = 0) = v_1$, so $cte = v_1$, therefore $\dot{x} = -\frac{F}{m}t + v_1$.
- $t = T$, $\dot{x}(t = T) = 0$, so $0 = -\frac{F}{m}T + v_1$, hence: $F_0 = \frac{mv_1}{T}$.
  $N.A.F = \frac{1300.\frac{100}{3,6}}{7} = 5159N$.
  To determine the stopping distance, the differential equation of motion must be integrated a second time with respect to time:
  $x = -\frac{F_0}{m}\frac{t^2}{2} + v_1 t + cte$.

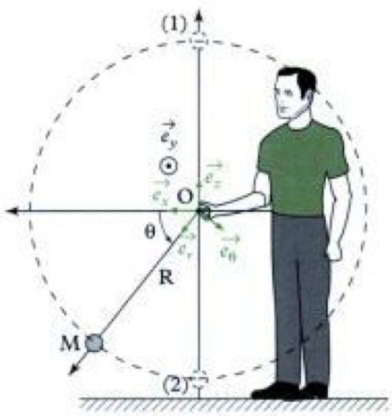
  - $t = 0, x(t = 0) = 0$ so $cte = 0$, hence $x = -\frac{F_0}{m}\frac{t^2}{2} + v_1 t$.
  - $t = T, x = d\, so\, d = -\frac{F_0}{m}\frac{T^2}{2} + v_1 T$
    $d = v_1\frac{mv_1}{F_0} - \frac{F_0}{m}\frac{1}{2}(\frac{mv_1}{F_0})^2 = \frac{mv_1^2}{F_0} - \frac{1}{2}\frac{mv_1^2}{F_0}$.
    $d = \frac{1}{2}\frac{mv_1^2}{F_0}$
    2) **Curve** $d = f(v_1)$



The distance varies with the square of the speed, hence the importance of speed limits (both in town and on the motorway).

- **Task 04**

A man rotates a ball (assimilated to a material point M of mass m=100g), attached to a wire of length $R = oM = 1m$ and of negligible mass. The trajectory of the ball is a circle with center O and radius R which is in the vertical plane $(\overrightarrow{e}_x.\overrightarrow{e}_y).\overrightarrow{e}_z$ is vertical upwards.We neglect the possible small movements of the man's hand, thus:$R(\overrightarrow{e}_x, \overrightarrow{e}_y, \overrightarrow{e}_z)$) is assumed to be Galilean. We neglect all friction



1. Determine the differential equation of the movement of the ball.
2. Determine the expression of the wire tension.
3. Determine the minimum velocity vm that the ball must have in position (1).
4. It is assumed that the ball passes into the position (1) with speed v_{min}. Indicate in which position the wire tension is maximum.

Solution

1- We try to write the differential equation of motion; we will therefore apply PFD. System studied: the ball M of mass m.. **System studied:** the ball M of mass m:

Force applied to the system:

$\overrightarrow{P} = m\overrightarrow{g} = -m\overrightarrow{g}\overrightarrow{e}_z$

wire tension $\overrightarrow{T} = -T\overrightarrow{e}_r$. The PFD gives:$m\overrightarrow{a}(M) = \overrightarrow{P} + \overrightarrow{T}$

$\overrightarrow{OM} = R\overrightarrow{e}_r$; ; velocity vector : $\overrightarrow{v}(M/R) = R\dot{\theta}\overrightarrow{e}_\theta$

2- acceleration vector: $\overrightarrow{a}(M/R) = -R\dot{\theta}^2\overrightarrow{e}_r + R\ddot{\theta}\overrightarrow{e}_\theta$. The vector equation from the PFD is :$m(-R\dot{\theta}^2\overrightarrow{e}_r + R\ddot{\theta}\overrightarrow{e}_\theta) = -mg\overrightarrow{e}_z - T\overrightarrow{e}_r$. As $\overrightarrow{T}$ is unknown and we looks for the diffirential equation of motion, we will project the previous vector relationship on $\overrightarrow{e}_\theta$:

$\overrightarrow{e}_\theta[-mR\dot{\theta}^2\overrightarrow{e}_r + mR\ddot{\theta}\overrightarrow{e}_\theta] = \overrightarrow{e}_\theta[-mg\overrightarrow{e}_z - T\overrightarrow{e}_r]$, d'où $mR\ddot{\theta} = -mg() - \cos\theta$+0

$\ddot{\theta} = \frac{g}{R}\cos\theta$.

3- To determine the expression of the tension of the wire, we use the vector equation of the PFD that we project on $\overrightarrow{e}_r$: $\overrightarrow{e}_r[-mR\dot{\theta}^2\overrightarrow{e}_r + mR\ddot{\theta}\overrightarrow{e}_\theta] = \overrightarrow{e}_\theta[-mg\overrightarrow{e}_z - T\overrightarrow{e}_r]$,soit $-mR\dot{\theta}^2 = mg\sin\theta - T$

$T = mg\sin\theta + mR\dot{\theta}^2$

- In order for the ball to pass into position (1) while remaining on the circle of radius R, the wire must not relax, that is to say that $T \geq 0$ soit$mg\sin\theta + mR\dot{\theta}^2 \geq 0$, with $\theta = -\frac{\pi}{2}$rad. Or $v(m/R) = R\dot{\theta}\overrightarrow{e}_\theta = v\overrightarrow{e}_\theta$

  Aussi:

