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THÈSE

Pour l'obtention du diplôme de doctorat LMD

Spécialité: Théorie des operateurs

Sur certains problèmes aux limites impliquant des équations différentielles d'ordre fractionnaire

Présentée Par : Houari BOUZID

Devant le jury composé de :

Boudjema Djeffal Khaled	MCA	Univ. Chlef	Président
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Hassiba Ben Bouali University Chlef
Faculty of Exact Sciences and Computer Science
Mathematics Department

THESIS

For the fulfillment of the requirements for the LMD Doctorate degree

Specialty: Operator Theory

On certain boundary value problems involving fractional-order differential equations

Presented by Houari BOUZID

In front of the jury composed of:

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Academic Year: 2025–2026

Dedication

I dedicate this modest work to my relatives, particularly

my dear parents, to my dear sisters

and to everyone who holds a special place in my heart.

Thank you.

- BOUZID Houari

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Publications

1. **H. Bouzid**, A. Benali, A. Salim and I. M. Erhan, A study on some classes of pantograph Langevin fractional differential problems with nonlocal conditions, *J Math Sci.* (2025), 1-18. https://doi.org/10.1007/s10958-025-07573-y.

2. **H. Bouzid**, A. Benali, A. Salim, G. Reny and E. Sina, On Solutions of the Nonlocal Generalized Coupled Langevin-Type Pantograph Systems, *Journal of Mathematics*, 20 pages, 2025. https://doi.org/10.1155/jom/8760264.

3. **H. Bouzid**, A. Benali, L. Tabharit, A. Salim, A Study on a Class of -Caputo Fractional Hybrid Equations with Langevin and Pantograph Arguments (**submitted**).

4. **H. Bouzid**, A. Benali, A. Salim, L. Tabharit, A Study on Some Classes of Hybrid Langevin Pantograph ψ-Caputo Fractional Coupled Systems, *Pan-Amer. J. Math.* vol(4), 14, (2025).https://doi.org/10.28919/cpr-pajm/4-14.

5. **H. Bouzid**, A. Benali, A. Salim and I. M. Erhan, Existence and Ulam stability results of hybrid Langevin pantograph ψ-fractional coupled systems.*Filomat*, vol. **39**(10), 3401–3424, **(2025)**. https://doi.org/10.2298/FIL2510401B.

6. **H. Bouzid**, A. Benali, A. Salim and A. Jehad, Qualitative Results on Hybrid Langevin Pantograph ψ-Fractional Differential Equations with Multiple Point Boundary Conditions, (**submitted**).

Abstract

This thesis presents several results on the existence, uniqueness, and stability of nonlocal and boundary value problems for differential equations involving the generalized Caputo fractional derivatives. In addition, we investigate coupled systems of nonlinear fractional differential equations within the same framework. The analysis relies on fixed point theorems, including those of Krasnoselskii, Dhage, Schaefer, and the Banach contraction principle. Moreover, the study extends to Banach spaces, employing Darbo's fixed-point theorem in conjunction with the measure of noncompactness technique. Each chapter is considered a continuation of the previous one and ends with illustrations to show the applicability of the results.

Keywords: Boundary value problem, existence, measure of noncompactness, fixed point, Banach space, Ulam-Hyers-Rassias stability, Ψ-Caputo fractional derivative, fractional integral, Ulam stability, nonlocal problem, hybrid fractional differential equations...

Résumé

Cette thèse présente plusieurs résultats sur l'existence, l'unicité et la stabilité de problèmes aux limites locales et non locales pour des équations différentielles impliquant les dérivées fractionnaires généralisées de Caputo. Nous étudions également des systèmes couplés d'équations différentielles fractionnaires non linéaires dans le même cadre. L'analyse repose sur les théorèmes du point fixe, notamment ceux de Krasnoselskii, Dhage, Schaefer, ainsi que sur le principe de contraction de Banach. En outre, l'étude est étendue aux espaces de Banach, en appliquant le théorème du point fixe de Darbo associé à la technique de mesure de la non-compacité. Chaque chapitre est considéré comme une continuité du précédent et se termine par des illustrations visant à montrer l'applicabilité des résultats obtenus.

Mots clés : Problème aux limites, existence, mesure de non-compacité, point fixe, espace de Banach, stabilité de Ulam-Hyers-Rassias, dérivée fractionnaire de Ψ-Caputo, intégrale fractionnaire, stabilité d'Ulam, problème non local, équations différentielles fractionnaires hybrides...

ملخص

تُقَدِّم هذه الأطروحة مجموعة من النتائج المهمة المتعلّقة بوجود الحل، وحدانيته، واستقراره لمسائل القيم الحدية للمعادلات التفاضلية التي تتضمن المشتقات الكسرية المعرفة وفقًا لمفهوم Ψ- كابوتو (Caputo).

بالإضافة إلى ذلك، تتناول الدراسة الأنظمة المرتبطة بالمعادلات التفاضلية الكسرية غير الخطية ضمن نفس الإطار. تركز التحليلات على مجموعة من مبرهنات النقطة الصامدة، من بينها مبرهنات Krasnoselskii، وDhage، وSchaefer، إلى جانب مبدأ الانكماش في فضاءات Banach . كما تمتدّ هذه الدراسة لتشمل فضاءات Banach من خلال توظيف مبرهنات النقطة الصامدة مع الاستفادة من تقنية اللاتراص (noncompactness).

ويُعدّ كل فصل من فصول الأطروحة امتدادًا لما سبقه، حيث يُختتم كل فصل برسوم توضيحية تهدف إلى إبراز قابلية تطبيق النتائج المحصَّل عليها.

الكلمات والمصطلحات المفتاحية: مسألة القيم الحدية، الوجود، مقياس اللاتراص، نقطة صامدة، فضاء باناش، استقرارية أولام-هايرس-راسياس، مشتقة كسرية Ψ- كابوتو، التكامل الفركتي، استقرارية أولام، مسألة غير محلية، معادلات تفاضلية كسرية هجينة...

Contents

Introduction

Fractional calculus is a field in mathematical analysis that generalizes classical integer-order differential calculus that involves real or complex order derivatives and integrals [1, 3, 9, 24, 32–35]. The concept of fractional differential calculus has a long and rich history. The idea of a derivative of fractional order, represented as $\frac{\mathrm{d}^n y}{\mathrm{d}\mathrm{x}^n}$, where n is a fraction, has intrigued mathematicians for centuries. This notion can be traced back to a correspondence between L'Hôpital and Leibniz in 1695. L'Hôpital posed the question, "What if $n = \dfrac{1}{2}$?" Leibniz responded, suggesting that $d^{\frac{1}{2}}\mathrm{x}$ would equal $\mathrm{x}\sqrt{\mathrm{dx} : \mathrm{x}}$, acknowledging it as a paradox from which valuable insights might eventually emerge. This exchange is regarded as the birth of fractional calculus.

Over time, numerous prominent mathematicians have contributed to the development of fractional calculus. The formal inception of this field is often attributed to 30 September 1695. Foundational work was carried out by Leibniz and L'Hôpital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Weyl (1917), Riesz (1922), P. Lévy (1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964), among others, who have significantly advanced the theoretical framework of fractional calculus.

Ross organized the first conference on fractional calculus at the University of New Haven in June 1974 and edited its proceedings [88]. Subsequently, Spanier published the first monograph dedicated to "Fractional Calculus" in the same year [76]. In recent years, fractional integrals and derivatives of non-integer order, as well as fractional integro-differential equations, have found extensive applications in theoretical physics, mechanics, and applied mathematics. A highly detailed encyclopedic monograph by Samko, Kilbas, and Marichev was initially published in Russian in 1987 and later translated into English in 1993 [106] (for additional details, see [68]). Several notable works focused on fractional differential equations include those by Miller and Ross (1993) [71], Podlubny (1999) [80], Kilbas *et al.* (2006) [64], Diethelm (2010) [52], Ortigueira (2011) [77], Abbas et al. (2012) [2], and Baleanu et al. (2012) [23].

The origins of fixed point theory, as it is very well known, go to the system of successive approximations (or the iterative method of Picard) used to solve certain differential equations. Roughly speaking, from the process of successive approximations, Banach obtained the fixed point theorem. The fixed point theory has been immense and independent of the differential equations in the last few decades. But, lately, the outcomes of fixed points have turned out to be the instruments for the solutions of differential equations. Recently, differential fractional equations have been shown to be an effective instrument for researching multiple phenomena in diverse fields of science and engineering, such as electrochemistry, electromagnetics, viscoelasticity, economics, etc. It is very popular in the literature to suggest a solution to fractional differential equations by adding various forms of fractional derivatives, see e.g. [1, 2, 7, 9, 16, 24, 29, 43, 63, 115]. On the other hand, there are more findings concerned with the issues of boundary value for fractional differential equations [9, 20, 44, 82, 93, 95, 97–101, 103–105, 115].

In 1940, Ulam [109, 110] posed a fundamental question concerning the stability of functional equations related to group homomorphisms: *"Under what conditions does there exist an additive mapping close to an approximately additive mapping?"*.

A partial resolution of this problem was initially provided by Hyers [60] in 1941. Later, between 1982 and 1998, Rassias [85, 86] extended these results by establishing the Hyers-Ulam stability for both linear and nonlinear mappings. Following these foundational contributions, numerous studies have explored various generalizations of Hyers' results, leading to a broad spectrum of developments in this domain (see, for instance, [1, 25, 38–40, 60, 66, 83, 84, 89, 110]).

The term "pantograph" refers to an articulated mechanism enabling electric vehicles, such as locomotives or trams, to draw power through friction with an overhead catenary system. The pantograph differential equation serves as a mathematical representation of the dynamics of mechanical systems with pantograph linkages, often seen in electric transportation. This equation provides insights into the dynamic behavior and performance of such systems. Its development traces back to the work of Ockendon and Tayler [75], who analyzed the dynamics of current collection systems for electric locomotives. Pantograph equations have since been applied in numerous scientific domains to model diverse problems [12, 56, 79]. In recent years, researchers have explored fractional-order pantograph equations, incorporating various aspects and distinct fractional derivative operators (see, for example, [6, 49, 56, 114]).

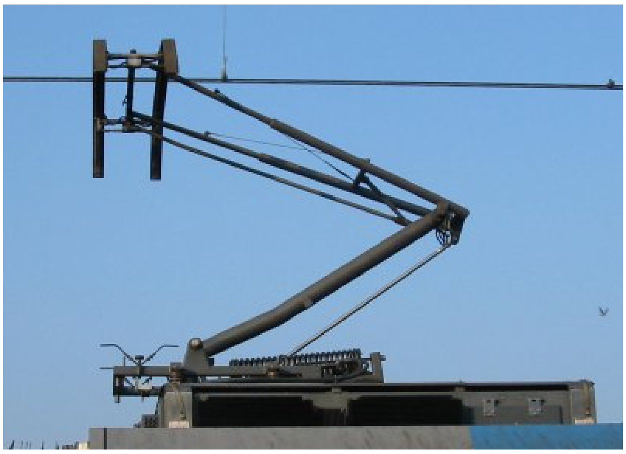

Figure 1: Example of a pantograph in transport.

The concept of the measure of noncompactness, which serves as a crucial tool in the theory of nonlinear analysis, was initially introduced through the pioneering works of Alvàrez [17] and Mönch [73], and subsequently developed further by Banaś and Goebel [28], along with various other researchers. The measure of noncompactness has found extensive applications in applied mathematics, particularly in the theory of differential equations (see [8, 78] and references therein). More recently, the measure of noncompactness has been employed to study certain classes of differential equations in Banach spaces, as demonstrated in [1, 17, 19, 28].

The concept of nonlocal conditions was introduced by Byszewski [45], who established the existence and uniqueness of mild and classical solutions for nonlocal Cauchy problems. Nonlocal conditions can provide a more effective framework than standard initial conditions for modeling certain physical phenomena. Fractional differential equations with nonlocal conditions have been examined in detail in [5, 10, 18, 41, 42, 57, 67, 74, 87, 102] and related works.

In 1908, Paul Langevin [65] introduced the classical Langevin equation, which provided a mathematical framework to describe the Brownian motion of particles. While this classical formulation has played a crucial role in mathematical physics, including applications to fractional reaction-diffusion systems, correlated noise sources, harmonic oscillators, and quantum noise phenomena, it has limitations in certain fractal disorder domains. To address these challenges, the fractional Langevin equation has proven to be more effective, particularly in scenarios where the microscopic time scale differential equations fail or macroscopic system descriptions are unavailable (see references [16, 22, 47, 59, 70, 111, 112]).

Hybrid fractional differential equations have been the subject of investigation by numerous researchers. A hybrid differential equation refers to an equation in which the terms are perturbed either linearly, quadratically, or through a combination of both types. A perturbation is classified as linear when it appears as a sum or difference of terms within the equation. Conversely, when the perturbation arises from the product or quotient of terms, it is referred to as quadratic. Consequently, the study of hybrid differential equations provides a more general framework that encompasses various dynamic systems as special cases. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers (see [11, 16, 21, 36, 37, 46, 48, 58, 94, 96, 113]).

In the following we give an outline of our thesis organization, which consists of four chapters defining the contributed work.

Chapter 1: This chapter provides an outline of the notations, foundational results, descriptions, theorems, and supporting concepts essential to this study. The first section presents the notations and definitions associated with the functional spaces employed throughout the thesis. The second section focuses on key aspects of fractional calculus theory, including its definitions, relevant lemmas, theorems, and properties. The third section discusses various properties associated with the Measure of Noncompactness. Finally, the chapter concludes with a summary of the fixed-point theorems applied throughout the thesis.

Chapter 2: This chapter presents a comprehensive analysis of fractional pantograph differential equations that incorporate the ψ-Caputo fractional derivative. The equations under consideration are formulated as follows:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}\left({}^{C}D_{0^+}^{\beta,\psi}+\mu\right)\mathrm{x}(\mathrm{t})=\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})), \quad \mathrm{t}\in J=[0,b], \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0, \\ \mathfrak{s}_1\mathrm{x}(\mathrm{t})\big|_{\mathrm{t}=\varepsilon_1}+\mathfrak{s}_2\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_2}+\cdots+\mathfrak{s}_m\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_m}=0. \end{cases} \tag{1}$$

Here, ${}^{C}D_{0^+}^{\alpha,\Psi}$ and ${}^{C}D_{0^+}^{\beta,\Psi}$ denote the Ψ-Caputo fractional derivatives of orders $\alpha,\beta\in(0,1]$. The term $\mu\in\mathbb{R}\setminus\{0\}$, and $0<\zeta<1$. The constants $\mathfrak{s}_i\neq 0$ for $i=1,\ldots,m$, while the points ε_i satisfy $0<\varepsilon_1\leq\cdots\leq\varepsilon_m<b$. The function $\Psi\in\mathcal{C}^1(J,\mathbb{R}^+)$ is an increasing differentiable function such that $\Psi'(t)\neq 0$ for all $t\in J$, and $\mathfrak{f}:J\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

This study focuses on the analysis of a class of coupled Langevin fractional pantograph differential equations involving the Ψ-Caputo fractional derivative with nonlocal boundary conditions in Banach spaces. To establish the existence and uniqueness of solutions, we apply several mathematical tools, including the Banach, Schaefer, and Darbo fixed-point theorems, along with the concept of the measure of noncompactness. Finally, we provide illustrative examples to demonstrate the applicability of our results.

Chapter 3: This chapter study the existence and uniqueness of solutions for the hybrid Langevin fractional pantograph differential equations involving ψ-Caputo fractional derivatives, specifically addressing the following problem:

$${}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}\left(\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)+\mu\mathrm{x}(\mathrm{t})\right]=\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})), \quad \mathrm{t}\in J=[0,b], \tag{2}$$

subject to the initial and boundary conditions:

$$\mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0,\ \mathrm{x}'(\mathrm{t})\mid_{\mathrm{t}=0}=0,\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=\kappa}=0,\ 0<\kappa\leq b, \tag{3}$$

where ${}^{C}D_{0^+}^{\alpha,\Psi}$ and ${}^{C}D_{0^+}^{\beta,\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha\in(0,1]$, $\beta\in(1,2]$, $\mu\in\mathbb{R}\setminus\{0\}$ and $0<\zeta<1$. The given functions $\mathfrak{f}:J\times\mathbb{R}^2\to\mathbb{R}$ and $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuous.

The results are based on the Banach and Dhage fixed-point theorems. Next, we investigate the stability of these solutions in the sense of Ulam-Hyers and it's generalized form. To illustrate our findings, we present several examples to demonstrate our results.