  $mg(-1) + m\frac{v^2}{R} \geq 0$, so $v^2 \geq Rg$ d'où $v \geq \sqrt{Rg}$

  $v_{min} = \sqrt{Rg}$

4- Using the differential equation of motion: $\ddot{\theta} = \frac{g}{R}\cos\theta$,

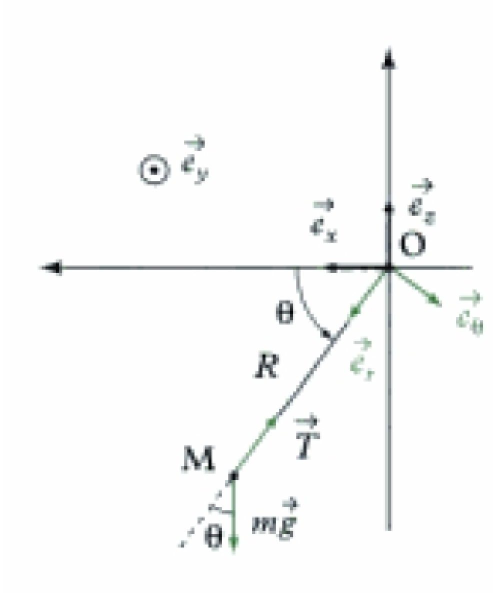
we see that as $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$rad, $0 \leq \cos\theta \leq 1$,so $\ddot{\theta} \geq 0$.

In addition, the ball has the speed v_{min} in the position (1), so the ball accelerates when θ changes from $-\frac{\pi}{2}$ à $\frac{\pi}{2}$ $((1) \to (2))$.

Conversely, it decelerates when θ goes from $\frac{\pi}{2}$ à $\frac{3\pi}{2}$ $((2) \to (1))$.

Also $\dot{\theta}$ is maximum when $\theta = \frac{\pi}{2}$rad (position (2)).

In addition $\sin\theta = 1$ when $\theta = \frac{\theta}{2}$ rad.
And since $T = mg\sin\theta + mR\dot{\theta}^2$, And since T T is maximum in position (2).



4

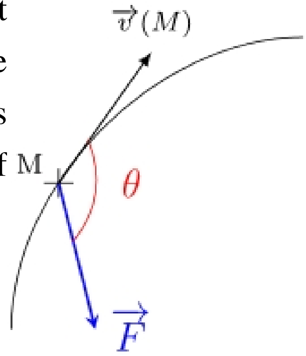
Work and Energy

4.1 Work and Energy

4.1.1 Work and power of a force

4.1.1.1 Power of a force

Let a point M move along its trajectory at a velocity $\overrightarrow{V}(M)$ relative to the study frame of reference, It is subjected to a force $\overrightarrow{F}$ as shown in the diagram opposite. The power of the force.



$\overrightarrow{F}$ is written as:
$P(\overrightarrow{F}) = \overrightarrow{F}\overrightarrow{v} = \|\overrightarrow{F}\|\|\overrightarrow{v}\| \cos\theta$

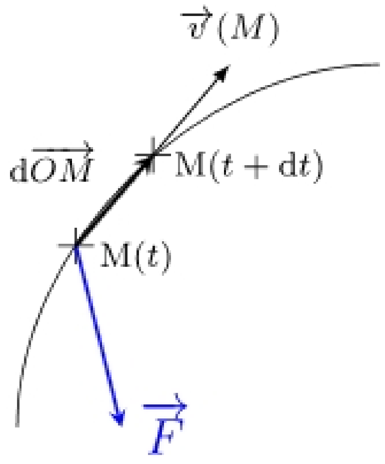
4.1.1.2 Elementary work of a force

We calculate the elementary work of force $\overrightarrow{F}$ as follows:
$\delta W = \overrightarrow{F} d\overrightarrow{OM}$
The elementary displacement vector $d\overrightarrow{OM}$ will be expressed as a function of the chosen coordinate system.
Remarks
The concept δ means that it is the calculation of a variation in the course of a displacement. most of the time, the work of a force between two points depends on the path followed between these two points.Nous We would have noted a d if it was a difference in size between the two points.

4.1.1.3 Work of a force

Generally speaking, the work done by a force depends on the path it follows, which is why this elementary work is necessary. To obtain the work on a displacement AB , this elementary work will be integrated:
$W_{A\to B} = \int_{AB} \delta W = \int_{AB} \overrightarrow{F} d\overrightarrow{OM}$

- Case of constant forces



$W_{A\to B} = \overrightarrow{F}.\overrightarrow{AB} = \|\overrightarrow{F}\|\|\overrightarrow{AB}\|\cos(\widehat{\overrightarrow{F},\overrightarrow{AB}})$

4.1.1.4 Relation between work and power

From the relation between the elementary displacement vector and the velocity, we can relate power and work : $P = \overrightarrow{F}.\overrightarrow{v} = \overrightarrow{F}.\frac{d\overrightarrow{OM}}{dt} = \frac{\overrightarrow{F}.d\overrightarrow{OM}}{dt} = \frac{\delta W}{dt}$ so $P = \frac{\delta W}{dt}$.

4.1.2 Kinetic energy

4.1.2.1 Definition

Kinetic energy is the energy, in Joules, that a body possesses as a result of its speed:
$E_c = \frac{1}{2}mv^2$

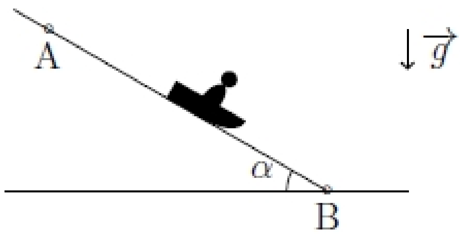
4.1.2.2 Kinetic energy theorem

We place ourselves in a galilean frame of reference, which allows us to use the Newton's second law:
$\sum \overrightarrow{F}_{ext} = m\overrightarrow{\gamma} = m\frac{d\overrightarrow{v}}{dt}$
$\Leftrightarrow \sum \overrightarrow{F}_{ext}.d\overrightarrow{OM} = m\frac{d\overrightarrow{v}}{dt}$
$\Leftrightarrow \sum \delta W(\overrightarrow{F}_{ext}) = m\frac{d\overrightarrow{v}}{dt}.\overrightarrow{v}dt$
$\Leftrightarrow \sum \delta W(\overrightarrow{F}_{ext}) = d(\frac{1}{2}m\overrightarrow{v}^2) = dE_c$

This gives us the differential form of the kinetic energy theorem:
$\delta W(\overrightarrow{F}) = dE_c$
Or by integrating it between two positions (two instants) A and B:
$\sum W_{A\to B}(\overrightarrow{F}_{ext}) = \Delta E_c = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$
Finally, we can express this theorem by considering the power of the forces :
$P(\overrightarrow{F}_{ext}) = \frac{dE_c}{dt}$.

Example

A toboggan run is made on a slope $L = 100m$ between A and B.The slope is characterised by an angle α. The initial speed is zero, what is the velocity at B? All friction will be neglected. We study the sledge system in the galilean frame of reference linked to the slope. system in the galilean reference frame linked to the slope. subject only to its point.
We apply the kinetic energy theorem:
$W_{A\to B}(\overrightarrow{P}) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \Leftrightarrow \overrightarrow{P}.\overrightarrow{AB} = \frac{1}{2}mv_B^2$
$\Leftrightarrow mgL\sin\alpha = \frac{1}{2}mv_B^2 \Leftrightarrow v_B = \sqrt{2gL\sin\alpha} = 31,3m.s^{-1}$



4.1.3 Conservative forces and potential energy

4.1.3.1 Definition

a force is conservative when its work between two points A and B does not depend on the path followed but only on the position of these two points.
It is then derived from a potential energy, a quantity that energetically characterises point M in each position. We write:
$W_{A\to B} = E_p(A) - E_p(B) = -\Delta E_p$
Or differentially:
$\delta W = -dE_p$

4.1.3.2 Another definition of a conservative force

A conservative force can be defined as follows:
$\overrightarrow{F} = -\overrightarrow{grad} E_p$
where the gradient is a mathematical operator that depends on the system of coordinate.
If we work in Cartesian coordinates, f being a scalar function :
$\overrightarrow{grad} f = \frac{\partial f}{\partial x}\overrightarrow{i} + \frac{\partial f}{\partial y}\overrightarrow{j} + \frac{\partial f}{\partial z}\overrightarrow{k}.$
Let's write the differential of the potential energy in two different ways :
from its relation to elementary work:
$dE_p = -\delta W = -\overrightarrow{F}.\overrightarrow{OM} = -F_x dx - F_y dy - F_z dz \quad (1)$
from the definition of a differential :
$dE_p = \frac{\partial E_p}{\partial x}dx + \frac{\partial E_p}{\partial y}dy + \frac{\partial E_p}{\partial z}dz \quad (2)$

By identifying the relation (1) and (2) we obtain:

$$\begin{matrix} F_x = -\frac{dE_p}{dx} \\ F_y = -\frac{dE_p}{dy} \\ F_z = -\frac{dE_p}{dz} \end{matrix} \Leftrightarrow \overrightarrow{F} = -\overrightarrow{grad} E_p$$

4.1.3.2.1 Potential energy of a point M in free fall Fall Free fall: is movement under the sole effect of gravity. An object in Free fall is therefore subject to a single force, its own weight. Other forces acting are either non-existent or neglected, be neglected.Among the forces frequently neglected, we count the resistance of the air in the environment.

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the point M subject only to its own$\overrightarrow{p}$, so the potential energy of M in the reference frame $R(O,\overrightarrow{i},\overrightarrow{j},\overrightarrow{k})$: is derived by its weight ($\overrightarrow{p}$ est toujours une force conservative).We calculate the potential energy of $\overrightarrow{p}$ on $R(O,\overrightarrow{i},\overrightarrow{j},\overrightarrow{k})$:

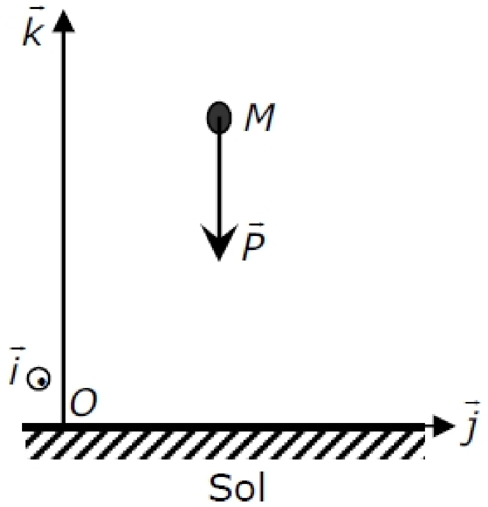
1. 1^{st} **method**
   $dE_p(M) = -dW(\overrightarrow{F}) = -\overrightarrow{P}.d\overrightarrow{OM} = mg\overrightarrow{k}(dx\overrightarrow{i}+dy\overrightarrow{j}+dz\overrightarrow{k}) = mgdz$
   $\rightarrow E_p(M) = mgz + cte = E_p(z)$
   Si $E_p(z=0) = 0 \rightarrow cte = 0 \Rightarrow E_p(M) = mgz$
2. 2^{nd} **method**
   $\overrightarrow{p} = -\overrightarrow{grad}(E_p) \rightarrow -mg\overrightarrow{k} = -\frac{\partial E_p}{\partial x}\overrightarrow{i} - \frac{\partial E_p}{\partial y}\overrightarrow{j} - \frac{\partial E_p}{\partial z}\overrightarrow{k} \Rightarrow$
   $\frac{\partial E_p}{\partial z} = mg \; (1)$
   $\frac{\partial E_p}{\partial x} = 0 \; (2)$
   $\frac{\partial E_p}{\partial y} = 0 \; (3)$
   $(1) \Rightarrow E_p = mgz + cte(x,y)$
   $(2) \Rightarrow \frac{\partial E_p}{\partial x} = \frac{\partial cte(x,y)}{\partial y} = 0 \Rightarrow cte(x,y) = cte(y) \Rightarrow E_p = mgz + cte(y)$
   $(3) \Rightarrow \frac{\partial E_p}{\partial y} = \frac{\partial cte(y)}{\partial y} = 0 \Rightarrow cte(y) = cte$

   $\Rightarrow E_p = mgz + cte$

4.1.3.2.2 Potential energy of a charge placed in an electrostatic field

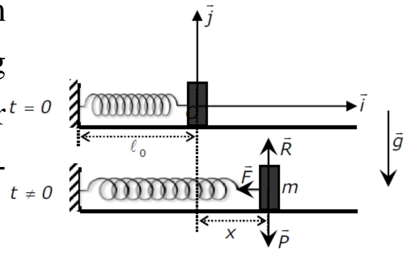
Consider a charge q placed in an electrostatic field. $\overrightarrow{E}$, so q so q is subjected to the electrostatic force $\overrightarrow{F} = q\overrightarrow{E}$

As $\overrightarrow{E} = -\overrightarrow{grad}(v)$ where V is the electrostatic potential, we have$\overrightarrow{F} = -\overrightarrow{grad}(qV)$,So $\overrightarrow{F}$ so derives from potential energy E_p such that $E_p = qV+cte$.



4.1.3.2.3 Potential energy derived by the restoring force or spring force

Consider a spring at rest (non-tensioned) arranged horizontally along the axis (Ox). Let M be a mass attached to its free end (the other end is fixed). Let's move the mass M away from its equilibrium position by a distance x. The spring will therefore exert a restoring force on the operator :$\overrightarrow{F} = -k\Delta l\,\overrightarrow{i} = -kx\,\overrightarrow{i}$ $\overrightarrow{rot}\,\overrightarrow{F} = \overrightarrow{0} \Rightarrow$ is a conservative force.



$dE_p = -dW(\overrightarrow{F}) = -\overrightarrow{F}d\overrightarrow{OM} = kxdx$

$\Rightarrow E_p = \frac{1}{2}kx^2 + cte$

In general, we take this constant to be zero so as to make $E_p = 0$ for $x = 0$, which gives :

$E_p = \frac{1}{2}kx^2$

4.1.4 Example of a non-conservative force

Friction forces are not conservative.Indeed, for a fluid friction force, for example:

$\delta W = \overrightarrow{F}.d\overrightarrow{OM} = -\alpha v_x dx = -\alpha v_x^2 dt$ (one dimensional example)

This relation cannot be written as a differential, so potential energy cannot be defined.

- A force is conservative if its work on a displacement AB, depends only on the position of points A and B, not on the path between A and B.
- Any constant force is conservative (e.g. weight, electric force, etc.), but not all conservative forces are constant (eg: spring or restoring force).
- Among other things, friction forces are non-conservative

4.1.5 Mechanical energy

4.1.5.1 Definition

The mechanical energy of a system is defined as the sum of its kinetic and potential energy
$E_M = E_c + E_p$

4.1.5.2 Mechanical energy theorem

4.1.5.2.1 Non conservation of mechanical energy If there is a variation in mechanical energy, this is due to non-conservative forces. We can therefore write:
$\frac{dE_M}{dt} = P_{non\ cons}$
We can call this expression the mechanical energy theorem..