Chapter 4: This chapter focuses on deriving results concerning the solutions of a coupled system of hybrid Langevin fractional pantograph differential equations that incorporate Ψ-Caputo fractional derivatives within the framework of Banach spaces, as detailed below

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1,\Psi}\left[{}^{C}D_{0^+}^{\beta_1,\Psi}\left(\dfrac{\mathrm{x}(\mathrm{t})-\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)+\mu_1\mathrm{x}(\mathrm{t})\right]=\mathfrak{f}_1(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\zeta\mathrm{t})), \quad \mathrm{t}\in J=[0,b], \\ {}^{C}D_{0^+}^{\alpha_2,\Psi}\left[{}^{C}D_{0^+}^{\beta_2,\Psi}\left(\dfrac{\mathrm{y}(\mathrm{t})-\mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}\right)+\mu_2\mathrm{y}(\mathrm{t})\right]=\mathfrak{f}_2(\mathrm{t},\mathrm{y}(\mathrm{t}),\mathrm{x}(\tilde{\zeta}\mathrm{t})), \quad \mathrm{t}\in J=[0,b], \end{cases}$$

under the given boundary conditions

$$\begin{cases} \left.\left(\dfrac{\mathrm{x}(\mathrm{t})-\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right|_{\mathrm{t}=0}=\mathcal{V}_1, \quad \left.\left(\dfrac{\mathrm{y}(\mathrm{t})-\mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}\right)\right|_{\mathrm{t}=0}=\mathcal{V}_2, \\ \left.\left(\dfrac{\mathrm{x}(\mathrm{t})-\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)'\right|_{\mathrm{t}=0}=\left.\left(\dfrac{\mathrm{y}(\mathrm{t})-\mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}\right)'\right|_{\mathrm{t}=0}=0, \\ \left.\left(\dfrac{\mathrm{x}(\mathrm{t})-\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)\right|_{\mathrm{t}=\epsilon_1}=\mathcal{V}_3, \quad \left.\left(\dfrac{\mathrm{y}(\mathrm{t})-\mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}\right)\right|_{\mathrm{t}=\epsilon_2}=\mathcal{V}_4, \ 0<\epsilon_1,\epsilon_2\leq b, \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha_\mathrm{i},\Psi}$, ${}^{C}D_{0^+}^{\beta_\mathrm{i},\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha_\mathrm{i}\in(0,1]$, $\beta_\mathrm{i}\in(1,2]$, for $i=1,2$, $\mu_1,\mu_2\in\mathbb{R}\setminus\{0\}$, $0<b$, $\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4\in\mathbb{R}$, ($\mathcal{V}_1\neq\mathcal{V}_3$ and $\mathcal{V}_2\neq\mathcal{V}_4$) and $0<\zeta,\tilde{\zeta}<1$ and $\Psi\in\mathcal{C}^1(J,\mathbb{R}^+)$ be an increasing differentiable function such that $\Psi'(\mathrm{t})\neq 0$, for all $\mathrm{t}\in J$. The given functions $\mathfrak{f}_\mathrm{j}:J\times\mathbb{R}^2\to\mathbb{R}$, $\mathcal{W}_\mathrm{j}:J\times\mathbb{R}\to\mathbb{R}$, and $\mathfrak{g}_\mathrm{j}:J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuous with $\mathrm{j}=1,2$.

The results based on Banach's fixed-point theorem, which ensures the existence of solutions. This is further validated through Dhage's hybrid fixed-point theorem applied to the sum of three operators. In addition, the stability of these solutions is examined in both the Ulam-Hyers sense and its generalized form. The theoretical results are also illustrated with several examples to demonstrate their applicability.

In conclusion, we wrap up the thesis by providing a summary and offering some future perspectives.

chapter

1 Preliminaries

In this chapter, an overview of the key mathematical tools, notations, and concepts that will be used in the subsequent chapters are provided. We examine important properties of fractional differential operators and revisit the fundamental properties of measures of noncompactness and fixed-point theorems, which are essential for the results related to fractional differential equations.

1.1 Notations and Definitions

Let $0 < b$ and $J = [0, b]$, we denote by $\mathcal{C}(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|\mathrm{x}\|_\infty = \sup\{|\mathrm{x}(\mathrm{t})| : \mathrm{t} \in [0, b]\}.$$

And, the set $\mathcal{C}^n(J; \mathbb{R})$ is the space of functions that are n-times continuously differentiable on J.

Now, we consider the following Banach space

$$\Upsilon = \{(\mathrm{x}, \mathrm{y}) : \ \mathrm{x}, \mathrm{y} \in \mathcal{C}(J, \mathbb{R})\},$$

endowed with the norm

$$\|(\mathrm{x}, \mathrm{y})\|_\Upsilon = \|\mathrm{x}\| + \|\mathrm{y}\|.$$

Consider the space $X_c^p(0, b)$, $(c \in \mathbb{R}, \ 1 \leq p \leq \infty)$ of those complex-valued Lebesgue measurable functions g on $[0, b]$ for which $\|\mathrm{g}\|_{X_c^p} < \infty$, where the norm is defined by

$$\|\mathrm{g}\|_{X_c^p} = \left(\int_0^b |\mathrm{t}^c \mathrm{g}(\mathrm{t})|^p \frac{\mathrm{dt}}{\mathrm{t}}\right)^{\frac{1}{p}}, \quad (1 \leq p < \infty, c \in \mathbb{R}).$$

In particular, when $c = \frac{1}{p}$, the space $X_c^p(0,b)$ coincides with the $L_p(0,b)$ space: $X_{\frac{1}{p}}^p(0,b) = L_p(0,b)$.
Let $(\mathbb{E}, \|\cdot\|)$ be a Banach space. By $\mathcal{C}_{\mathbb{E}}([0,b])$ we denote the Banach space of all continuous functions from $[0,b]$ into E with the norm

$$\|\mathrm{x}\|_{\mathbb{E}} = \sup\{\|\mathrm{x}(\mathrm{t})\| : \mathrm{t} \in [0,b]\}.$$

By $L^1([0,b])$, we denote the space of Bochner–integrable functions $\mathfrak{f} : J \longrightarrow \mathbb{E}$ with the norm

$$\|\mathfrak{f}\|_1 = \int_0^b \|\mathfrak{f}(\mathrm{t})\| dt.$$

1.2 Special Functions of the Fractional Calculus

1.2.1 Gamma function

Undoubtedly, one of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(\mathrm{z})$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

Definition 1.1 ([80]). *The gamma function $\Gamma(\alpha)$ is defined by the integral :*

$$\Gamma(\alpha) = \int_0^{+\infty} \mathrm{t}^{\alpha-1} e^{-\mathrm{t}} \mathrm{dt},$$

which converges in the right half of the complex plane $\mathrm{Re}(\alpha) > 0$.
One of the basic properties of the gamma function is that it satisfies the following functional equation:

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha),$$

so, for positive integer values n, the Gamma function becomes $\Gamma(n) = (n-1)!$ and thus can be seen as an extension of the factorial function to real values.
A useful particular value of the function: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, is used throughout many examples in this thesis.

1.3 Elements From Fractional Calculus Theory

In this section, we present the definitions of the fractional integral and fractional differential operators used throughout this thesis. The section concludes with key lemmas, theorems, and properties.

1.3.1 Fractional integrals

Definition 1.2 (**Ψ-Riemann-Liouville Fractional Integral** [64]). *Let* $(0, b)$ $(-\infty \leq 0 < b \leq \infty)$ *be a finite or infinite interval of the real line* $\mathbb{R}$*,* $\alpha > 0$*,* $c \in \mathbb{R}$ *and* $\mathcal{A} \in X_c^p(0, b)$*. Also let* $\Psi(\mathrm{t})$ *be an increasing and positive monotone function on* $[0, b]$*, having a continuous derivative* $\Psi'(\mathrm{t})$ *on* $(0, b)$*. The left and right sided R-L fractional integrals of a function* $\mathcal{A}$ *of order* α *with respect to another function* Ψ *on* J *are defined by*

$$\left(I_{0^+}^{\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) = \int_0^{\mathrm{t}} \Psi'(\tau)\left(\Psi(\mathrm{t}) - \Psi(\tau)\right)^{\alpha-1} \frac{\mathcal{A}(\tau)}{\Gamma(\alpha)} \mathrm{d}\tau,$$

and

$$\left(I_{b^-}^{\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) = \int_{\mathrm{t}}^{b} \Psi'(\tau)\left(\Psi(\mathrm{t}) - \Psi(\tau)\right)^{\alpha-1} \frac{\mathcal{A}(\tau)}{\Gamma(\alpha)} \mathrm{d}\tau.$$

1.3.2 Fractional derivatives

Definition 1.3 (**Ψ-Riemann-Liouville fractional derivative** [64]). *Let* $\Psi'(\mathrm{t}) \neq 0$ $(-\infty \leq 0 < \mathrm{t} < b \leq \infty)$*,* $\alpha > 0$ *and* $n \in \mathbb{N}$*. The Riemann-Liouville derivatives of a function* $\mathcal{A}$ *of order* α *with respect to another function* Ψ *on* $[0, b]$ *are defined by*

$$\begin{aligned}\left(D_{0^+}^{\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) &= \mathfrak{f}^n \left(I_{0^+}^{n-\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) \\ &= \mathfrak{f}^n \int_0^{\mathrm{t}} \Psi'(\tau)\left(\Psi(\mathrm{t}) - \Psi(\tau)\right)^{n-\alpha-1} \frac{\mathcal{A}(\tau)}{\Gamma(n-\alpha)}\, \mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}\left(D_{b^-}^{\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) &= (-1)^n \mathfrak{f}^n \left(I_{b^-}^{n-\alpha;\Psi}\mathcal{A}\right)(\mathrm{t}) \\ &= (-1)^n \mathfrak{f}^n \int_{\mathrm{t}}^{b} \Psi'(\tau)\left(\Psi(\tau) - \Psi(\mathrm{t})\right)^{n-\alpha-1} \frac{\mathcal{A}(\tau)}{\Gamma(n-\alpha)}\, \mathrm{d}\tau.\end{aligned}$$

where $n = [\alpha] + 1$ *and* $\mathfrak{f}^n = \left(\frac{1}{\Psi'(\mathrm{t})}\frac{\mathrm{d}}{\mathrm{dt}}\right)^n$.

1.3.3 Necessary lemma, Theorems and properties

Lemma 1.1 ([15]). *For* $\alpha > 0$*, we obtain*

i) $\left({}^C D_{0^+}^{\alpha,\Psi} \; I_{0^+}^{\alpha,\Psi}\mathcal{A}\right)(\mathrm{t}) = \mathcal{A}(\mathrm{t})$ *for all functions* $\mathcal{A} \in \mathcal{C}(J,\mathbb{R})$.

ii) If $\mathcal{A} \in \mathcal{C}^n(J,\mathbb{R})$*, then* $I_{0^+}^{\alpha,\Psi}\; {}^C D_{0^+}^{\alpha,\Psi}\mathcal{A}(\mathrm{t}) = \mathcal{A}(\mathrm{t}) - \sum_{k=0}^{n-1}\frac{h_{\Psi}^{[k]}(0)}{k!}[\Psi(\mathrm{t}) - \Psi(0)]^k$.

Lemma 1.2 ([15]). *Consider the functions* $\mathcal{A},\ \Psi \in \mathcal{C}(J,\mathbb{R})$ *and* $\alpha > 0$*, we have*

i) $I_{0^+}^{\alpha,\Psi}(.)$ *is linear and bounded form* $\mathcal{C}(J,\mathbb{R})$ *to* $\mathcal{C}(J,\mathbb{R})$.

ii) $I_{0^+}^{\alpha,\Psi}\mathcal{A}(0) = \lim_{\mathrm{t}\to 0^+} I_{0^+}^{\alpha,\Psi}\mathcal{A}(\mathrm{t}) = 0$.

Lemma 1.3 ([15]). *Let* $\alpha,\ \beta > 0$ *and* $\mathcal{A} \in \mathcal{C}(J,\mathbb{R})$*. Then for each* $\mathrm{t} \in J$*, we have*

(C1) $I_{0^+}^{\alpha,\Psi}[\Psi(\mathrm{t}) - \Psi(0)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}[\Psi(\mathrm{t}) - \Psi(0)]^{\alpha+\beta-1}$,

(C2) if $\beta > n \in \mathbb{N}$*, then* ${}^C D_{0^+}^{\alpha,\Psi}[\Psi(\mathrm{t}) - \Psi(0)]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}[\Psi(\mathrm{t}) - \Psi(0)]^{\beta-\alpha-1}$,

(C3) $\forall k \in \{0,1,\cdots,n-1\}$*,* n *is a positive integer, then* ${}^C D_{0^+}^{\alpha,\Psi}[\Psi(\mathrm{t}) - \Psi(0)]^k = 0$,

(C4) for any constant ρ*, we always have* ${}^C D_{0^+}^{\alpha,\Psi}\rho = 0$.

Remark 1.1 ([15]). *The fractional derivative* ${}^c D_{0^+}^{\alpha,\Psi}$ *interpolate the following fractional derivatives: Caputo* ($\Psi(\mathrm{t}) = \mathrm{t}$)*, Caputo-Hadamard* ($\Psi(\mathrm{t}) = \ln \mathrm{t}$)*, Caputo–Erdély–Kober* ($\Psi(\mathrm{t}) = \mathrm{t}^\sigma,\ \sigma > 0$) *and Caputo–Katugampola* ($\Psi(\mathrm{t}) = \frac{\mathrm{t}^\sigma}{\sigma},\ \sigma > 0$).

1.4 Kuratowski Measure of Noncompactness

To proceed, we introduce fundamental notions associated with the measure of noncompactness. Let $\mathcal{B}_{\mathbb{X}}$ denote the family of all bounded subsets of a metric space $\mathbb{X}$.

Definition 1.4 ([28]). *A mapping* $\mathcal{S} : \mathcal{B}_{\mathbb{X}} \to [0,\infty)$ *is defined as a measure of noncompactness in* $\mathbb{X}$ *if it satisfies the following properties for every* $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}_{\mathbb{X}}$*:*

(a) Regularity: $\mathcal{S}(\mathcal{B}) = 0$ *if and only if* $\mathcal{B}$ *is relatively compact.*

(b) Invariance under closure: $\mathcal{S}(\mathcal{B}) = \mathcal{S}(\overline{\mathcal{B}})$.

(c) Semi-additivity: $\mathcal{S}(\mathcal{B}_1 \cup \mathcal{B}_2) = \max\{\mathcal{S}(\mathcal{B}_1), \mathcal{S}(\mathcal{B}_2)\}$.

Definition 1.5 ([28]). *Let* $\mathbb{X}$ *be a Banach space. The Kuratowski measure of non-compactness is a function* $\mathcal{S} : \mathcal{B}_{\mathbb{X}} \to [0, \infty)$ *defined as*

$$\mathcal{S}(\mathcal{M}) = \inf \left\{ \epsilon > 0 \mid \mathcal{M} \subset \bigcup_{j=1}^{m} \mathcal{M}_j, \ \mathrm{diam}(\mathcal{M}_j) \leq \epsilon \right\},$$

for any $\mathcal{M} \in \mathcal{B}_{\mathbb{X}}$.

The function μ *satisfies the following properties:*

- $\mathcal{S}(\mathcal{M}) = 0$ *if and only if* $\overline{\mathcal{M}}$ *is compact (i.e.,* $\mathcal{M}$ *is relatively compact).*
- $\mathcal{S}(\mathcal{M}) = \mathcal{S}(\overline{\mathcal{M}})$.
- *If* $\mathcal{M}_1 \subset \mathcal{M}_2$, *then* $\mathcal{S}(\mathcal{M}_1) \leq \mathcal{S}(\mathcal{M}_2)$.
- $\mathcal{S}(\mathcal{M}_1 + \mathcal{M}_2) \leq \mathcal{S}(\mathcal{M}_1) + \mathcal{S}(\mathcal{M}_2)$.
- $\mathcal{S}(c\mathcal{M}) = |c|\mathcal{S}(\mathcal{M})$ *for any* $c \in \mathbb{R}$.
- $\mathcal{S}(\mathrm{conv}\, \mathcal{M}) = \mathcal{S}(\mathcal{M})$.

1.5 Some Fixed Point Theorems

Theorem 1.1 (**Banach's Fixed Point Theorem** [54]). *Let* $\mathcal{B}$ *be a closed and non-empty subset of a Banach space* $\mathbb{X}$. *If* $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ *is a contraction mapping, then* $\mathcal{T}$ *admits a unique fixed point in* $\mathcal{B}$.

Theorem 1.2 (**Schaefer's Fixed Point Theorem** [54]). *Let* $\mathbb{X}$ *be a Banach space, and consider a completely continuous operator* $\mathcal{T} : \mathbb{X} \to \mathbb{X}$. *If the set*

$$\mathcal{B} = \{\mathrm{x} \in \mathbb{X} \mid \mathrm{x} = \mathrm{z}\mathcal{T}\mathrm{x} \text{ for some } \mathrm{z} \in (0, 1)\}$$

is bounded, then $\mathcal{T}$ *admits at least one fixed point in* $\mathbb{X}$.

Theorem 1.3 (**Krasnoselskii's Fixed Point Theorem** [54]). *Let $\mathcal{B}$ be a nonempty, closed, and convex subset of a Banach space $\mathbb{X}$. Consider two operators $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{B} \to \mathbb{X}$ satisfying the following conditions:*

1) *For any $\mathrm{x}, \mathrm{y} \in \mathcal{B}$, the combination $\mathcal{T}_1\mathrm{x} + \mathcal{T}_2\mathrm{y}$ remains within $\mathcal{B}$;*

2) *The operator $\mathcal{T}_1$ is compact and continuous;*

3) *The operator $\mathcal{T}_2$ is a contraction mapping.*

Then, there exists at least one fixed point $\mathrm{x} \in \mathcal{B}$ such that

$$\mathrm{x} = \mathcal{T}_1\mathrm{x} + \mathcal{T}_2\mathrm{x}.$$

Theorem 1.4 (**Dhage's Fixed-Point Theorem** [50]). *Let $\mathcal{B}$ be a nonempty, closed, convex, and bounded subset of a Banach algebra $(\mathbb{X}, \|\cdot\|)$. Consider two operators $\mathcal{T}_1 : \mathbb{X} \to \mathbb{X}$ and $\mathcal{T}_2 : \mathcal{B} \to \mathbb{X}$ satisfying the following conditions:*

1) *$\mathcal{T}_1$ is a Lipschitz operator with a Lipschitz constant denoted by $\mathfrak{a}^*$.*

2) *$\mathcal{T}_2$ is completely continuous.*

3) *For every $\mathrm{y} \in \mathcal{B}$, if $\mathrm{x} = \mathcal{T}_1\mathrm{x}\mathcal{T}_2\mathrm{y}$, then $\mathrm{x} \in \mathcal{B}$.*

4) *The condition $\mathfrak{a}^*\mathfrak{b}^* < 1$ holds, where $\mathfrak{b}^*$ is given by*

$$\mathfrak{b}^* = \|\mathfrak{b}(\mathcal{B})\| = \sup\{\|\mathfrak{b}(\mathrm{y})\| : \mathrm{y} \in \mathcal{B}\}.$$

Under these assumptions, the operator equation

$$\mathcal{T}_1\mathrm{x}\mathcal{T}_2\mathrm{x} = \mathrm{x}$$

admits at least one solution in $\mathcal{B}$.

Theorem 1.5 (**Dhage's Fixed Point Theorem for Three Operators** [51]). *Let $\mathcal{B}$ be a closed, convex, bounded, and nonempty subset of a Banach algebra $(\mathbb{X}, \|\cdot\|)$. Consider the operators $\mathcal{P}, \mathcal{R} : \mathbb{X} \to \mathbb{X}$ and $\mathcal{Q} : \mathcal{B} \to \mathbb{X}$ satisfying the following conditions:*

1) *$\mathcal{P}$ and $\mathcal{R}$ are Lipschitz continuous with constants η_1 and η_2, respectively.*

2) *$\mathcal{Q}$ is compact and continuous.*

3) *For any* $\mathrm{y} \in \mathcal{B}$*, if* x *satisfies*

$$\mathrm{x} = \mathcal{P}\mathrm{x}\mathcal{Q}\mathrm{y} + \mathcal{R}\mathrm{x},$$

then it follows that $\mathrm{x} \in \mathcal{B}$.

4) *The inequality* $\eta_1 \mathfrak{s} + \eta_2 < 1$ *holds, where*

$$\mathfrak{s} = \sup\{\|\mathcal{Q}(\mathrm{y})\| : \mathrm{y} \in \mathcal{B}\}.$$

Then, the operator equation

$$\mathcal{P}\mathrm{x}\mathcal{Q}\mathrm{x} + \mathcal{R}\mathrm{x} = \mathrm{x}$$

admits at least one solution in $\mathcal{B}$.

Theorem 1.6 (**Darbo's Fixed Point Theorem** [53])**.** *Let* $\mathcal{B}$ *be a non-empty, closed, bounded, and convex subset of a Banach space* $\mathbb{X}$*. Suppose that* $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ *is a continuous operator satisfying the following condition for any non-empty subset* $C \subseteq \mathcal{B}$:

$$\mathcal{S}(\mathcal{T}(C)) \leq k\mathcal{S}(C), \tag{1.1}$$

where $\mathcal{S}$ *denotes the Kuratowski measure of noncompactness, and* $0 \leq k < 1$*. Then, the mapping* $\mathcal{T}$ *admits at least one fixed point in* $\mathcal{B}$.

chapter 2

Nonlocal Langevin Pantograph Ψ-Caputo Fractional Problem in Banach Spaces

2.1 Introduction

In the present chapter, we prove some results concerning the existence of solutions for a class of nonlinear Langevin fractional pantograph systems with nonlocal conditions. This study aims to prove the uniqueness of the solution using Banach's fixed-point theorem and to confirm the existence of solutions using Schaefer's and Darbo's fixed-point theorems. Darbo's theorem, in particular, is useful due to its reliance on the measure of noncompactness for the Ψ-Caputo type of the following problem. There are numerous books and articles focused on linear and nonlinear problems for fractional differential equations involving different kinds of fractional derivatives. One can refer to [1, 2, 9, 24, 26, 30] for instance and references therein.

In this section, we prove the uniqueness of the solution using Banach's fixed-point theorem and to confirm the existence of solutions using Schaefer's and Darbo's fixed-point theorems. Darbo's theorem, in particular, is interesting for using the "measure of noncompactness" for the Ψ-Caputo type of the following problem:

[1] **H. Bouzid**, A. Benali, A. Salim and I. M. Erhan, A study on some classes of pantograph Langevin fractional differential problems with nonlocal conditions, *J Math Sci.* (2025), 1-18.

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\Psi}\left({}^{C}D_{0^+}^{\beta,\Psi}+\mu\right)\mathrm{x}(\mathrm{t})=\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})), \quad \mathrm{t}\in J=[0,b], \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0, \\ \mathfrak{s}_1\mathrm{x}(\mathrm{t})\big|_{\mathrm{t}=\varepsilon_1}+\mathfrak{s}_2\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_2}+\cdots+\mathfrak{s}_m\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_m}=0, \end{cases} \tag{2.1}$$

where ${}^{C}D_{0^+}^{\alpha,\Psi}$ and ${}^{C}D_{0^+}^{\beta,\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha,\beta\in(0,1]$, $\mu\in\mathbb{R}\setminus\{0\}$, $0<\zeta<1$, $\mathfrak{s}_i,\ i=1,\dots,m$, they are real constants that are not zero, $\varepsilon_i,\ i=1,\dots,m$, are pre-fixed points satisfying $0<\varepsilon_1\leq\cdots\leq\varepsilon_m<b$, $\Psi\in\mathcal{C}^1(J,\mathbb{R}^+)$ be an increasing differentiable function such that $\Psi'(\mathrm{t})\neq 0$, for all $\mathrm{t}\in J$ and $\mathfrak{f}: J\times\mathbb{R}^2\to\mathbb{R}$ is a continuous function.

2.2 First Problem with the Second Member Defined in ℝ.

2.2.1 Existence results

Definition 2.1. *A function $\mathrm{x}\in\mathcal{C}^2(J,\mathbb{R})$ that satisfies the equation (2.2) on J and verifies the specified nonlocal conditions is considered a solution to the fractional boundary value problem (2.2).*

Lemma 2.1. *Let $\alpha,\beta\in(0,1]$, $\mu\in\mathbb{R}\setminus\{0\}$, $\mathfrak{s}_i,\ i=1,\dots,m$, are non-zero real numbers, $\varepsilon_i,\ i=1,\dots,m$, are pre-fixed points satisfying $0<\varepsilon_1\leq\cdots\leq\varepsilon_m<b$ and $\omega\in\mathcal{C}(J,\mathbb{R})$. Then, the unique solution x of problem*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}+\mu\right]\mathrm{x}(\mathrm{t})=\omega(\mathrm{t}), \quad \mathrm{t}\in J, \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0, \\ \mathfrak{s}_1\mathrm{x}(\mathrm{t})\big|_{\mathrm{t}=\varepsilon_1}+\mathfrak{s}_2\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_2}+\cdots+\mathfrak{s}_m\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_m}=0, \end{cases} \tag{2.2}$$

is given by

$$\mathrm{x}(\mathrm{t}):=\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\omega(s)\mathrm{d}s-\mu\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\mathrm{x}(s)\mathrm{d}s$$

$$+\frac{\Phi(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\sum_{i=1}^{m}\mathfrak{s}_i\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\mathrm{x}(s)\mathrm{d}s$$
$$-\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\sum_{i=1}^{m}\mathfrak{s}_i\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\omega(s)\mathrm{d}s. \quad (2.3)$$

Proof. Let $\mathrm{x}\in\mathcal{C}^2(J,\mathbb{R})$ be a solution of the problem (2.2), then by using Lemma 1.1, we have

$$\mathrm{x}(\mathrm{t}):=I_{0^+}^{\alpha+\beta,\Psi}\omega(\mathrm{t})+\varsigma_0\frac{[\Psi(\mathrm{t})-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}-\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\mathrm{t})+\varsigma_1, \quad (2.4)$$

such that $\varsigma_0,\varsigma_1\in\mathbb{R}$. By the condition $\mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0$, we obtain $\varsigma_1:=0$.

Next, we substitute $\mathrm{t}=\varepsilon_i$ into (2.4) with $\varsigma_1=0$. Then we multiply $\mathfrak{s}_i$ to both sides, we obtain

$$\mathfrak{s}_i\mathrm{x}(\varepsilon_i)=\mathfrak{s}_i I_{0^+}^{\alpha+\beta,\Psi}\omega(\varepsilon_i)+\mathfrak{s}_i\varsigma_0\frac{[\Psi(\varepsilon_i)-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}-\mathfrak{s}_i\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\varepsilon_i), \quad (2.5)$$

Then by using condition $\mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=\sum_{i=1}^{m}\mathfrak{s}_i\mathrm{x}(\mathrm{t})\Big|_{\mathrm{t}=\varepsilon_i}$, we have

$$\varsigma_0 \quad := \quad \frac{\Gamma(\beta+1)}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\left(\mu\sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\beta,\Psi}\mathrm{x}(\varepsilon_i)-\sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\alpha+\beta,\Psi}\omega(\varepsilon_i)\right). \quad (2.6)$$

Finally, replacing these constants into (2.4), we get (2.3).

Reversely, let us now prove that if (2.3) satisfies Eq (2.2), then the aforementioned equation can be formulated as

$$\mathrm{x}(\mathrm{t}):=I_{0^+}^{\alpha+\beta,\Psi}\omega(\mathrm{t})+\frac{\Gamma(\beta+1)}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}$$
$$\times\left(\mu\sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\beta,\Psi}\mathrm{x}(\varepsilon_i)-\sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\alpha+\beta,\Psi}\omega(\varepsilon_i)\right)I_{a_1^+}^{\beta,\Psi}1-\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\mathrm{t}).$$

Using the Ψ-Caputo derivative, ${}^{C}D_{0^+}^{\beta,\Psi}$ on both sides and applying Lemma 1.1, we obtain

$$
\begin{aligned}
{}^{C}D_{0^+}^{\beta,\Psi}\mathrm{x}(\mathrm{t}) :=& I_{0^+}^{\alpha,\Psi}\omega(\mathrm{t}) + \frac{\Gamma(\beta+1)}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}} \\
&\times\left(\mu\sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\beta,\Psi}\mathrm{x}(\varepsilon_i) - \sum_{i=1}^{m}\mathfrak{s}_i I_{0^+}^{\alpha+\beta,\Psi}\omega(\varepsilon_i)\right) - \varPhi\mathrm{x}(\mathrm{t}).
\end{aligned}
$$

Applying ${}^{C}D_{0^+}^{\alpha,\Psi}$ to the above equation again, we obtain

$$
{}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi} + \mu\right]\mathrm{x}(\mathrm{t}) = \omega(\mathrm{t}).
$$

Finally, it is evident that the function in (2.3) satisfies the related nonlocal conditions. This completes the proof. □

Next, we present the solution for the problem (2.1).

Lemma 2.2. *Let* $j=1,2,\ 0<\alpha,\beta\le 1,\ \mu\in\mathbb{R}\setminus\{0\}$, $\mathfrak{s}_i,\ i=1,\dots,m$, *they represent non-zero real numbers,* $\varepsilon_i,\ i=1,\dots,m$, *are pre-fixed points satisfying* $0<\varepsilon_1\le\cdots\le\varepsilon_m<b$ *and let* $\mathfrak{f}: J\times\mathbb{R}^2\to\mathbb{R}$ *be continuous functions. Then* x *satisfies the problem* (2.1) *if and only if* x(t) *is the fixed point of the operator* $\mathcal{T}:\mathcal{C}(J,\mathbb{R})\longrightarrow\mathcal{C}(J,\mathbb{R})$ *defined by:*

$$
\mathcal{T}\mathrm{x}(\mathrm{t}) := \varPhi\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\sum_{i=1}^{m}\mathfrak{s}_i\varPhi\mathrm{x}(\varepsilon_i),
$$

where

$$
\begin{aligned}
\varPhi\mathrm{x}(\mathrm{t}) :=& \int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s))\mathrm{d}s \\
&-\mu\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\mathrm{x}(s)\mathrm{d}s,
\end{aligned}
$$

The next result relies on the application of the Banach fixed-point theorem. Furthermore, we assume the following conditions for this outcome:

(Ax_1) The functions $\mathfrak{f}: J\times\mathbb{R}^2\to\mathbb{R}$, $i=1,2$, are continuous.

$(Ax_{\mathbf{2}})$ There exist positive functions $\mathfrak{p} \in \mathcal{C}(J, \mathbb{R}^+)$ such that

$$|\mathfrak{f}(\mathrm{t}, \mathrm{x}, \tilde{\mathrm{x}}) - \mathfrak{f}(\mathrm{t}, \mathrm{y}, \tilde{\mathrm{y}})| \leq \mathfrak{p}(\mathrm{t})(|\mathrm{x} - \mathrm{y}| + |\tilde{\mathrm{x}} - \tilde{\mathrm{y}}|),$$

for all $\mathrm{t} \in J$ and $\mathrm{x}, \mathrm{y}, \tilde{\mathrm{x}}, \tilde{\mathrm{y}} \in \mathbb{R}$, where

$$\mathfrak{p}^* = \sup_{\mathrm{t} \in J} \mathfrak{p}(\mathrm{t}).$$

$(Ax\mathbf{3})$ There exists a positive constant $\mathcal{F}$, where $|\mathfrak{f}(\mathrm{t}, \cdot, \cdot)| < \mathcal{F}$, for all $\mathrm{t} \in J$ and $\cdot \in \mathbb{R}$.

For the sake of clarity, we denote

$$\begin{aligned}
\Lambda_1 &= \frac{[(\Psi(b) - \Psi(0)]^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}, \\
\Lambda_2 &= \frac{|\mu|[(\Psi(b) - \Psi(0)]^{\beta}}{\Gamma(\beta+1)}, \\
\Lambda_3 &= \max_{1 \leq i \leq m} |\mathfrak{s}_i| \frac{((\Psi(b) - \Psi(0))^{\beta} \sum_{i=1}^{m} (\Psi(\varepsilon_i) - \Psi(0))^{\alpha+\beta}}{\mathcal{K}\ \Gamma(\alpha+\beta+1)}, \\
\mathscr{W} &= \mathfrak{p}^* \left(\Lambda_1 + \Lambda_3\right) + \left(1 + \frac{\max\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}{\min\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}\right) \Lambda_2, \\
\mathcal{E} &= (\mathtt{M}_2^* + \mathtt{M}_3^*) \left(\Lambda_1 + \Lambda_3\right) + \left(1 + \frac{\max\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}{\min\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}\right) \Lambda_2,
\end{aligned}$$

where $\mathcal{K} = \min\limits_{1 \leq i \leq m} |\mathfrak{s}_i| \sum\limits_{i=1}^{m} [\Psi(\varepsilon_i) - \Psi(0)]^{\beta}$.

Theorem 2.1. *Suppose that (Ax_{1})–(Ax_{3}) hold. If*

$$\mathscr{W} < 1, \tag{2.7}$$

then, the problem (2.1) has a unique solution on J.