- **Demonstration**
  $dE_c = \delta W_{totale}$
  $dE_c = \delta W_{cos} + \delta W_{non\ cos}$
  $dE_c = -dE_p + \delta W_{non\ cos}$
  $\Leftrightarrow d(E_c + E_p) = \delta W_{non\ con}$
  $\frac{dE_m}{dt} = \frac{\delta W_{non\ cons}}{dt} = P_{non\ cons}$

4.1.5.2.2 conservation of mechanical energy A system subjected to conservative forces or who don't work has a constant mechanical energy..
$\frac{dE_m}{dt} = 0$

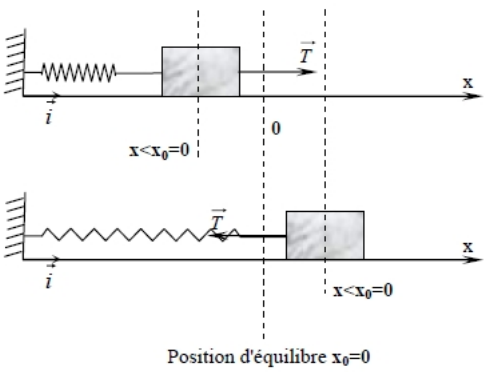
4.1.6 System stability

4.1.6.1 Definition of stability

For a system subject only to a conservative force $\overrightarrow{F}_c$, it is important to know whether or not equilibrium states exist.The local form of potential energy is as follows :
$\overrightarrow{F}_c = -\overrightarrow{grad}E_p$
In the case where energy depends on only one variable x, that means that:
$\overrightarrow{F}_c = -\frac{dE_p}{dx}\overrightarrow{i}$
the equilibrium condition, for any system subjected to a set of forces, is that the sum or resultant of all the forces is zero $(\sum\overrightarrow{F} = \overrightarrow{0})$. In the case of a system subject only to a conservative force $\overrightarrow{F}_c$,it should be zero, it comes:
$F_c = 0 \Rightarrow \frac{dE_p}{dx} = 0$.
An equilibrium position therefore translates into an extremum of the potential energy function.In other words, the potential energy should be maximum or minimum for the system to be in equilibrium.

4.1.6.2 Stability condition

Consider the case shown in the figure below, where potential energy does not depend on only one variable x x. Assuming that the derivative of potential



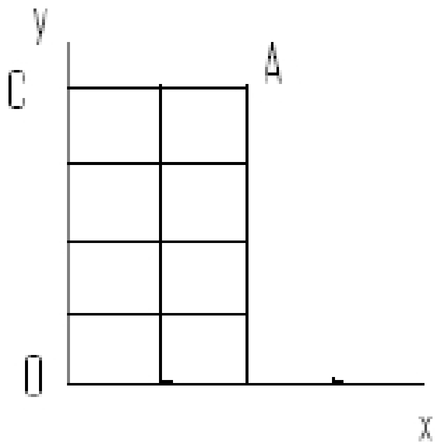
energy at x_0 is zero $\frac{dE_p}{dx}\mid_{x=x_0}= 0$.For a disturbance causing the system to $x < x_0$.the algebraic value of the force must be positive to bring the system back to x_0 $F_c > 0$ so:

$\frac{dE_p}{dx} < 0$ because $F_c = -\frac{dE_p}{dx}$.
Otherwise $x > x_0$.The force must be negative and therefore $\frac{dE_p}{dx} > 0$.
Potential energy E_p decreases before x_0 and increases after x_0.It presents a minimum for $x = x_0$.
In this case, the function $\frac{dE_p}{dx}$is an increasing function that cancels for $x = x_0$.The stability condition, i.e,E_p minimum, can be translated as $\frac{d^2E_p}{dx^2} > 0$.in the vicinity of x_0 and therefore for $x = x_0$.Otherwise, the position will be one of unstable equilibrium..
Stable equilibrium for $x = x_0 \Leftrightarrow E_p(x_0)$ **minimum**
$\Downarrow$
$\frac{dE_p}{dx} \mid_{x=x_0}= 0$ et $\frac{d^2E_p}{dx^2} \mid_{x=x_0}> 0$
Unstable equilibrium for $x = x_0 \Leftrightarrow E_p(x_0)$ **maximum**
$\Downarrow$
$\frac{dE_p}{dx} \mid_{x=x_0}= 0$ et $\frac{d^2E_p}{dx^2} \mid_{x=x_0}< 0$
A system, left to itself, spontaneously evaluates towards a state of equilibrium which corresponds to a position for which the potential energy is minimal..

4.1.7 Corrected exercises

- **Exercice 01:**
  Let a material point M be subjected to a force field $\overrightarrow{F}$ such that
  $\overrightarrow{F} = (x - ay)\overrightarrow{i} + (3y - 2x)\overrightarrow{j}$

  1. Calculate the work of force $\overrightarrow{F}$ for the displacement of M from point $O(0,0)$ to point $A(2,4)$ envia point $C(0,4)$.
  2. Find the value of a for $\overrightarrow{F}$ to be conservative, deduce the potential energy Ep resulting from this force field.
  3. Determine the work done by $\overrightarrow{F}$ to move M along a circular trajectory of radius R and centre $O(0,0)$.



- **Solution:**

  1- $\overrightarrow{F} = (x - ay)\overrightarrow{i} + (3y - 2x)\overrightarrow{j}$

  $dW = \overrightarrow{F}.\overrightarrow{dr} = F_x dx + F_y dy = (x - ay)dx + (3y - 2x)dy$

  $W_{OCA} = W_{OC} + W_{CA}$

  The path from O to C (x=0, dx=0, y varies from 0 to 4

  $W_{OC} = \int_0^3 3ydy = \frac{3y^2}{2}|_0^4 = 24j$

  The path from O to C (x varies from 0 to 2, y=4, dy=0)

  $W_{OCA} = \int_0^2 (x - 4a)dx = \frac{x^2}{2} - 4ax|_0^2 = (2 - 8a)j$

  so $W_{OCA} = 26 - 8a$.