Proof. Setting

$$\delta \geq \frac{\mathcal{F}(\Lambda_3 + \Lambda_1)}{1 - \left(1 + \frac{\max\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}{\min\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}\right) \Lambda_2},$$

with

$$0 \leq \left(1 + \frac{\max\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}{\min\limits_{1 \leq i \leq m} |\mathfrak{s}_i|}\right) \Lambda_2 < 1.$$

We show that $\mathcal{T}\mathcal{B}_\delta \subset \mathcal{B}_\delta$, where

$$\mathcal{B}_\delta = \left\{ \mathrm{x} \in \mathcal{C}^2(J,\mathbb{R}) : \|\mathrm{x}\|_\infty \leq \delta \right\}.$$

For $\mathrm{x} \in \mathcal{B}_\delta$ and for each $\mathrm{t} \in J$, from the definition of $\mathcal{T}$ and hypothesis $(Ax_{\mathbf{1}})$ with $(Ax_{\mathbf{3}})$, we obtain

$$\begin{aligned}
&|\mathcal{T}\mathrm{x}(\mathrm{t})| \\
&\leq \left| \varPhi\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^\beta}{\sum_{i=1}^m \mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^\beta} \sum_{i=1}^m \mathfrak{s}_i \varPhi\mathrm{x}(\varepsilon_i) \right| \\
&\quad \leq \mathcal{F}\frac{[(\Psi(b)-\Psi(0)]^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\mu|\frac{[(\Psi(b)-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}\delta \\
&\quad + \frac{[(\Psi(b)-\Psi(0)]^{\beta}}{\sum_{i=1}^m |\mathfrak{s}_i|[\Psi(\varepsilon_i)-\Psi(0)]^\beta} \sum_{i=1}^m |\mathfrak{s}_i| \left[\mathcal{F}\frac{[(\Psi(\varepsilon_i)-\Psi(0)]^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\mu|\frac{[(\Psi(\varepsilon_i)-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}\delta \right] \\
&\leq \mathcal{F}\frac{[(\Psi(b)-\Psi(0)]^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\mu|\frac{[(\Psi(b)-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}\delta \\
&\quad + \frac{[(\Psi(b)-\Psi(0)]^{\beta}}{\min\limits_{1\leq i\leq m} |\mathfrak{s}_i| \sum_{i=1}^m [\Psi(\varepsilon_i)-\Psi(0)]^\beta} \max_{1\leq i\leq m} |\mathfrak{s}_i| \sum_{i=1}^m \left[\mathcal{F}\frac{[(\Psi(\varepsilon_i)-\Psi(0)]^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\mu|\frac{[(\Psi(\varepsilon_i)-\Psi(0)]^{\beta}}{\Gamma(\beta+1)}\delta \right] \\
&\leq \mathcal{F}\left[\Lambda_1 + \Lambda_3\right] + \left(1 + \frac{\max\limits_{1\leq i\leq m} |\mathfrak{s}_i|}{\min\limits_{1\leq i\leq m} |\mathfrak{s}_i|} \right) \Lambda_2 \delta \\
&\leq \delta.
\end{aligned}$$

Hence,

$$\|\mathcal{T}\mathrm{x}\|_\infty \leq \delta,$$

which implies that $\mathcal{T}\mathcal{B}_\delta \subset \mathcal{B}_\delta$. Let for $\mathrm{x}, \tilde{\mathrm{x}} \in \mathcal{C}^2(J,\mathbb{R})$ and for any $\mathrm{t} \in J$. By $(Ax_{\mathbf{2}})$, we get

$$\begin{aligned}
&|\varPhi(\mathrm{x})(\mathrm{t}) - \varPhi(\tilde{\mathrm{x}})(\mathrm{t})| \\
&\quad \leq \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |\mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s)) - \mathfrak{f}(s,\tilde{\mathrm{x}}(s),\tilde{\mathrm{x}}(\zeta s))| \mathrm{d}s
\end{aligned}$$

$$+|\mu|\int_0^{t}\frac{\Psi'(s)(\Psi(t)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}|x(s)-\tilde{x}((s)|ds$$
$$\leq\left(\frac{\mathfrak{p}^*(\Psi(t)-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(t)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|x-\tilde{x}\|_\infty.$$

Therefore,

$$|\mathcal{T}(x)(t)-\mathcal{T}(\tilde{x})(t)|$$
$$\leq|\Phi(x)(t)-\Phi(\tilde{x})(t)|+\frac{(\Psi(t)-\Psi(0))^{\beta}}{\sum_{i=1}^{m}|\mathfrak{s}_i|(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\left(|\sum_{i=1}^{m}\mathfrak{s}_i\Phi(x)(\varepsilon_i)\right.$$
$$\left.-\sum_{i=1}^{m}\mathfrak{s}_i\Phi(\tilde{x})(\varepsilon_i)|\right).$$

We conclude that

$$|\mathcal{T}(x)(t)-\mathcal{T}(\tilde{x})(t)|$$
$$\leq\left(\frac{2\mathfrak{p}^*(\Psi(t)-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(t)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|x,\tilde{x}\|_\infty$$
$$+\frac{(\Psi(t)-\Psi(0))^{\beta}}{\sum_{i=1}^{m}|\mathfrak{s}_i|(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\left(\sum_{i=1}^{m}|\mathfrak{s}_i|\frac{2\mathfrak{p}^*(\Psi(\varepsilon_i)-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right.$$
$$\left.+\sum_{i=1}^{m}|\mathfrak{s}_i|\frac{|\mu|(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|x-\tilde{x}\|_\infty.$$

After taking the supremum over J and simplifying, we get

$$\|\mathcal{T}(x)-\mathcal{T}(\tilde{x})\|_\infty\leq\mathscr{W}\|x-\tilde{x}\|_\infty. \tag{2.8}$$

Since $\mathscr{W}<1$, $\mathcal{T}$ is a contraction operator. Consequently by utilizing Banach's fixed point theorem, the problem (2.1) has a unique solution. □

The next existence result is based on the Schaefer's fixed point theorem.

Theorem 2.2. *Let $\mathfrak{f}: J\times\mathbb{R}^2\to\mathbb{R}$ be a continuous function satisfying :*

(Ax_4) *There exists positive functions $\mathtt{M}_1,\mathtt{M}_2,\mathtt{M}_3\in\mathcal{C}(J,\mathbb{R}^+)$, for all $t\in J$ and for any $x,y\in\mathbb{R}$, we have*

$$|\mathfrak{f}(t,x,\ y)|\leq\mathtt{M}_1(t)+\mathtt{M}_2(t)|x|+\mathtt{M}_3(t)|y|, \tag{2.9}$$

such that

$$\mathtt{M}_i^* = \sup_{\mathrm{t}\in J} \mathtt{M}_i(\mathrm{t}), \;\; i = 1,2,3.$$

If

$$\mathcal{E} < 1. \tag{2.10}$$

Then the problem (2.1) *has at least one mild solution on* J.

Proof. In this proof, we will use Schaefer's fixed point theorem. The proof will be given in several steps.

Step 1: We show that $\mathcal{T}$ is continuous. Let $\{(\mathrm{x}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^2(J,\mathbb{R})$ that converges to a point $\mathrm{x} \in \mathcal{C}^2(J,\mathbb{R})$. Then, for each $\mathrm{t} \in J$, we have

$$\begin{aligned}
&|\mathcal{T}(\mathrm{x}_n)(\mathrm{t}) - \mathcal{T}(\mathrm{x})(\mathrm{t})| \\
&\leq \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |\mathfrak{f}(s,\mathrm{x}_n(s),\mathrm{x}_n(\zeta s)) - \mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s))| \mathrm{d}s \\
&\quad + |\mu| \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)} |\mathrm{x}_n(s) - \mathrm{x}(s)| \mathrm{d}s + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}} \\
&\quad \times \left(\sum_{i=1}^{m} |\mathfrak{s}_i| \int_0^{\varepsilon_i} \frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} |\mathfrak{f}(s,\mathrm{x}_n(s),\mathrm{x}_n(\zeta s)) - \mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s))| \mathrm{d}s \right. \\
&\quad \left. + |\mu| \sum_{i=1}^{m} |\mathfrak{s}_i| \int_0^{\varepsilon_i} \frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)} |\mathrm{x}_n(s) - \mathrm{x}(s)| \mathrm{d}s \right).
\end{aligned}$$

Since $\mathrm{x}_n \longrightarrow \mathrm{x}$ as $n \to 0$ and $\mathcal{T}$, is continuous, by the Lebesgue dominated convergence theorem

$$\|\mathcal{T}\mathrm{x}_n - \mathcal{T}\mathrm{x}\|_\infty \longrightarrow 0, \text{ as } n \longrightarrow 0.$$

Step 2: We prove that the set $\mathcal{T}(\mathcal{B}_{\mathfrak{e}})$ is a uniformly bounded in $\mathcal{B}_{\mathfrak{e}}$.

Define the set

$$\mathcal{B}_{\mathfrak{e}} = \left\{ \mathrm{x} \in \mathcal{C}^2(J,\mathbb{R}) \; : \; \|\mathrm{x}\|_\infty \leq \mathfrak{e} \right\},$$

whit $\mathfrak{e}$ satisfying

$$\mathfrak{e} \geq \frac{\mathtt{M}_1^*\left(\Lambda_1+\Lambda_3\right) + \mathtt{N}_1^*\left(\nabla_1+\nabla_3\right)}{1-\mathcal{E}}.$$

Applying condition $(Ax_{\mathbf{3}})$, for any $\mathrm{x} \in \mathcal{B}_{\mathfrak{e}}$ and $\mathrm{t} \in J$, we have

$$|\Phi \mathrm{x}(\mathrm{t})| \leq \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+(\mathtt{M}_2^*+\mathtt{M}_3^*)\mathfrak{e}\right)}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}\mathfrak{e}}{\Gamma(\beta+1)}.$$

Therefore,

$$\begin{aligned}
|\mathcal{T}\mathrm{x}(\mathrm{t})| &\leq |\Phi \mathrm{x}(\mathrm{t})|+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\sum_{i=1}^{m}|\mathfrak{s}_i||\Phi \mathrm{x}(\varepsilon_i)| \\
&\leq \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+(\mathtt{M}_2^*+\mathtt{M}_3^*)\mathfrak{e}\right)}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}\mathfrak{e}}{\Gamma(\beta+1)} \\
&\quad+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\left[\sum_{i=1}^{m}|\mathfrak{s}_i|\frac{(\Psi(\varepsilon_i)-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+(\mathtt{M}_2^*+\mathtt{M}_3^*)\mathfrak{e}\right)}{\Gamma(\alpha+\beta+1)}\right. \\
&\quad\left.+\sum_{i=1}^{m}|\mathfrak{s}_i|\frac{|\mu|(\Psi(\varepsilon_i)-\Psi(0))^{\beta}\mathfrak{e}}{\Gamma(\beta+1)}\right],
\end{aligned}$$

which implies that

$$\|\mathcal{T}\mathrm{x}\|_{\infty} \leq \mathtt{M}_1^*\left(\Lambda_1+\Lambda_3\right)+\mathcal{E}\mathfrak{e}.$$

Thus $\|\mathcal{T}\mathrm{x}\|_{\infty} \leq +\infty$, for all $\mathrm{x} \in \mathcal{B}_{\mathfrak{e}}$. This shows that $\mathcal{T}$ is uniformly bounded on $\mathcal{B}_{\mathfrak{e}}$.
Step 3: Next, we demonstrate that $\mathcal{T}$ is an equicontinuous set in $\mathcal{B}_{\mathfrak{e}}$. Let $\mathrm{t}_1, \mathrm{t}_2 \in [0,b]$ with $\mathrm{t}_1 < \mathrm{t}_2$ and $\mathrm{x} \in \mathcal{B}_{\mathfrak{e}}$. Then we have

$$\begin{aligned}
&|\mathcal{T}\mathrm{x}(\mathrm{t}_2)-\mathcal{T}\mathrm{x}(\mathrm{t}_1)| \\
&\leq \int_0^{\mathrm{t}_1}\frac{\Psi'(s)[(\Psi(\mathrm{t}_2)-\Psi(s))^{\alpha+\beta-1}-(\Psi(\mathrm{t}_1)-\Psi(s))^{\alpha+\beta-1}]}{\Gamma(\alpha+\beta)}|\mathfrak{f}(s,\mathrm{x}(s),\mathrm{y}(\zeta s))|\mathrm{d}s \\
&\quad+\int_{\mathrm{t}_1}^{\mathrm{t}_2}\tfrac{\Psi'(s)(\Psi(\mathrm{t}_2)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|\mathfrak{f}(s,\mathrm{x}(s),y(\zeta s))|\mathrm{d}s \\
&\quad+\int_0^{\mathrm{t}_1}\frac{|\lambda|\Psi'(s)[(\Psi(\mathrm{t}_2)-\Psi(s))^{\beta-1}-(\Psi(\mathrm{t}_1)-\Psi(s))^{\beta-1}]}{\Gamma(\beta)}|\mathrm{x}(s)|\mathrm{d}s \\
&\quad+\int_{\mathrm{t}_1}^{\mathrm{t}_2}\frac{|\lambda|\Psi'(s)(\Psi(\mathrm{t}_2)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}|\mathrm{x}(s)|\mathrm{d}s \\
&\quad+\frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta}-(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}]}{\mathcal{K}} \\
&\quad\times\left(\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|\mathfrak{f}(s,\mathrm{x}(s),y(\zeta s))|\mathrm{d}s\right. \\
&\quad\left.+|\mu|\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}|\mathrm{x}(s)|\mathrm{d}s\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&|\mathcal{T}\mathrm{x}(\mathrm{t}_2) - \mathcal{T}\mathrm{x}(\mathrm{t}_1)| \\
&\leq |\Phi\mathrm{x}(\mathrm{t}_2) - \Phi\mathrm{x}(\mathrm{t}_1)| \\
&\quad + \frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta} - (\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}]}{\mathcal{K}} \sum_{i=1}^{m} |\mathfrak{s}_i||\Phi\mathrm{x}(\varepsilon_i)| \qquad (2.11)\\
&\leq \frac{[(\Psi(\mathrm{t}_1)-\Psi(0))^{\alpha+\beta} - (\Psi(\mathrm{t}_2)-\Psi(0))^{\alpha+\beta}]}{\Gamma(\alpha+\beta+1)}[\mathtt{M}_1^* + (\mathtt{M}_2^* + \mathtt{M}_3^*)\mathfrak{c}] \\
&\quad + \left(\frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta} - (\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}] \max\limits_{1\leq i\leq m}\{|\mathfrak{s}_i|\} \sum\limits_{i=1}^{m} (\Psi(\varepsilon_i)-\Psi(0))^{\alpha+\beta}}{\mathcal{K}\ \ \Gamma(\alpha+\beta+1)} \right) \\
&\qquad \times [\mathtt{M}_1^* + (\mathtt{M}_2^* + \mathtt{M}_3^*)\mathfrak{c}]. \qquad (2.12)
\end{aligned}
$$

Then, when $\mathrm{t}_1 \to \mathrm{t}_2$, the right-hand side of the inequalities (2.12) tend to zero. Therefore, the operator $\mathcal{T}$ is equicontinuous, and thus, the operator $\mathcal{T}$ is completely continuous and finally $\mathcal{T}$ is compact on $\mathcal{B}_k$.

Step 4: We show that the set

$$
\mathcal{B} = \{\mathrm{x} \in \mathcal{C}^2(\mathrm{T}, \mathbb{R}) : \ \mathrm{x} = \delta\mathcal{T}\mathrm{x} \text{ for some } 0 < \delta < 1\}
$$

is bounded. Let $\mathrm{x} \in \mathcal{B}$ and $\delta \in (0,1)$ be such that $\mathrm{x} = \delta\mathcal{T}\mathrm{x}$. By Step 2, then for each $\mathrm{t} \in T$, we have

$$
\begin{aligned}
\mathcal{T}\mathrm{x}(\mathrm{t}) &\leq (\mathtt{M}_1^* + (\mathtt{M}_2^* + \mathtt{M}_3^*)\|\mathrm{x}\|_{\infty})\,(\Lambda_1 + \Lambda_3) \\
&\quad + \left(1 + \frac{\max\limits_{1\leq i\leq m} |\mathfrak{s}_i|}{\min\limits_{1\leq i\leq m} |\mathfrak{s}_i|}\right) \Lambda_2\|\mathrm{x}\|_{\infty}.
\end{aligned}
$$

As $\delta \in (0,1)$ then $\mathrm{x} < \mathcal{T}\mathrm{x}$, and

$$
\begin{aligned}
\|\mathrm{x}\|_{\infty} &< \|\mathcal{T}\mathrm{x}\|_{\infty} \\
&\leq (\mathtt{M}_1^* + (\mathtt{M}_2^* + \mathtt{M}_3^*)\|\mathrm{x}\|_{\infty}) \\
&\quad \times (\Lambda_1 + \Lambda_3) + \left(1 + \frac{\max\limits_{1\leq i\leq m} |\mathfrak{s}_i|}{\min\limits_{1\leq i\leq m} |\mathfrak{s}_i|}\right) \Lambda_2\|\mathrm{x}\|_{\infty}. \qquad (2.13)
\end{aligned}
$$

Thus, we obtain

$$\|x\|_\infty \leq \frac{M_1^*(\Lambda_1 + \Lambda_3)}{1 - \mathscr{W}},$$

which implies that $\mathcal{B}_\delta$ is bounded. By the Schaefer's fixed point theorem, the operator $\mathcal{T}$ has at least a fixed point. Hence, the problem (2.1) has at least one solution $[0, b]$.

□

2.3 Second Problem with the Second Member Defined in a Banach Space.

This section is devoted to the study of existence of a problem similar to problem (2.1) in a Banach space

2.3.1 Existence results

Consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{\alpha,\Psi} \left({}^C D_{0^+}^{\beta,\Psi} + \mu \right) x(t) = \mathfrak{f}(t, x(t), x(\zeta t)), \quad t \in J = [0, b], \\ x(t) \mid_{t=0} = 0, \\ \mathfrak{s}_1 x(t)\big|_{t=\varepsilon_1} + \mathfrak{s}_2 x(t)\Big|_{t=\varepsilon_2} + \cdots + \mathfrak{s}_m x(t)\Big|_{t=\varepsilon_m} = 0, \end{cases} \tag{2.14}$$

where ${}^C D_{0^+}^{\alpha,\Psi}$ and ${}^C D_{0^+}^{\beta,\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha, \beta \in (0, 1]$,, $\mu \in \mathbb{R} \setminus \{0\}$, $0 < \zeta < 1$, $\mathfrak{s}_i$ $i = 1, \ldots, m$, they are real constants that are not zero, ε_i, $i = 1, \ldots, m$, are pre-fixed points satisfying $0 < \varepsilon_1 \leq \cdots \leq \varepsilon_m < b$ and $\mathfrak{f} : J \times \Theta^2 \to \Theta$ are continuous functions.

Let $(\mathbb{E}, \|.\|)$ be a Banach space. By $\mathcal{C}(J, \mathbb{E})$ we denote the Banach space of all continuous functions from J into Θ with the norm

$$\|x\|_{\mathcal{C}(J,\mathbb{E})} = \sup\{\|x(t)\|_{\mathbb{E}} : \ t \in J\}.$$

Lemma 2.3. *Let* $0 < \alpha, \beta \leq 1$, $\mu \in \mathbb{R} \setminus \{0\}$, $\mathfrak{s}_i$, $i = 1, \ldots, m$, *are non-zero real numbers,* ε_i, $i = 1, \ldots, m$, *are pre-fixed points satisfying* $0 < \varepsilon_1 \leq \cdots \leq \varepsilon_m < b$ *and* *let* $\mathfrak{f} : J \times \Theta^2 \to \Theta$ *are continuous functions. Then* $\mathrm{x} \in \mathcal{C}^2(J, \Theta)$ *satisfies the problem* (2.14) *if and only if* x *is the fixed point of the operator* $\mathcal{T} : \mathcal{C}(J, \Theta) \longrightarrow \mathcal{C}(J, \Theta)$ *defined by:*

$$\mathcal{T}\mathrm{x}(\mathrm{t}) := \mu\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{\sum_{i=1}^{m} \mathfrak{s}_i(\Psi(\varepsilon_i) - \Psi(0))^\beta} \sum_{i=1}^{m} \mathfrak{s}_i \mu\mathrm{x}(\varepsilon_i),$$

where

$$\begin{aligned}\varPhi\mathrm{x}(\mathrm{t}) :=& \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \mathfrak{f}(s, \mathrm{x}(s), y(\zeta s))\mathrm{d}s \\ &- \mu \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\beta-1}}{\Gamma(\beta)} \mathrm{x}(s)\mathrm{d}s,\end{aligned}$$

Let us set the following conditions:

($Ax_{\mathbf{5}}$) The functions $\mathfrak{f} : J \times \mathbb{E}^2 \to \mathbb{E}$ is continuous.

($Ax_{\mathbf{6}}$) There exist positive functions $\mathfrak{p} \in \mathcal{C}(J, \mathbb{R}^+)$ such that

$$\|\mathfrak{f}(\mathrm{t}, \mathrm{x}, \tilde{\mathrm{x}}) - \mathfrak{f}(\mathrm{t}, \mathrm{y}, \tilde{\mathrm{y}})\|_{\mathbb{E}} \leq \mathfrak{p}(\mathrm{t})(\|\mathrm{x} - \mathrm{y}\|_{\mathbb{E}} + \|\tilde{\mathrm{x}} - \tilde{\mathrm{y}}\|_{\mathbb{E}}),$$

for all $\mathrm{t} \in J$ and $\mathrm{x}, \mathrm{y}, \tilde{\mathrm{x}}, \tilde{\mathrm{y}} \in \mathbb{E}$, where

$$\mathfrak{p}^* = \sup_{\mathrm{t}\in J} \mathfrak{p}(\mathrm{t}).$$

($Ax_{\mathbf{7}}$) For each $\mathrm{t} \in J$ and bounded sets $\mathcal{B}_1, \mathcal{B}_2 \in \mathbb{E}$, we have

$$\mathcal{S}(\mathfrak{f}(\mathrm{t}, \mathcal{B}_1, \mathcal{B}_2)) \leq \mathfrak{p}^*(\mathcal{S}(\mathcal{B}_1) + \mathcal{S}(\mathcal{B}_2)),$$

where $\mathcal{S}$ is a measure of noncompactness on the Banach space $\mathbb{E}$.

Remark 2.1. *It is worth noting that the hypotheses* ($Ax_{\mathbf{6}}$) *and* ($Ax_{\mathbf{7}}$) *are equivalent.*

Theorem 2.3. *Assume* ($\mathcal{H}5$)-($\mathcal{H}7$) *are verified. If*

$$\mathscr{W} < 1, \tag{2.15}$$

then, the problem (2.14) *has at least one solution.*

To prove the existence of solution of the problem (2.14), we will use the concept of measures of noncompactness and Darbo's fixed point theorem.

Proof. Consider the operator $\mathcal{T}$ defined on $\mathcal{C}(J,\mathbb{E})$ by such as

$$\mathcal{T}\mathrm{x}(\mathrm{t}) := \mu\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\sum_{i=1}^{m}\mathfrak{s}_i(\Psi(\varepsilon_i)-\Psi(0))^{\beta}}\sum_{i=1}^{m}\mathfrak{s}_i\mu\mathrm{x}(\varepsilon_i),$$

for any $\mathrm{x}\in\mathcal{C}^2(J,\mathbb{E})$, and $\mathrm{t}\in[0,b]$. In virtue of Lemma 2.3, a fixed point of $\mathcal{T}$ gives us the desired result.

For a better readability, we divide the proof in several steps.

Step 1: The operator $\mathcal{T}:\mathcal{C}(J,\mathbb{E})\longrightarrow\mathcal{C}(J,\mathbb{E})$ is continuous. Let $\{\mathrm{x}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(J,\mathbb{E})$ that converges to a point $\mathrm{x}\in\mathcal{E}(J,\mathbb{E})$. Then, for each $\mathrm{t}\in J$, we have

$$\begin{aligned}
&\|\mathcal{T}\mathrm{x}_n(\mathrm{t})-\mathcal{T}\mathrm{x}(\mathrm{t})\|_{\mathbb{E}}\\
&\leq\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|\mathfrak{f}(s,\mathrm{x}_n(s),\mathrm{x}_n(\zeta s))-\mathfrak{f}(s,\mathrm{x}(s),y(\zeta s))\|_{\mathbb{E}}\mathrm{d}s\\
&\quad+|\mu|\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\|\mathrm{x}_n(s)-\mathrm{x}(s)\|_{\mathbb{E}}\mathrm{d}s+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\\
&\quad\times\Bigg(\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|\mathfrak{f}(s,\mathrm{x}_n(s),\mathrm{x}_n(\zeta s))-\mathfrak{f}(s,\mathrm{x}(s),y(\zeta s))\|_{\mathbb{E}}\mathrm{d}s\\
&\quad+|\mu|\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\|\mathrm{x}_n(s)-\mathrm{x}(s)\|_{\mathbb{E}}\mathrm{d}s\Bigg)
\end{aligned}$$

Since $\mathrm{x}_n\longrightarrow\mathrm{x}$ as $n\to 0$ and $\mathcal{T}$, is continuous, by the Lebesgue dominated convergence theorem

$$\|\mathcal{T}\mathrm{x}_n-\mathcal{T}\mathrm{x}\|_{\mathcal{C}(J,\mathbb{E})}\longrightarrow 0,\ \text{as } n\longrightarrow 0.$$

This shows that $\mathcal{T}\mathrm{x}$ is a continuous operator on $\mathcal{C}(J,\mathbb{E})$.

Let

$$\gamma\geq\frac{\mathtt{M}_1^*(\Lambda_1+\Lambda_3)}{1-\mathcal{A}},$$

where

$$\mathtt{M}_1(\mathrm{t})=\|\mathfrak{f}(\mathrm{t},0,0)\|_{\mathcal{E}},$$

with

$$\mathtt{M}_1^* = \sup_{\mathrm{t}\in J} \mathtt{M}_1(\mathrm{t}), \ \mathtt{N}_1^* = \sup_{\mathrm{t}\in J} \mathtt{N}_1(\mathrm{t}).$$

Define

$$\mathcal{B}_\gamma = \left\{ \mathrm{x} \in \mathcal{C}^2(J,\Theta) \ : \ \|\mathrm{x}\|_{\mathcal{C}(J,\mathbb{E})} \leq \gamma \right\},$$

It is clear that $\mathcal{B}_\gamma$ is a bounded, closed, and convex subset of $\mathcal{C}^2(J,\mathbb{E})$.

Step 2: $\mathcal{T}(\mathcal{Q}_\gamma) \subset \mathcal{B}_\gamma$. Let $\mathrm{x} \in \mathcal{B}_\gamma$. If $\mathrm{t} \in J$. From hypothesis $(Ax_{\mathbf{5}})$, we have

$$\|\varPhi\mathrm{x}(\mathrm{t})\|_{\mathbb{E}} \leq \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+\mathfrak{p}^*\gamma\right)}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}\gamma}{\Gamma(\beta+1)}.$$

Therefore,

$$\begin{aligned}
\|\mathcal{T}\mathrm{x}(\mathrm{t})\|_{\mathbb{E}} &\leq \|\varPhi\mathrm{x}(\mathrm{t})\|_{\mathbb{E}} + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}} \sum_{i=1}^{m} |\mathfrak{s}_i| \|\varPhi\mathrm{x}(\varepsilon_i)\|_{\mathbb{E}} \\
&\leq \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+\mathfrak{p}^*\gamma\right)}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}\gamma}{\Gamma(\beta+1)} \\
&\quad + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}} \left[\sum_{i=1}^{m} |\mathfrak{s}_i| \frac{(\Psi(\varepsilon_i)-\Psi(0))^{\alpha+\beta}\left(\mathtt{M}_1^*+\mathfrak{p}^*\gamma\right)}{\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. + \sum_{i=1}^{m} |\mathfrak{s}_i| \frac{|\mu|(\Psi(\varepsilon_i)-\Psi(0))^{\beta}\gamma}{\Gamma(\beta+1)} \right].
\end{aligned}$$

which implies that

$$\|\mathcal{T}\mathrm{x}\|_{\mathcal{C}(J,\mathbb{E})} \leq \mathtt{M}_1^*\left(\Lambda_1+\Lambda_3\right) + \Lambda_2\gamma.$$

Consequently

$$\begin{aligned}
\|\mathcal{T}\mathrm{x}\|_{\mathcal{C}(J,\mathbb{E})} &= \sup_{t\in J} \|\mathcal{T}\mathrm{x}(\mathrm{t})\|_{\mathbb{E}} \\
&\leq \mathtt{M}_1^*\left(\Lambda_1+\Lambda_3\right) + \Lambda_2\gamma.
\end{aligned}$$

Thus $\|\mathcal{T}\mathrm{x}\|_{\mathcal{C}(J,\mathbb{E})} \leq \gamma$, for all $\mathrm{x} \in \mathcal{B}_\gamma$. So $\mathcal{T}(\mathcal{B}_\gamma) \subset \mathcal{B}_\gamma$.

Step 3: $\mathcal{T}(\mathcal{B}_\gamma)$ is equicontinuous. Let $\mathrm{t}_1, \mathrm{t}_2 \in [0,b]$ with $\mathrm{t}_1 < \mathrm{t}_2$ and $\mathrm{x} \in \mathcal{B}_\gamma$. By $(Ax_{\mathbf{5}})$ we have

$$\|\mathcal{T}\mathrm{x}(\mathrm{t}_2) - \mathcal{T}\mathrm{x}(\mathrm{t}_1)\|_{\mathbb{E}}$$

$$
\begin{aligned}
&\leq \int_0^{t_1} \frac{\Psi'(s)[(\Psi(t_2)-\Psi(s))^{\alpha+\beta-1}-(\Psi(t_1)-\Psi(s))^{\alpha+\beta-1}]}{\Gamma(\alpha+\beta)}\|\mathfrak{f}(s,x(s),y(\zeta s))\|_{\mathbb{E}}ds \\
&\quad+\int_{t_1}^{t_2} \frac{\Psi'(s)(\Psi(t_2)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|\mathfrak{f}(s,x(s),y(\zeta s))\|_{\mathbb{E}}ds \\
&\quad+\int_0^{t_1} \frac{|\lambda|\Psi'(s)[(\Psi(t_2)-\Psi(s))^{\beta-1}-(\Psi(t_1)-\Psi(s))^{\beta-1}]}{\Gamma(\beta)}\|x(s)\|_{\mathbb{E}}ds \\
&\quad+\int_{t_1}^{t_2} \frac{|\lambda|\Psi'(s)(\Psi(t_2)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\|x(s)\|_{\Theta}ds \\
&\quad+\frac{[(\Psi(t_2)-\Psi(0))^{\beta}-(\Psi(t_1)-\Psi(0))^{\beta}]}{\mathcal{K}} \\
&\quad\times\left(\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\|\mathfrak{f}(s,x(s),y(\zeta s))\|_{\mathbb{E}}ds\right. \\
&\quad\left.+|\mu|\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\|x(s)\|_{\mathbb{E}}ds\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\mathcal{T}x(t_2)-\mathcal{T}x(t_1)\|_{\mathbb{E}} \\
&\leq \|\Phi x(t_2)-\Phi x(t_1)\|_{\mathbb{E}} \\
&\quad+\frac{[(\Psi(t_2)-\Psi(0))^{\beta}-(\Psi(t_1)-\Psi(0))^{\beta}]}{\mathcal{K}}\sum_{i=1}^{m}|\mathfrak{s}_i|\|\Phi x(\varepsilon_i)\|_{\mathbb{E}} \qquad (2.16)\\
&\leq \frac{[(\Psi(t_1)-\Psi(0))^{\alpha+\beta}-(\Psi(t_2)-\Psi(0))^{\alpha+\beta}]}{\Gamma(\alpha+\beta+1)}[\mathtt{M}_1^*+\mathfrak{p}^*\gamma] \\
&\quad+\left(\frac{[(\Psi(t_2)-\Psi(0))^{\beta}-(\Psi(t_1)-\Psi(0))^{\beta}]m\max_{1\leq i\leq m}\{|\mathfrak{s}_i|\}(\Psi(b)-\Psi(0))^{\alpha+\beta}}{\mathcal{K}\ \ \Gamma(\alpha+\beta+1)}\right) \\
&\qquad\times[\mathtt{M}_1^*+\mathfrak{p}^*\gamma]. \qquad (2.17)
\end{aligned}
$$

Then, when $t_1 \to t_2$, the right-hand side of the inequalitie (2.17) tend to zero. Hence, $\mathcal{T}(\mathcal{B}_\gamma)$ is equicontinuous.

Step 4: $\mathcal{T}$ is a contraction with respect to the measure of noncompactness. let $\mathcal{B}_\gamma$ be a subset of $\mathbb{E}$. If $t \in J$, then

$$
\begin{aligned}
&\mathcal{S}(\mathcal{T}\mathcal{B}_\gamma(t)) \\
&\quad=\mathcal{S}\{\mathcal{T}x(t),\ x\in\mathcal{B}_\gamma\} \\
&\quad\leq\{\mathcal{S}(\Phi x(t)),\ x\in\mathcal{B}_\gamma\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\sum_{i=1}^{m}|\mathfrak{s}_i|\{\mathcal{S}(\Phi\mathrm{x}(\varepsilon_i)),\ \mathrm{x}\in\mathcal{B}_\gamma\}\\
\leq&\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\{\mathcal{S}(\mathfrak{f}(s,\mathrm{x}(s),y(\zeta s))),\ \mathrm{x}\in\mathcal{B}_\gamma\}\,\mathrm{d}s\\
&+|\mu|\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\{\mathcal{S}(\mathrm{x}(s)),\ \mathrm{x}\in\mathcal{B}_\gamma\}\,\mathrm{d}s\\
&+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\left(\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right.\\
&\{\mathcal{S}(\mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s))),\ \mathrm{x}\in\mathcal{B}_\gamma\}\,\mathrm{d}s\\
&\left.+|\mu|\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\{\mathcal{S}(\mathrm{x}(s)),\ \mathrm{x}\in\mathcal{B}_\gamma\}\,\mathrm{d}s\right)
\end{aligned}
$$

By the hypotheses $(Ax_{\mathbf{6}})$, we get

$$
\begin{aligned}
\mathcal{S}(\mathcal{T}\mathcal{Q}_\gamma(\mathrm{t}))=&\ \mathcal{S}\{\mathcal{T}\mathrm{x}(\mathrm{t}),\ \mathrm{x}\in\mathcal{B}_\gamma\}\\
\leq&\ \mathfrak{p}^*\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\{\mathcal{S}(\mathrm{x}(s))\mathrm{d}s,\ \mathrm{x}\in\mathcal{B}_\gamma\}\\
&+|\mu|\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\{\mathcal{S}(\mathrm{x}(s))\mathrm{d}s,\ \mathrm{x}\in\mathcal{B}_\gamma\}+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\mathcal{K}}\\
&\times\left(\mathfrak{p}^*\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\{\mathcal{S}(\mathrm{x}(s))\mathrm{d}s,\ \mathrm{x}\in\mathcal{B}_\gamma\}\right.\\
&\left.+|\mu|\sum_{i=1}^{m}|\mathfrak{s}_i|\int_0^{\varepsilon_i}\frac{\Psi'(s)(\Psi(\varepsilon_i)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\{\mathcal{S}(\mathrm{x}_n(s))\mathrm{d}s,\ \mathrm{x}\in\mathcal{B}_\gamma\}\right).
\end{aligned}
$$

Therefore,

$$
\mathcal{S}_c(\mathcal{T}\mathcal{B}_\gamma)\leq\mathscr{A}\ \mathcal{S}_c(\mathcal{B}_\gamma).
$$

Thus by condition (2.15), the operator $\mathcal{T}$ is a contraction, as a consequence of Darbo's fixed point theorem, we deduce that $\mathcal{T}$ has a fixed point that is a solution of problem (2.14). □

2.4 An Example

Firstly, we give an example to the problem (2.1).