  2-For the force to be conservative it must verify $\overrightarrow{rot}\overrightarrow{F} = \overrightarrow{0}$

  where $\overrightarrow{rot}\overrightarrow{F} = (-2 + a)\overrightarrow{k}$.

  i.e. $a = 2$.

  3- $\overrightarrow{F} = (x - ay)\overrightarrow{i} + (3y - 2x)\overrightarrow{j}$ which is a conservative force.So it's a function derived from a potential E_p

  $\overrightarrow{F} = -\overrightarrow{grad}E_p(x, y) = -\frac{\partial E_p}{\partial x}\overrightarrow{i} - \frac{\partial E_p}{\partial y}\overrightarrow{j}$

  $\frac{\partial E_p}{\partial x} = -(x - 2y)$ (1)

  $\frac{\partial E_p}{\partial y} = -(3y - 2x)$ (2)

  (1) $\Rightarrow E_p = -\int(x - 2y)dx = -\frac{x^2}{2} + 2yx + C(y)$

  (2) $\Rightarrow \frac{\partial E_p}{\partial y} = 2x + \frac{\partial C(y)}{\partial y} = -(3y - 2x) \Rightarrow C(y) = -\frac{3y^2}{2} + C$

  So $E_p(x, y) = -\frac{x^2}{2} - \frac{3y^2}{2} + 2yx + C$

- **Exercice 02**

  The force $\overrightarrow{F} = 2xz\,\overrightarrow{i} + yz\,\overrightarrow{j} + (ax^2 + by^2 + cz^2)\,\overrightarrow{k}$.

  1. Find the constants a, b et c pour que $\overrightarrow{F}$ so that F is conservative $(1,1,1)$ the force $\overrightarrow{F} = 2\overrightarrow{i} + \overrightarrow{j} - 3\overrightarrow{k}$
  2. Reduce the potential $E_p(x, y, z)$.

- **Solution**

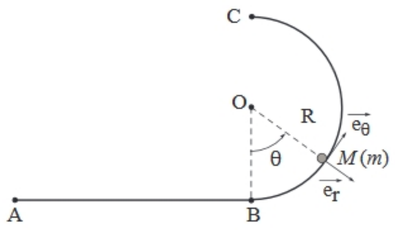
  1-$\overrightarrow{rot}\overrightarrow{F} = (2by - y)\,\overrightarrow{i} - (2ax - 2x)\,\overrightarrow{j} = \overrightarrow{0}$ (1)
  and $\overrightarrow{F}(1,1,1) = 2\overrightarrow{i} + \overrightarrow{j} + (a + b + c)\overrightarrow{k} = 2\overrightarrow{i} + \overrightarrow{j} - 3\overrightarrow{k}$ (2)

  of (1)we get $2by - y = 0 \Rightarrow b = \frac{1}{2}$ $2ax - 2x = 0 \Rightarrow a = 1$
  from equation (2) we get $a + b + c = -3 \Rightarrow c = -\frac{9}{2}$
  So $\overrightarrow{F}$ So F can be written in this form
  $\overrightarrow{F} = 2xz\,\overrightarrow{i} + yz\,\overrightarrow{j} + (x^2 + \frac{1}{2}y^2 - \frac{9}{2}z^2\,\overrightarrow{k}$

  2-$\overrightarrow{F} = -\overrightarrow{grad}E_p(x, y, z) = -\frac{\partial E_p}{\partial x}\overrightarrow{i} - \frac{\partial E_p}{\partial y}\overrightarrow{j} - \frac{\partial E_p}{\partial z}\overrightarrow{k}$
  $\frac{\partial E_p}{\partial x} = -(2xz)$ (1)
  $\frac{\partial E_p}{\partial y} = -(yz)$ (2)
  $\frac{\partial E_p}{\partial z} - (x^2 + \frac{1}{2}y^2 - \frac{9}{2}z^2)$ (3)
  From equation (1)$E_p = -\int(2xz)dx = -2zx^2 + C(y, z)$
  From equation (1)(2)$\frac{\partial E_p}{\partial y} = \frac{\partial C(y,z)}{\partial y} = -(yz) \Rightarrow C(y, z) = -\frac{y^2 z}{2} + C_1(z)$
  By replacing the expression for $C(y, z)$ dans E_p and using the equation (3). We will have
  $\frac{\partial E_p}{\partial z} = -2x^2 - \frac{y^2}{2} + \frac{\partial C_1(z)}{\partial z} = -(x^2 + \frac{1}{2}y^2 - \frac{9}{2}z^2) \Rightarrow C_1(z) = \int(x^2 - \frac{9}{2}z^2)dz = x^2 z - \frac{9}{4}z^3 + C$

- **Exercice 03**

A point solid of mass m is thrown at A on a horizontal track extended by a semicircle $AB = 1m$,$R = 1m$, $m = 0.5kg$ and $g = 9.81ms^2$ vertical with radius R R.