Example 2.1. *Taking* $\alpha = 0.75, \beta = 0.45,\ J = [0,1],\ b = 1,\ \psi(\mathrm{t}) = \mathrm{t},\ \mathfrak{s}_1 = 2,\ \mathfrak{s}_2 = 3,\ \mathfrak{s}_3 = 4,\ \varepsilon_1 = \frac{2}{3},\ \varepsilon_2 = \frac{10}{17}, \varepsilon_3 = \frac{10}{19},\ \zeta = \frac{7}{10}$ *and* $\mu = \frac{25}{100}$, *we obtain nonlocal initial value problem which is a particular case of problem (2.18) with* ψ*-Caputo fractional derivative, given by*

$$\begin{cases} {}^{C}\mathscr{D}_{0^+}^{0.75,\psi}\left[{}^{C}\mathscr{D}_{0^+}^{0.45,\psi}\mathrm{x}(\mathrm{t}) + \frac{25}{100}\mathrm{x}(\mathrm{t})\right] = \mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\frac{7}{10}\mathrm{t})), \quad \mathrm{t} \in J = [0,1], \\ \mathrm{x}(\mathrm{t}) \mid_{\mathrm{t}=0} = 0, \\ 2\mathrm{x}(\varepsilon_1) \mid_{\varepsilon_1=\frac{2}{3}} + 3\mathrm{x}(\varepsilon_2) \mid_{\varepsilon_2=\frac{10}{17}} + 4\mathrm{x}(\varepsilon_3) \mid_{\varepsilon_3=\frac{10}{19}} = 0,\ 0 < \varepsilon_1, \varepsilon_2, \varepsilon_3 < 1, \end{cases} \tag{2.18}$$

Set

$$\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\frac{7}{10}\mathrm{t})) = \frac{\sqrt{\mathrm{t}}\left(\mathrm{x}(\frac{7}{10}\mathrm{t}) + \mathrm{x}(\mathrm{t})\right) + 0.01}{15e^{2+\mathrm{t}}(|\mathrm{x}(\mathrm{t})|\sqrt{\mathrm{t}} + 1)}.$$

Clearly the function $\mathfrak{f} \in \mathcal{C}([0,1] \times \mathbb{R}^2, \mathbb{R})$ *and we have*

$$|\mathfrak{f}(\mathrm{t}, x_1, x_2) - \mathfrak{f}(\mathrm{t}, y_1, y_2)| \leq \frac{1}{15e^2}(|x_1 - y_1| + |x_2 - y_2|),$$

Thus, conditions (Ax_{2})–(Ax_{3}) *are satisfied with*

$$p^* = 9.0224 \times 10^{-3},\ \mathcal{L} = 2.4525 \times 10^{-2},$$

and

$$\Lambda_1 = 1.0398, \quad \Lambda_2 = 0.43523, \quad \Lambda_3 = 2.6986 \times 10^{-2}.$$

Also, we can show that:

$$\mathcal{A} = 0.86373 < 1.$$

Given that all the conditions of Theorem 3.3 are satisfied, it follows that fractional boundary value problem (2.18) has a unique solution on $[0,1]$.

Now, we give an example to the problem (2.14).

Example 2.2. *Consider the Banach space*

$$\mathbb{E} = L^1 = \{\mathrm{x} = (\mathrm{x}, \mathrm{y}, ..., \mathrm{x}_n, ...),\ \sum_{n=1}^{\infty} |\mathrm{x}_n| < \infty\},$$

with the norm $\|\mathrm{x}\| = \sum_{n=1}^{\infty} |\mathrm{x}_n|$.

Consider the following problem:

$$\begin{cases} {}^{C}\mathscr{D}_{0^+}^{0.35,\psi}\left[{}^{C}\mathscr{D}_{0^+}^{0.55,\psi}\mathrm{x}_n(\mathrm{t}) + \frac{4}{10}\mathrm{x}_n(\mathrm{t})\right] = \mathfrak{f}(\mathrm{t}, \mathrm{x}_n(\mathrm{t}), \mathrm{x}_n(\frac{1}{2}\mathrm{t})), \quad \mathrm{t} \in \mathrm{T} = [0,1], \\ \mathrm{x}_n(\mathrm{t}) \mid_{\mathrm{t}=1} = 0, \\ 1\mathrm{x}_n(\varepsilon_1) \mid_{\varepsilon_1=\frac{5}{6}} + 2\mathrm{x}_n(\varepsilon_2) \mid_{\varepsilon_2=\frac{5}{7}} + 5\mathrm{x}_n(\varepsilon_3) \mid_{\varepsilon_3=\frac{5}{8}} = 0, \ 1 < \varepsilon_1, \varepsilon_2, \varepsilon_3 < 2, \end{cases} \tag{2.19}$$

where $\alpha = 0.35,\ \beta = 0.55,\ J = [0,1],\ b = 1,\ \psi(\mathrm{t}) = \mathrm{t}^2,\ \mathfrak{s}_1 = -1,\ \mathfrak{s}_2 = 1,\ \mathfrak{s}_3 = 5,\ \varepsilon_1 = \frac{5}{6},\ \varepsilon_2 = \frac{5}{7}, \varepsilon_3 = \frac{5}{8},\ \theta = \frac{1}{2}$ *and* $\mu = \frac{4}{10}$. *Let*

$$\mathfrak{f}(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{x}(\frac{1}{2}\mathrm{t})) = \frac{(2\mathrm{t}^3 + 3e^{-1})(2 + |\mathrm{x}_n(\mathrm{t})| + |\mathrm{x}_n(\frac{\mathrm{t}}{2})|)}{116e^{2-\mathrm{t}}(\|\mathrm{x}(\mathrm{t})\|_{\mathbb{E}} + \|\mathrm{x}(\frac{\mathrm{t}}{2})\|_{\mathbb{E}} + 1)},\ \mathrm{t} \in \mathrm{T},\ \mathrm{u}(\mathrm{t}) \in \mathbb{E},$$

with $n \in \mathbb{N}$. *Clearly, the continuous function* $\mathfrak{f} \in \mathcal{C}(\mathrm{T} \times \Theta^2, \Theta)$.

For each $\mathrm{x}_i,\ \bar{\mathrm{y}}_i \in \Theta,\ i = 1,2$ *and* $\mathrm{t} \in \mathrm{T}$, *we have*

$$\|\mathfrak{f}(\mathrm{t}, \mathrm{x}, \mathrm{y}) - \mathfrak{f}(\mathrm{t}, \bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\mathbb{E}} \le \frac{16 + 3e^{-1}}{116e^2}(\|\mathrm{x} - \bar{\mathrm{x}}\|_{\mathbb{E}} + \|\mathrm{y} - \bar{\mathrm{y}}\|_{\mathbb{E}}),$$

then the assumption $(Ax_{\mathbf{5}})$ *is satisfied with*

$$p^* = 1.9955 \times 10^{-2},$$

and

$$\mathcal{L} = 2.4525 \times 10^{-2},\ \Lambda_1 = 0.9076, \quad \Lambda_2 = 0.28227, \quad \Lambda_3 = 0.03018.$$

The condition (2.15) *of Theorem 2.3 is satisfied for*

$$\begin{aligned} &(p_1^* + p_2^*)(\Lambda_1 + \Lambda_3) + \left(1 + \frac{\max\limits_{1 \le i \le m} |\mathfrak{s}_i|}{\min\limits_{1 \le i \le m} |\mathfrak{s}_i|}\right)\Lambda_2 \\ &= 0.86373, \\ &< 1. \end{aligned}$$

As all the assumptions of Theorem 2.3 are satisfied, the problem (2.19) has at least one solution.

2.5 Notes and Remarks

In addition to the main fractional problem addressed in this chapter, we have also explored a related system formulated within the broader framework of fractional calculus. This complementary study aims to enrich the qualitative analysis of solutions, focusing on essential aspects such as existence, uniqueness, and stability. Without delving into the technical details of the associated system, we note that the results were obtained by employing fundamental fixed-point theorems. Moreover, illustrative examples have been provided to highlight the practical relevance and effectiveness of the theoretical findings. This general investigation contributes to a deeper understanding of fractional systems and opens avenues for future research in more complex or applied setting.

[2] **H. Bouzid**, A. Benali, A. Salim, G. Reny and E. Sina, On Solutions of the Nonlocal Generalized Coupled Langevin-Type Pantograph Systems, Journal of Mathematics, 20 pages, 2025.

chapter 3

Investigation of a Class of Ψ-Caputo Fractional Hybrid Equations Incorporating Langevin and Pantograph Terms

3.1 Introduction and Motivations

This chapter presents results on the existence, uniqueness, and Ulam-Hyers-Rassias stability of hybrid Langevin fractional pantograph differential equations involving Λ-Caputo fractional derivatives in Banach spaces. The findings are derived using Dhage's and Banach's fixed-point theorems, along with the contraction principle. Examples are provided to demonstrate the applicability of the results. This work builds on studies in [27, 72, 107], which focus on linear and nonlinear initial and boundary value problems for fractional differential equations with various fractional derivatives. The stability results, particularly those regarding Ulam, are inspired by the monographs of Abbas et al. [1] and the papers [26, 30], which extensively explore Ulam-Hyers and Ulam-Hyers-Rassias stability for different classes of functional equations.

The focus of this study is on analyzing solutions to the following problem involving

[3] **H. Bouzid**, A. Benali, L. Tabharit, A. Salim, A Study on a Class of -Caputo Fractional Hybrid Equations with Langevin and Pantograph Arguments. (**submitted**).

the Λ-Caputo type fractional derivatives:

$$ {}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}\left(\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right)+\mu\mathrm{x}(\mathrm{t})\right]=\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})),\quad \mathrm{t}\in J=(0,b], \tag{3.1} $$

$$ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0,\ \mathrm{x}'(\mathrm{t})\mid_{\mathrm{t}=a}=0,\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=\kappa}=0,\ 0<\kappa\leq b, \tag{3.2} $$

where ${}^{C}D_{0^+}^{\alpha,\Psi}$ and ${}^{C}D_{0^+}^{\beta,\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha\in(0,1]$, $\beta\in(1,2]$, $\mu\in\mathbb{R}\setminus\{0\}$ and $0<\zeta<1$. The given functions $\mathfrak{f}:J\times\mathbb{R}^2\to\mathbb{R}$ and $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuous.

3.2 Existence Results

We consider the following linear fractional differential equation

$$ {}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}\left[\frac{\mathrm{x}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right]+\mu\mathrm{x}(\mathrm{t})\right]=\omega(\mathrm{t}),\quad \mathrm{t}\in J, \tag{3.3} $$

where $0<\alpha<1, 1<\beta\leq 1, \rho>0,\ \mu\in\mathbb{R}$ and $(\omega;\mathcal{G})\in\mathcal{C}(J,\mathbb{R})\times\mathcal{C}^3(J,\mathbb{R}\setminus\{0\})$ with the boundary condition

$$ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0}=0,\ \mathrm{x}'(\mathrm{t})\mid_{\mathrm{t}=0}=0,\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=\kappa}=0,\ 0<\kappa\leq b, \tag{3.4} $$

The following theorem shows that the problem (3.3)–(3.4) has a unique solution given by

$$ \begin{aligned} \mathrm{x}(\mathrm{t}) := \mathcal{G}(\mathrm{t})\Bigg[&\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\omega(s)\mathrm{d}s-\mu\int_0^{\mathrm{t}}\frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\mathrm{x}(s)\mathrm{d}s \\ &+\frac{\Phi(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{(\Psi(\kappa)-\Psi(0))^{\beta}}\int_0^{\kappa}\frac{\Psi'(s)(\Psi(\kappa)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}\mathrm{x}(s)\mathrm{d}s \\ &-\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{(\Psi(\kappa)-\Psi(0))^{\beta}}\int_0^{\kappa}\frac{\Psi'(s)(\Psi(\kappa)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\omega(s)\mathrm{d}s\Bigg]. \end{aligned} \tag{3.5} $$

Theorem 3.1. *A solution to the fractional boundary value problem (3.3)-(3.4), is defined as a function* $\mathrm{x}\in\mathcal{C}^3(J,\mathbb{R})$ *that fulfills the equation (3.3) on* J *along with the specified boundary conditions (3.4).*

Proof. Let $\mathrm{x} \in \mathcal{C}^3(J, \mathbb{R})$ be a solution of the problem (3.3)-(3.4), then by using Lemma 1.1, we have

$$\mathrm{x}(\mathrm{t}) := \mathcal{G}(\mathrm{t}) \left[\mathcal{J}_{0^+}^{\alpha+\beta,\Psi} \omega(\mathrm{t}) + \varsigma_0 \frac{[\Psi(\mathrm{t}) - \Psi(0)]^{\beta}}{\Gamma(\beta+1)} - \mu \mathcal{J}_{0^+}^{\beta,\Psi} \mathrm{x}(\mathrm{t}) + \varsigma_1 + \varsigma_2 [\Psi(\mathrm{t}) - \Psi(0)] \right], \tag{3.6}$$

such as $\varsigma_i \in \mathbb{R}$, with $i = 0, 1, 2$. By the condition $\mathrm{x}(\mathrm{t}) \mid_{\mathrm{t}=0} = 0$ and $\mathrm{x}'(\mathrm{t}) \mid_{\mathrm{t}=0} = 0$, we obtain $\varsigma_1 := 0$ and $\varsigma_2 := 0$.

On the other hand by $\mathrm{x}(\mathrm{t}) \mid_{\mathrm{t}=\kappa} = 0$, we have

$$\varsigma_0 := \frac{\Gamma(\beta+1)}{(\Psi(\kappa) - \Psi(0))^{\beta}} \left(\mu \mathcal{J}_{0^+}^{\beta,\Psi} \mathrm{x}(\kappa) - \mathcal{J}_{0^+}^{\alpha+\beta,\Psi} \omega(\kappa) \right). \tag{3.7}$$

Finally, replacing these constants into (3.6), we get (3.5).

Conversely, let us now demonstrate that if (3.5) satisfies Eq (3.3), then the aforementioned equation can be expressed as

Applying the Λ-Caputo derivative, ${}^{C}D_{0^+}^{\beta,\Psi}$ on both sides and utilizing Lemma 1.1, we obtain

$${}^{C}D_{0^+}^{\beta,\Psi} \left(\frac{\mathrm{x}(\mathrm{t})}{\mathcal{G}(\mathrm{t})} \right) := \mathcal{J}_{0^+}^{\alpha,\Psi} \omega(\mathrm{t}) + \frac{\Gamma(\beta+1)}{(\Psi(\kappa) - \Psi(0))^{\beta}} \left(\mu \mathcal{J}_{0^+}^{\beta,\Psi} \eta(\kappa) - \mathcal{J}_{0^+}^{\alpha+\beta,\Psi} \omega(\kappa) \right) - \mu \eta(\mathrm{t}).$$

Reapplying, ${}^{C}D_{0^+}^{\alpha,\Psi}$ to the above equation, we obtain

$${}^{C}D_{0^+}^{\alpha,\Psi} \left[{}^{C}D_{0^+}^{\beta,\Psi} \left[\frac{\mathrm{x}(\mathrm{t})}{\mathcal{G}(\mathrm{t})} \right] + \varPhi \mathrm{x}(\mathrm{t}) \right] = \omega(\mathrm{t}).$$

Lastly, it is clear that the function in (3.5) meets the associated boundary conditions. This completes the proof. □

As a consequence of Theorem 3.1, we have the following result

Lemma 3.1. *Let $\alpha \in (0, 1]$ and $\beta \in (1, 2]$, let $\mathfrak{f} : J \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathfrak{g} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous. If the function $t \longrightarrow \left(\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t}))} \right) \in \mathcal{C}^3(J, \mathbb{R})$, then x is a solution to the problem (3.1)-(3.2) if and only if x is the fixed point of the operator $\mathcal{T} : \mathcal{C}(J, \mathbb{R}) \longrightarrow \mathcal{C}(J, \mathbb{R})$ defined by*

$$\mathcal{T}\mathrm{x}(\mathrm{t}) \quad := \quad \mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t})) \left[\varPhi \mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta}}{(\Psi(\kappa) - \Psi(0))^{\beta}} \varPhi \mathrm{x}(\kappa) \right],$$

where

$$\Phi \mathrm{x}(\mathrm{t}) := \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \omega(s) \mathrm{d}s - \mu \int_0^{\mathrm{t}} \mathrm{t} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\beta-1}}{\Gamma(\beta)} \mathrm{x}(s) \mathrm{d}s.$$

where $\omega : J \to \mathbb{R}$ *be a function satisfying the functional equation*

$$\omega(\mathrm{t}) = \mathfrak{f}(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{x}(\zeta \mathrm{t})).$$

Clearly, $\mathfrak{f} \in \mathcal{C}(J, \mathbb{R})$.

We are now in a position to state and prove our existence result for the problem (3.1)-(3.2) based on Theorem 3.2. For the sake of clarity, we denote

$$\begin{aligned}
\Lambda_1 =& \frac{(\Psi(b) - \Psi(0))^{\beta+\alpha}}{\Gamma(\alpha+\beta+1)}, \\
\Lambda_2 =& 2|\mu| \frac{(\Psi(b) - \Psi(0))^{\beta}}{\Gamma(\beta+1)}, \\
\Lambda_3 =& \frac{(\Psi(b) - \Psi(0))^{\beta}(\Psi(\kappa) - \Psi(0))^{\alpha}}{\Gamma(\alpha+\beta+1)}.
\end{aligned}$$

Theorem 3.2. *Assume that the following hypotheses hold.*

(Ax_1) *The function* $\mathfrak{f} : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is continuous on* J.