1. Since friction is negligible,calculate at A the minimum speed $v_{A,min}$ that the mass must have for it to reach the point C

2. Same question when friction between the object and the track are comparable a constant force of norm $f = 1N$

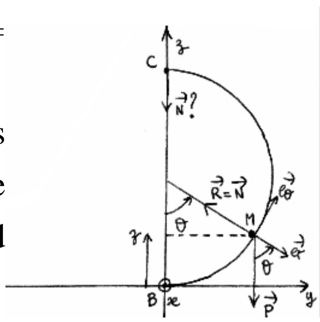
Solution

- If we arrive at C , there must be contact between M and the support, i.e. :$N(\theta=\pi)\geq 0$
- N must therefore be expressed as a function of θ.
- To do this, let's apply the PDF to M in R assumed to be Galilean.
 $m\overrightarrow{a}(M/R)=\overrightarrow{P}+\overrightarrow{R}$
 when M is on the $\widehat{BC}$

$$P=\begin{cases} mg\cos\theta, \\ -mg\sin\theta, \end{cases}$$

$$R=\begin{cases} -N, \\ 0, \end{cases}$$

hence, projected according to $\overrightarrow{e}_r$: $-mR\dot{\theta}^2 = mg\cos\theta - N$ or $N = mg\cos\theta + mR\dot{\theta}^2 \quad (1)$.
This relation (1) is valid even if there is friction or not Or, on the portion $\widehat{BC}$, the movement is circular,so $\overrightarrow{v} = R\dot{\theta}\overrightarrow{e}_\theta$, and thus $R\dot{\theta}^2 = \frac{v^2}{R}$ (2)



To make v^2, appear, we need only to apply the energy theorem between point M and another point. //As we load v_A ,we will apply the energy theorem between A and M:
$\frac{1}{2}mv^2-\frac{1}{2}mv_A^2 = W(\overrightarrow{F})_{A\to B}+W(\overrightarrow{P})_{B\to M}+W(\overrightarrow{R}) = mg(z_B - z_M) = -mgz$ taking B as the origin of the space frame $(Bxyz)$
Like $W(\overrightarrow{F})_{A\to B}=0$

$W(\overrightarrow{R})=0$ and
$z=R(1-\cos\theta)$ so
$m\frac{v^2}{R} = m\frac{v_1^2}{R} - 2mg(1-\cos\theta)$ (3)
En remplaçons (1) et (2) dans (3)
$N=m(\frac{v_1^2}{R}+g(-2+3\cos\theta))$ (4)
Pour que $M=C(\theta=\pi)$ il faut $\frac{v_A^2}{R}+g(-2+$

This time $\overrightarrow{R} = \overrightarrow{R}_N + \overrightarrow{R}_T = -N\overrightarrow{e}_r - f\overrightarrow{e}_\theta$ with $N = \|\overrightarrow{R_N}\|$ and $f = \|\overrightarrow{R_T}\|$.
We still have to find v_A/ $N(\theta = \pi) \geq 0$ et avec $N = mg\cos\theta + mR\dot{\theta}^2$ soit$N = mg\cos\theta + m\frac{v^2}{R}$.
Let's apply the kinetic energy theorem between A and M:
$\frac{1}{2}mv^2 - \frac{1}{2}mv_A^2 = N_{A\to M}(\overrightarrow{P} + W_{A\to M}(\overrightarrow{R}_N) + W_{A\to M}(\overrightarrow{R}_T) = -mgz + \int_A^M \overrightarrow{R}_T.d\overrightarrow{BM} = -mgz + \int_A^B -f\overrightarrow{e}_y.dy.\overrightarrow{e}_y + \int_B^M -f\overrightarrow{e}_\theta.Rd\theta\overrightarrow{e}_\theta$.
hence $\frac{1}{2}mv^2 - \frac{1}{2}mv_A^2 = -mg(1-\cos\theta) - f(AB + R\theta)$ hence $m\frac{v^2}{R} = m\frac{v_A^2}{R} - 2mg(1-\cos\theta) - 2\frac{f}{R}(AB + R\theta)$ (5)
So $N = m(\frac{v_A^2}{R} + g(-2 + 3\cos\theta) - \frac{2f}{mR}(AB + R\theta))$.
For M to reach the point C, it must $N_{\theta=\pi} \geq 0$ or:
$\frac{v_A^2}{R} + g(-2 + 3\cos\pi) - \frac{2f}{mR}(AB + R\pi) \geq 0$.
hence $v_A \geq \sqrt{5gR + \frac{2f}{m}(AB + R\pi)}$.
N.A $v_A \geq 8,2ms^{-1}$

- **Exercice 04:**

A punctual mass $m = 200g$ is launched upwards from point A with an initial speed $v_A = 10m.s^{-1}$.
Assuming the friction force is vertical, of constant intensity $f = 0.50N$, calculate

1. The height $h = AB$ at which it is mounted
2. Its speed v'_A when it passes back through the launch point.
   Data: The vertical Oz is oriented upwards. $g = 10m.s^{-2}$

Solution:
Let A be the starting point and B any point on the trajectory.
In the presence of friction, the mechanical energy theorem written as:
$E_M(B) - E_M(A) = W_{A\to B}(\overrightarrow{f})$.
So $E_c(B) + E_p(B) - (E_c(A) + E_p(A)) = W_{A\to B(\overrightarrow{f})}$
let $(\frac{1}{2}mv_B^2 + mgz_B) - (\frac{1}{2}mv_A^2 + mgz_A) = W_{A\to B}(\overrightarrow{f})$
and finally:$\frac{1}{2}v_B^2 + gz_B - \frac{1}{2}v_A^2 - gz_A) = \frac{1}{m}W_{A\to B}(\overrightarrow{f})$.

a.If B is the point of maximum altitude, then $v_B = 0$
and $W_{A\to B}(\overrightarrow{f}) = \overrightarrow{f}.\overrightarrow{AB}\cos(\overrightarrow{f}.\overrightarrow{AB}) = f.AB.\cos\pi = -f.AB = -f(z_B - z_A)$
(Car $\overrightarrow{f} = \overrightarrow{cte}$ de A à B) .
hence:$g(z_B - z_A) - \frac{v_A^2}{2} = -\frac{f}{m}(z_B - z_A)$
so: $z_B - z_A = \frac{v_A^2}{2(g+\frac{f}{m})}$ (1).
N.A $z_B - z_A = 4,0m$

b.Let's apply the same formula between B And A:
We obtain (with $v_B = 0$ also):
$g(z_B - z_A) + \frac{v_A^2}{2} = \frac{1}{m}W_{B\to A}(\overrightarrow{f}) = -\frac{f}{m}.BA' = -\frac{f}{m}(z_B - z_A)$
or $z_{A'} - z_B = \frac{v_{A'}^2}{2(-g+\frac{f}{m})}$ (2)
$(1) + (2)$ with $z_A = z_{A'}$ becomes:
$0 = \frac{v_A^2}{2(g+\frac{f}{m})} + \frac{v_{A'}^2}{2(-g+\frac{f}{m})}$
hence: $v_{A'} = v_A\sqrt{\frac{m.g-f}{m.g+f}}$
A.N:$v_{A'} = 7,7m.s^{-1}$

- **Exercice 05**

A car is travelling on a motorway at a speed of $v_0' = 130km.h^{-1}$.It is assumed that there is solid friction between the car and the road..
Remember that the reaction of the road is divided into a normal component $\overrightarrow{R}_N$ and a tangential component. $\overrightarrow{R}_T$ in the opposite direction to the speed and whose norm verifies $R_T = fR_N$ we note f the coefficient of friction.
It must $D' = 500m$ to stop the vehicle when no braking force is applied.

1. Calculate the braking distance D if the initial speed was $v_0 = 110km.h^{-1}$.
2. Does the result change if the road makes an angle α with the horizontal? (the car going up or down the slope)?

Solution

Car system considered as a material point M of mass m studied in the reference frame assumed to be Galilean.

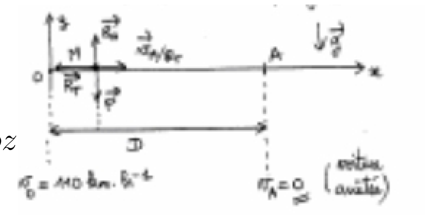
Balance of forces: the point $\overrightarrow{P} = m\overrightarrow{g}$

the reaction of the road on the car : $\overrightarrow{R} = \overrightarrow{R}_N + \overrightarrow{R}_T$ with $R_T = fR_N$.

According to kinetic energy theory between $O(\overrightarrow{v}_0)$ and $A(\overrightarrow{v}_A = \overrightarrow{0})$.

$\frac{1}{2}v_A^2 - \frac{1}{2}mv_0^2 = W(\overrightarrow{P} + \overrightarrow{R}_N) + W(\overrightarrow{R}_T)$

$W(\overrightarrow{P} + \overrightarrow{R}_N) = 0$ car $\overrightarrow{P}$ et $\overrightarrow{R}_N$ are perpendicular to the movement (and thus to d $\overrightarrow{OM}$).



Moreover, the PFD :

$m\overrightarrow{a}(M/R) = \overrightarrow{P} + \overrightarrow{R}_N + \overrightarrow{R}_T$ projected along oz so :

$0 = -mg + R_N$

hence $R_N = mg$ et donc $R_T = fR_N = fmg$.

hence $W(\overrightarrow{R}_T) = \int_0^A \overrightarrow{R}_T.d\overrightarrow{OM} = \int_0^A -f.m.g\overrightarrow{e}_x.dx.\overrightarrow{e}_x = -fmgD$.

so $v_0^2 = 2fgD$ soit $D = \frac{v_0^2}{2fg}$ but we don't know f f, so we can't calculate D directly.

On the other hand, the statement tells us that for a speed $v_0' = 130km.h^{-1}$ the frienage distance is $D' = 500m$

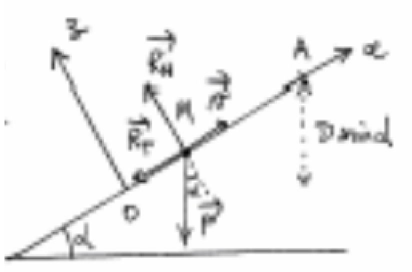
Let $D' = \frac{v_0'^2}{2fg}$ d'où $D = (\frac{v_0}{v_0'})^2 D' = 360m$

When the road makes an angle α to the horizontal

The projection of the PFD along Oz gives:

$R_N = mg\cos\alpha$ and then $R_T = fR_N = fmg\cos\alpha$

the theory of energy between O and A :



$0 - \frac{1}{2}mv_0^2 = W(\overrightarrow{R}_N) + W(\overrightarrow{P}) + W(\overrightarrow{R}_T)$

Let :$-\frac{1}{2}mv_0^2 = -mgD\sin\alpha - fmg\cos\alpha D$.

$W(\overrightarrow{R}_N) = 0$

Hence

$v_0^2 = (2g\sin\alpha + 2fg\cos\alpha)D$

also $v_0'^2 = (2g\sin\alpha + 2fg\cos\alpha)D'$

hence $D = (\frac{v_0}{v_0'})^2 D'$

$\Rightarrow$ the result is identical whether the road is horizontal or not.

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