(Ax_2) *The function* $\mathfrak{g} : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ *is continuous and there exists a function* $\mathfrak{g} \in \mathcal{C}^3(J, [0, \infty))$ *such that*

$$|\mathfrak{g}(\mathrm{t}, \mathrm{x}) - \mathfrak{g}(\mathrm{t}, \bar{\mathrm{x}})| \leq l_{\mathfrak{g}}(\mathrm{t}) |\mathrm{x} - \bar{\mathrm{x}}|,$$

for any $\mathrm{x}, \bar{\mathrm{x}} \in \mathbb{R}$ *and* $t \in J$.

(Ax_3) *There exist functions* $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \in \mathcal{C}(J, \mathbb{R}^+)$ *such that*

$$|\mathfrak{f}(\mathrm{t}, \mathrm{x}, \mathrm{y})| < \mathfrak{p}_1(\mathrm{t}) + \mathfrak{p}_2(\mathrm{t})|\mathrm{x}| + \mathfrak{p}_3(\mathrm{t})|\mathrm{y}|,$$

for all $\mathrm{t} \in J$ *and* $\mathrm{x}_i \in \mathbb{R},\ i = 1, 2.$

(Ax_4) *There exists* $\gamma > 0$ *such that*

$$\frac{\mathfrak{g}_0 \Pi_\gamma}{1 - l_{\mathfrak{g}}^* \Pi_\gamma} \leq \gamma,$$

and

$$l_{\mathfrak{g}}^* \Pi_\gamma < 1,$$

where

$$\Pi_\gamma = (\Lambda_1 + \Lambda_3)\,\mathfrak{p}_1^* + \left[(\Lambda_1 + \Lambda_3)\,(\mathfrak{p}_2^* + \mathfrak{p}_3^*) + \Lambda_2\right]\gamma, \tag{3.8}$$

and

$$l_{\mathfrak{g}}^* = \sup_{\mathrm{t}\in J} l_{\mathfrak{g}}(\mathrm{t}),\ \ \mathfrak{g}_0 = \sup_{\mathrm{t}\in J} |\mathfrak{g}(\mathrm{t}, 0)|,\ \ \mathfrak{p}_i^* = \sup_{\mathrm{t}\in J} \mathfrak{p}_i(\mathrm{t}),\ \ i = 1, 2, 3.$$

Then the fractional boundary value problem (3.1)-(3.2) has at least one mild solution on J.

Proof. Define the set

$$\mathcal{B}_\gamma = \left\{\mathrm{x} \in \mathcal{C}^3(J, \mathbb{R})\ :\ \|\mathrm{x}\|_\infty \leq \gamma\right\}.$$

Next, to convert problem (3.1)-(3.2) into the operator equation $\mathrm{x} = \mathcal{R}\mathrm{x}\mathcal{P}\mathrm{x}$, we must define $\mathcal{R}$ and $\mathcal{P}$ as follows:

$$\mathcal{R}\mathrm{x}(\mathrm{t}) = \mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t})), \tag{3.9}$$

and

$$\mathcal{P}\mathrm{x}(\mathrm{t}) \;=\; \Phi\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{(\Psi(\kappa) - \Psi(0))^\beta}\Phi\mathrm{x}(\kappa).$$

We show that the operators $\mathcal{R}$ and $\mathcal{P}$ satisfy all the conditions of Theorem 1.4.

Step 1: Firstly, we show that $\mathcal{R}$ is Lipschitzian on $\mathcal{C}(J, \mathbb{R})$. Let $\mathrm{x}, \bar{\mathrm{x}} \in \mathcal{C}^3(J, \mathbb{R})$. Then by (Ax_2) we have

$$\begin{aligned} |\mathcal{R}\mathrm{x}(\mathrm{t}) - \mathcal{R}\bar{\mathrm{x}}(\mathrm{t})| &= |\mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t})) - \mathfrak{g}(\mathrm{t}, \bar{\mathrm{x}}(\mathrm{t}))| \\ &\leq l_{\mathfrak{g}}^* |\mathrm{x} - \bar{\mathrm{x}}|. \end{aligned}$$

Then,

$$\|\mathcal{R}\mathrm{x} - \mathcal{R}\bar{\mathrm{x}}\|_\infty \leq l^*_{\mathfrak{g}} \|\mathrm{x} - \bar{\mathrm{x}}\|_\infty, \tag{3.10}$$

for all $\mathrm{x}, \bar{\mathrm{x}} \in \mathcal{C}^3(J, \mathbb{R})$. Therefore $\mathcal{R}$ is Lipschitzian on $\mathcal{C}(J, \mathbb{R})$ with Lipschitz constant $l^*_{\mathfrak{g}}$.

Step 2: We demonstrate that the operator $\mathcal{P}$ is completely continuous on $\mathcal{B}_\gamma$. To achieve this, we first establish that the operator $\mathcal{P}$ is continuous on $\mathcal{C}(J, \mathbb{R})$. Let $\{\mathrm{x}_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}_\gamma$ that converges to a point $\mathrm{x} \in \mathcal{B}_\gamma$. Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{P}\mathrm{x}_n(\mathrm{t}) \\
&= \lim_{n\to+\infty} \left\{ \Phi\mathrm{x}_n(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{(\Psi(\kappa) - \Psi(0))^\beta} \Phi\mathrm{x}_n(\kappa) \right\} \\
&= \lim_{n\to+\infty} \left\{ \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \psi(s, \mathrm{x}_n(s), \mathrm{x}_n(\zeta s)) \mathrm{d}s \right. \\
&\quad -\mu \int_0^t \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\beta-1}}{\Gamma(\beta)} \mathrm{x}_n(s) \mathrm{d}s \\
&\quad - \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{(\Psi(\kappa) - \Psi(0))^\beta} \left(\int_0^\kappa \frac{\Psi'(s)(\Psi(\kappa) - \Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \psi(s, \mathrm{x}_n(s), \mathrm{x}_n(\zeta s)) \mathrm{d}s \right. \\
&\quad \left.\left. -\mu \int_0^\kappa \frac{\Psi'(s)(\Psi(\kappa) - \Psi(s))^{\beta-1}}{\Gamma(\beta)} \mathrm{x}_n(s) \mathrm{d}s \right) \right\} \\
&= \mathcal{P}\mathrm{x}(\mathrm{t}),
\end{aligned}$$

for all $\mathrm{t} \in J$. This shows that $\mathcal{P}$ is a continuous operator on $\mathcal{B}_\gamma$.

Next, we prove that the set $\mathcal{P}(\mathcal{B}_\gamma)$ is a uniformly bounded in $\mathcal{B}_\gamma$. For any $\mathrm{x} \in \mathcal{B}_\gamma$ and $\mathrm{t} \in J$, we have

$$|\Phi\mathrm{x}(\mathrm{t})| \leq \frac{(\Psi(\mathrm{t}) - \Psi(0))^{\alpha+\beta} \left(\|\mathfrak{p}_1\| + (\|\mathfrak{p}_2\| + \|\mathfrak{p}_3\|)\gamma \right)}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t}) - \Psi(0))^\beta \gamma}{\Gamma(\beta+1)}.$$

Therefore,

$$\begin{aligned}
&|\mathcal{P}\mathrm{x}(\mathrm{t})| \\
&\leq |\Phi\mathrm{x}(\mathrm{t})| + \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{(\Psi(\kappa) - \Psi(0))^\beta} |\Phi\mathrm{x}(\kappa)| \\
&\leq \frac{(\Psi(\mathrm{t}) - \Psi(0))^{\alpha+\beta} \left(\|\mathfrak{p}_1\| + (\|\mathfrak{p}_2\| + \|\mathfrak{p}_3\|)\gamma \right)}{\Gamma(\alpha+\beta+1)} + \frac{2|\mu|(\Psi(\mathrm{t}) - \Psi(0))^\beta \gamma}{\Gamma(\beta+1)}
\end{aligned}$$

$$+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}(\Psi(\kappa)-\Psi(0))^{\alpha}\left(\|\mathfrak{p}_1\|+(\|\mathfrak{p}_2\|+\|\mathfrak{p}_3\|)\gamma\right)}{\Gamma(\alpha+\beta+1)}$$
$$\leq(\Lambda_1+\Lambda_3)\,\mathfrak{p}_1^*+\left[(\Lambda_1+\Lambda_3)\,(\mathfrak{p}_2^*+\mathfrak{p}_3^*)+\Lambda_2\right]\gamma.$$

Thus $\|\mathcal{P}\mathrm{x}\|\leq\Pi_\gamma$, for all $\mathrm{x}\in\mathcal{B}_\gamma$ with Π_γ given in (3.8). This shows that $\mathcal{P}$ is uniformly bounded on $\mathcal{B}_\gamma$.

On the other hand, we demonstrate that $\mathcal{P}$ is an equicontinuous set in $\mathcal{B}_\gamma$. Let $\mathrm{t}_1,\mathrm{t}_2\in J$ with $\mathrm{t}_1<\mathrm{t}_2$ and $\mathrm{x}\in\mathcal{B}_\gamma$. Then we have

$$\begin{aligned}
&|\mathcal{P}\mathrm{x}(\mathrm{t}_2)-\mathcal{P}\mathrm{x}(\mathrm{t}_1)|\\
&\leq|\Phi\mathrm{x}(\mathrm{t}_2)-\Phi\mathrm{x}(\mathrm{t}_1)|+\frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta}-(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}]}{(\Psi(\kappa)-\Psi(0))^{\beta}}|\Phi\mathrm{x}(\kappa)|\\
&\leq\frac{[(\Psi(\mathrm{t}_1)-\Psi(0))^{\alpha+\beta}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\alpha+\beta}]}{\Gamma(\alpha+\beta+1)}[\mathfrak{p}_1^*+(\mathfrak{p}_2^*+\mathfrak{p}_3^*)\gamma]\\
&\quad+\left(\frac{(\Psi(\mathrm{t}_2)-\Psi(\mathrm{t}_1))^{\beta}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta}+(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)|\mu|\gamma\\
&\quad+\frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta}-(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta}]}{2}\\
&\quad\times\left(\int_0^{\kappa}\frac{\Psi'(s)(\Psi(\kappa)-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|\psi(s,\mathrm{x}(s),\mathrm{x}(\zeta s))|\mathrm{d}s\right.\\
&\quad\left.+|\mu|\int_0^{\kappa}\frac{\Psi'(s)(\Psi(\kappa)-\Psi(s))^{\beta-1}}{\Gamma(\beta)}|\mathrm{x}(s)|\mathrm{d}s\right).
\end{aligned}$$

This implies, $|\mathcal{P}\mathrm{x}(\mathrm{t}_2)-\mathcal{P}\mathrm{x}(\mathrm{t}_1)|\longrightarrow 0$ as $\mathrm{t}_1\to\mathrm{t}_2$, uniformly for all $\mathrm{x}\in\mathcal{B}_\gamma$.
Thus $\mathcal{P}$ has the equicontinuity specification on the Banach space $\mathcal{C}(J,\mathbb{R})$. As a consequence, $\mathcal{P}$ is relatively compact, and thus the Arzelà-Ascoli theorem yields that $\mathcal{P}$ is completely continuous and finally $\mathcal{P}$ is compact on $\mathcal{B}_\gamma$.

Step 3: we show that hypothesis (3) of Theorem 1.4 is satisfied. For $\mathrm{x}\in\mathcal{C}^3(J,\mathbb{R})$ and $\bar{\mathrm{x}}\in\mathcal{B}_\gamma$, where $\mathrm{x}=\mathcal{R}\mathrm{x}\mathcal{P}\bar{\mathrm{x}}$, we get

$$\begin{aligned}
|\mathrm{x}(\mathrm{t})| &= |\mathcal{R}\mathrm{x}(\mathrm{t})\mathcal{P}\bar{\mathrm{x}}(\mathrm{t})|\\
&\leq [|\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))+\mathfrak{g}(\mathrm{t},0)|+|\mathfrak{g}(\mathrm{t},0)|]\Pi_\gamma\\
&\leq [l_{\mathfrak{g}}^*\|\mathrm{x}\|+\mathfrak{g}_0]\Pi_\gamma,
\end{aligned}$$

which implies that

$$\|\mathrm{x}\|_\infty\leq\frac{\mathfrak{g}_0\Pi_\gamma}{1-l_{\mathfrak{g}}^*\Pi_\gamma}=\gamma.$$

This shows that condition (3) of Theorem 1.4 is satisfied.

Step 4: Finally, we have
$\mathfrak{b}_{\mathcal{P}} = \|\mathfrak{b}_{\mathcal{P}}(\mathcal{B}_\gamma)\| = \sup\{\|\mathcal{P}(\mathrm{x})\| : \mathrm{x} \in \mathcal{B}_\gamma\} \leq \Pi_\gamma$.
Fromabove estimate we obtain

$$\mathfrak{a}_{\mathcal{R}}\mathfrak{b}_{\mathcal{P}} \leq l_{\mathfrak{g}}^{*}\Pi_\gamma < 1.$$

Hence, all the conditions of Theorem 1.4 are satisfied, and therefore, the operator equation $\mathrm{x} = \mathcal{R}\mathrm{x}\mathcal{P}\mathrm{x}$ has a solution in $\mathcal{B}_\gamma$. Consequently, problem (3.1)–(3.2) has a solution on J. This completes the proof. □

Our next existence result for the problem (3.1)–(3.2) is based on Banach fixed point Theorem 1.1. Furthermore, to obtain the uniqueness result due to the nature of our problem, we must assume the following stronger hypotheses:

(Ax_5) The functions $\mathfrak{f} : J \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathfrak{g} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous.

(Ax_6) There exist positive functions $l_{\mathfrak{f}}, l_{\mathfrak{g}} \in \mathcal{C}(J, \mathbb{R}^+)$ such that

$$|\mathfrak{f}(\mathrm{t}, \mathrm{x}_1, \mathrm{x}_2) - \mathfrak{f}(\mathrm{t}, \bar{\mathrm{x}}_1, \bar{\mathrm{x}}_2)| \leq l_{\mathfrak{f}}(\mathrm{t}) \sum_{i=1}^{2} (|\mathrm{x}_i - \bar{\mathrm{x}}_i|),$$

and

$$|\mathfrak{g}(\mathrm{t}, \mathrm{x}) - \mathfrak{g}(\mathrm{t}, \bar{\mathrm{x}}_1)| \leq l_{\mathfrak{g}}(\mathrm{t})|\mathrm{x} - \bar{\mathrm{x}}_1|,$$

for all $\mathrm{t} \in J$ and $\mathrm{x}_i,\ \bar{\mathrm{x}}_i \in \mathbb{R},\ i = 1, 2$, where $l_{\mathfrak{f}}^{*} = \sup_{\mathrm{t}\in J} l_{\mathfrak{f}}(\mathrm{t})$ and $l_{\mathfrak{g}}^{*} = \sup_{\mathrm{t}\in J} l_{\mathfrak{g}}(\mathrm{t})$.

(Ax_7) There exist positive constants L and M, where $|\mathfrak{f}(\mathrm{t}, \cdot, \cdot)| < L$ and $|\mathfrak{g}(\mathrm{t}, \cdot)| < M$, for all $\mathrm{t} \in J$ and $\cdot \in \mathbb{R},\ i = 1, 2$.

Theorem 3.3. *Suppose that (Ax_5)–(Ax_7) holds. Assume also that*

$$\Lambda^{*} = (2l_{\mathfrak{f}}^{*}M + l_{\mathfrak{g}}^{*}L)(\Lambda_3 + \Lambda_1) + (M + l_{\mathfrak{g}}^{*})\Lambda_2\varepsilon < 1, \tag{3.11}$$

then the fractional boundary value problem (3.1)-(3.2) have a unique solution on J.

Proof. Setting $\varepsilon \geq \frac{M(\Lambda_3+\Lambda_1)}{1-M\Lambda_2}$, with $0 \leq M\Lambda_2 < 1$. we show that $\mathcal{T}\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$, where

$$\mathcal{B}_\varepsilon = \left\{\mathrm{x} \in \mathcal{C}^3(J; \mathbb{R}) : \|\mathrm{x}\| \leq \varepsilon\right\}.$$

For $\mathrm{x} \in \mathcal{B}_\varepsilon$ and for each $\mathrm{t} \in J$, from the definition of $\mathcal{T}$ and hypothesis $(Ax_{\mathbf{5}})$– $(Ax_{\mathbf{7}})$, we obtain

$$\begin{aligned}
&|\mathcal{T}\mathrm{x}(\mathrm{t})| \\
&\leq \left|\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\left[\mathit{\Phi}\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^\beta}{(\Psi(\kappa)-\Psi(0))^\beta}\mathit{\Phi}\mathrm{x}(\kappa)\right]\right| \\
&\leq |\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))|\left[\frac{(\Psi(\mathrm{t})-\Psi(0))^\beta}{(\Psi(\kappa)-\Psi(0))^\beta}\right. \\
&\quad\times\left(\frac{L(\Psi(\kappa)-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\kappa)-\Psi(0))^\beta}{\Gamma(\beta+1)}\varepsilon\right) \\
&\quad\left.+\frac{L(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^\beta}{\Gamma(\beta+1)}\varepsilon\right] \\
&\leq \mathcal{M}\left[\Lambda_3+\Lambda_1+\Lambda_2\mathfrak{f}\right] \\
&\leq \varepsilon.
\end{aligned}$$

Hence,

$$\|\mathcal{T}\mathrm{x}\| \leq \varepsilon,$$

which implies that $\mathcal{T}\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$.

Now for $\mathrm{x}, \bar{\mathrm{x}} \in \mathcal{B}_\varepsilon$ and for any $\mathrm{t} \in J$. By $(Ax_{\mathbf{6}})$, we get

$$\begin{aligned}
|\mathit{\Phi}\mathrm{x}(\mathrm{t}) - \mathit{\Phi}y(\mathrm{t})| &\leq \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|\mathfrak{f}(s,\mathrm{x}(s),\mathrm{x}(\zeta s)) - \mathfrak{f}(s,y(s),y(\zeta s))|\mathrm{d}s \\
&\quad + |\mu|\int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta-1}}{\Gamma(\beta)}|\mathrm{x}(s)-\bar{\mathrm{x}}(s)|\mathrm{d}s \\
&\leq \left(\frac{2l_{\mathfrak{f}}^*(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^\beta}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\|.
\end{aligned}$$

and by condition $(Ax_{\mathbf{7}})$, we obtain

$$|\mathit{\Phi}\mathrm{x}(\mathrm{t})| \leq \frac{L(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^\beta}{\Gamma(\beta+1)}\varepsilon.$$

By applying the triangle inequality, we obtain

$$\begin{aligned}
|\mathcal{T}\mathrm{x}(\mathrm{t}) - \mathcal{T}\bar{\mathrm{x}}(\mathrm{t})| &\leq |\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\mathit{\Phi}\mathrm{x}(\mathrm{t}) - \mathfrak{g}(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))\mathit{\Phi}\bar{\mathrm{x}}(\mathrm{t})| \\
&\quad + \frac{(\Psi(\mathrm{t})-\Psi(0))^\beta}{(\Psi(\kappa)-\Psi(0))^\beta}|\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\mathit{\Phi}\mathrm{x}(\kappa) - \mathfrak{g}(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))\mathit{\Phi}\bar{\mathrm{x}}(\kappa)|.
\end{aligned}$$

We conclude that

$$
\begin{aligned}
&|\mathcal{T}\mathrm{x}(\mathrm{t})-\mathcal{T}\bar{\mathrm{x}}(\mathrm{t})| \\
&\quad\leq M\left(\frac{2l_{\mathfrak{f}}^{*}(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad\quad+l_{\mathfrak{g}}^{*}\left(\frac{L(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\varepsilon\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad\quad+M(\Psi(\mathrm{t})-\Psi(0))^{\beta}\left(\frac{2l_{\mathfrak{f}}^{*}(\Psi(\kappa)-\Psi(0))^{\alpha}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad\quad+l_{\mathfrak{g}}^{*}(\Psi(\mathrm{t})-\Psi(0))^{\beta}\left(\frac{L(\Psi(\kappa)-\Psi(0))^{\alpha}}{\Gamma(\alpha+\beta+1)}+\frac{|\mu|}{\Gamma(\beta+1)}\varepsilon\right)\|\mathrm{x}-\bar{\mathrm{x}}\|.
\end{aligned}
$$

After taking the supremum over J and simplifying, we get

$$\|\mathcal{T}\mathrm{x}-\mathcal{T}\bar{\mathrm{x}}\|_{\infty} \leq \Lambda^{*}\|\mathrm{x}-\bar{\mathrm{x}}\|_{\infty}.$$

Consequently, by (3.11), $\mathcal{T}$ is a contraction, and by utilizing Banach's fixed point theorem, the fractional boundary value problem (3.1)-(3.2) has a unique solution. □

3.3 Ulam-Hyers Stability

We now address the Ulam-Hyers and generalized Ulam-Hyers stability of the equation (3.1).

Let $\varrho>0$, and consider the following inequality:

$$\left|{}^{C}D_{0^{+}}^{\alpha,\Psi}\left[{}^{C}D_{0^{+}}^{\beta,\Psi}\left[\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right]+\Phi\mathrm{x}(\mathrm{t})\right]-\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t}))\right|<\varrho,\quad \mathrm{t}\in J=[0,b]. \tag{3.12}$$

Definition 3.1. *([26, 30]) The fractional boundary value problem (3.1)–(3.2) is said to be Ulam-Hyers stable if there exists a constant $\varphi>0$ such that for any $\varrho>0$ and for each solution $\mathrm{x}\in\mathcal{C}^{3}(J,\mathbb{R})$ of inequality (3.12), there exists a solution $\bar{\mathrm{x}}\in\mathcal{C}^{3}(J,\mathbb{R})$ of the fractional boundary value problem (3.1)–(3.2) satisfying*

$$\|\mathrm{x}-\bar{\mathrm{x}}\|_{\infty}\leq\varrho\varphi.$$

Definition 3.2. *([26, 30]) The fractional boundary value problem (3.1)–(3.2) is generalized Ulam-Hyers stable if there exists a function $\mathfrak{J}\in\mathcal{C}(\mathbb{R}^{+})$, with $\mathfrak{J}(0)=0$, such* *that for any $\varrho > 0$ and each solution $\mathrm{x} \in \mathcal{C}^3(J,\mathbb{R})$ of inequality (3.12), there exists a solution $\bar{\mathrm{x}} \in \mathcal{C}^3(J,\mathbb{R})$ of the fractional boundary value problem (3.1)–(3.2) such that*

$$\|\mathrm{x} - \bar{\mathrm{x}}\|_\infty \leq \mathfrak{I}(\varrho).$$

Remark 3.1. *A function $\mathrm{x} \in \mathcal{C}^3(J,\mathbb{R})$ is a solution of inequality (3.12) if and only if there exists a function $\Omega \in \mathcal{C}(J,\mathbb{R})$ (depending on the solution x) such that:*

i) $|\Omega(\mathrm{t})| \leq \varrho$, *for all* $\mathrm{t} \in J$.

ii) ${}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}\left(\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right) + \mu\mathrm{x}(\mathrm{t})\right] - \mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})) = \Omega(\mathrm{t}), \quad \mathrm{t} \in J.$

To simplify the expressions in the subsequent results, let

$$\mathbb{S} := M(\Lambda_1 + \Lambda_3).$$

Theorem 3.4. *Suppose that assumptions $(Ax_{\boldsymbol{5}})$, $(Ax_{\boldsymbol{6}})$, and $(Ax_{\boldsymbol{7}})$ hold. Then the fractional boundary value problem (3.1)–(3.2) is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable under condition (3.11).*

Proof. Assume $\varrho > 0$, $\mathcal{B}_\varepsilon \subset \mathcal{C}^3(J,\mathbb{R})$ and $\mathrm{x} \in \mathcal{B}_\varepsilon$ is a function that fulfills the inequality (3.12), and let $\bar{\mathrm{x}} \in \mathcal{B}_\varepsilon$ be the sole solution of the fractional boundary value problem (3.1)-(3.2). Since $\mathrm{x} \in \mathcal{B}_\varepsilon$ is a function satisfes the inequality (3.12). It follows from Remark 3.1 that

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\Psi}\left[{}^{C}D_{0^+}^{\beta,\Psi}\left(\dfrac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right) + \varPhi\mathrm{x}(\mathrm{t})\right] - \mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})) = \Omega(\mathrm{t}),\ \mathrm{t} \in J, \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=0} = 0,\ \mathrm{x}'(\mathrm{t})\mid_{\mathrm{t}=0} = 0, \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=\kappa} = 0,\ 0 < \kappa \leq b\ . \end{cases}$$

Using Lemma 3.1 once more, we have

$$\mathrm{x}(\mathrm{t}) \quad := \quad \mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\left[\mathbb{F}\mathrm{x}(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^\beta}{(\Psi(\kappa) - \Psi(0))^\beta}\mathbb{F}\mathrm{x}(\kappa)\right],$$

where

$$\mathbb{F}\mathrm{x}(\mathrm{t}) := \mathrm{I}_{0^+}^{\alpha+\beta,\Psi}[\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})) + \Omega(\mathrm{t})] - \mu\mathrm{I}_{0^+}^{\alpha,\Psi}\mathrm{x}(\mathrm{t}).$$

Moreover, using part (I) of Remark 3.1 and $(Ax_{\mathbf{6}})$, we can obtain the following formula for each $\mathrm{t} \in J$.

$$\begin{aligned}
&|\mathbb{F}\mathrm{x}(\mathrm{t}) - \Phi\bar{x}(\mathrm{t})| \\
&\leq \mathrm{I}_{0^+}^{\alpha+\beta,\Psi}|\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\zeta\mathrm{t})) - \mathfrak{f}(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}),\bar{\mathrm{x}}(\zeta\mathrm{t}))| + \mu\mathcal{J}_{0^+}^{\alpha,\Psi}|\mathrm{x}(\mathrm{t}) - \bar{x}(\mathrm{t})| \\
&\quad + \mathcal{J}_{0^+}^{\alpha+\beta,\Psi}|\Omega(\mathrm{t})| \\
&\leq \left(\frac{2l_{\mathfrak{f}}^*(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\|. \\
&\quad + \frac{\varrho(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)},
\end{aligned}$$

and

$$\begin{aligned}
|\mathbb{F}\mathrm{x}(\mathrm{t})| &\leq \frac{(L+\varrho)(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\varepsilon, \\
|\Phi\mathrm{x}(\mathrm{t})| &\leq \frac{L(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\varepsilon,
\end{aligned}$$

Applying the triangle inequality, we have

$$\begin{aligned}
&|\mathrm{x}(\mathrm{t}) - \bar{\mathrm{x}}(\mathrm{t})| \\
&\leq |\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\mathbb{F}\mathrm{x}(\mathrm{t}) - \mathfrak{g}(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))\Phi\bar{\mathrm{x}}(\mathrm{t})| \\
&\quad + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{(\Psi(\kappa)-\Psi(0))^{\beta}}|\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))\mathbb{F}\mathrm{x}(\kappa) - \mathfrak{g}(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))\Phi\bar{\mathrm{x}}(\kappa)| \\
&\leq M\left(\frac{2l_{\mathfrak{f}}^*(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad + l_{\mathfrak{g}}^*\left(\frac{L(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|(\Psi(\mathrm{t})-\Psi(0))^{\beta}}{\Gamma(\beta+1)}\varepsilon\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad + M(\Psi(\mathrm{t})-\Psi(0))^{\beta}\left(\frac{2l_{\mathfrak{f}}^*(\Psi(\kappa)-\Psi(0))^{\alpha}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|}{\Gamma(\beta+1)}\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad + l_{\mathfrak{g}}^*(\Psi(\mathrm{t})-\Psi(0))^{\beta}\left(\frac{L(\Psi(\kappa)-\Psi(0))^{\alpha}}{\Gamma(\alpha+\beta+1)} + \frac{|\mu|}{\Gamma(\beta+1)}\varepsilon\right)\|\mathrm{x}-\bar{\mathrm{x}}\| \\
&\quad + \frac{M[(\Psi(\mathrm{t})-\Psi(0))^{\alpha+\beta} + ((\Psi(\mathrm{t})-\Psi(0))^{\beta}((\Psi(\kappa)-\Psi(0))^{\alpha}]}{\Gamma(\alpha+\beta+1)}\varrho.
\end{aligned}$$

In conclusion, we obtain

$$\|\mathrm{x}-y\|_{\infty} \leq \frac{\mathbb{S}}{1-\Lambda^*}\varrho = \varphi\varrho.$$

Consequently, the fractional boundary value problem (3.1)–(3.2) is stable in the sense of Ulam-Hyers. This completes the proof using Ulam-Hyers definition. □

Theorem 3.5. *Suppose the conditions of Theorem 3.4 hold. If there exists* $\mathfrak{J} \in \mathcal{C}(\mathbb{R}^+)$, *such that* $\mathfrak{J}(0) = 0$ *with* $\varrho > 0$. *Therefore, the fractional boundary value problem (3.1)–(3.2) is generalized Ulam-Hyers stable.*

Proof. For $\mathfrak{J}(\varrho) = \frac{\mathbb{S}}{1-\Lambda^*}\varrho = \varphi\varrho = \varphi\varrho$; $\mathfrak{J}(0) = 0$. We prove that the solution to the fractional boundary value problem (3.1)-(3.2) is also generalized Ulam-Hyers stable. □

3.4 An Example

Example 3.1. *We consider the following problem:*

$$\begin{cases} {}^{C}D_{0^+}^{0.75,\sqrt{e^{t}}}\left[{}^{C}D_{0^+}^{1.25,\sqrt{e^{t}}}\left[\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right]+\frac{4}{100}\mathrm{x}(\mathrm{t})\right] = \mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\frac{1}{5}\mathrm{t})), \quad \mathrm{t} \in J, \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=1} = 0, \ \mathrm{x}'(\mathrm{t})\mid_{\mathrm{t}=1} = 0, \\ \mathrm{x}(\mathrm{t})\mid_{\mathrm{t}=\frac{5}{4}} = 0, \ 0 < \kappa = \frac{5}{4} < 2, \end{cases} \tag{3.13}$$

where the interval is defined as $J = [0, 2]$ *with parameters* $b = 2$, $\zeta = \frac{1}{5}$, $\kappa = \frac{5}{4}$, *and* $\mu = \frac{4}{100}$. *Additionally, we set* $\Psi(\mathrm{t}) = \sqrt{\mathrm{e}^{\mathrm{t}}}$ *and define the function:*

$$\mathfrak{g}(\mathrm{t},\mathrm{x}(\mathrm{t})) = \frac{|\sin(\mathrm{t})|}{24^{\mathrm{t}} + 73e^{2+\mathrm{t}} + |\cos(\mathrm{t})| + 2\pi}(|\mathrm{x}(\mathrm{t})| + 2), \quad \mathrm{t} \in J, \quad \mathrm{x} \in \mathcal{C}^3(J,\mathbb{R}).$$

Moreover, the function ψ *is given by:*

$$\mathfrak{f}(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{x}(\tfrac{\mathrm{t}}{5})) = \frac{|\sin(\mathrm{t})|}{35e^{t+3}}\left(\frac{\cos(\mathrm{t})\mathrm{x}(\mathrm{t})}{\pi + |\mathrm{x}(\mathrm{t})|} + \frac{\mathrm{x}(\frac{1}{5}\mathrm{t})}{2\pi + |\mathrm{x}(\frac{1}{5}\mathrm{t})| + 3e^{\pi}}\right) + 0.02, \mathrm{t} \in J, \mathrm{x} \in \mathcal{C}^3(J,\mathbb{R}).$$

Since the function $\mathfrak{f} \in \mathcal{C}(J \times \mathbb{R}^2, \mathbb{R})$ *is continuous, it satisfies condition* (Ax_1). *Furthermore, for any* $\mathrm{x}, \bar{\mathrm{x}} \in \mathbb{R}$ *and* $\mathrm{t} \in J$, *we have:*

$$|\mathfrak{g}(\mathrm{t},\mathrm{x}) - \mathfrak{g}(\mathrm{t},\bar{\mathrm{x}})| \leq \frac{|\sin(\mathrm{t})|}{24^{\mathrm{t}}}|\mathrm{x} - \bar{\mathrm{x}}|.$$

Thus, condition (Ax_2) *holds with:*

$$l_{\mathfrak{g}}(\mathrm{t}) = \frac{|\sin(\mathrm{t})|}{24^{\mathrm{t}}}, \quad \textit{implying } l_{\mathfrak{g}}^* = \frac{1}{24}.$$

For any $\mathrm{x}, \mathrm{y} \in \mathbb{R}$*, we obtain:*

$$|\mathfrak{f}(\mathrm{t}, \mathrm{x}, \mathrm{y})| \leq \frac{|\sin(\mathrm{t})|}{35e^{\mathrm{t}+3}} (|\mathrm{x}| + |\mathrm{y}|) + 0.02, \quad \mathrm{t} \in J.$$

This confirms that condition (Ax_3) *is satisfied with:*

$$\mathfrak{p}_1(\mathrm{t}) = 0.02, \quad \mathfrak{p}_2(\mathrm{t}) = \mathfrak{p}_3(\mathrm{t}) = \frac{|\sin(\mathrm{t})|}{35e^{\mathrm{t}+3}}, \quad \textit{and} \quad \mathfrak{p}_2^* = \mathfrak{p}_3^* = \frac{1}{35e^4}.$$

Furthermore, we define:

$$\mathfrak{g}_0 = \frac{1}{12}. \tag{3.14}$$

Additionally, condition (Ax_4) *from Theorem 3.2 is met by selecting a constant* γ *in the range:*

$$3.9898 \times 10^{-3} \leq \gamma \leq 167.43. \tag{3.15}$$

By applying Theorem 3.2, we conclude that problem (3.13) admits at least one solution in the interval J*.*

Example 3.2. *Consider the following fractional boundary value problem defined on the interval* $J = [0, 2]$*:*

$$\begin{cases} {}^C D_{0^+}^{0.75,\psi} \left[{}^C D_{0^+}^{1.45,\psi} \left(\dfrac{\mathrm{x}(\mathrm{t})}{\mathrm{g}(\mathrm{t}, \mathrm{x}(\mathrm{t}))} \right) + \dfrac{33}{100}\mathrm{x}(\mathrm{t}) \right] = \mathfrak{f}(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{x}(\zeta\mathrm{t})), \quad \mathrm{t} \in J, \\ \mathrm{x}(1) = 0, \quad \mathrm{x}'(1) = 0, \quad \mathrm{x}(3/2) = 0, \quad \textit{where } 0 < \kappa = \dfrac{3}{2} \leq 2. \end{cases} \tag{3.16}$$

Here, the kernel function is given by $\psi(\mathrm{t}) = \frac{1}{2}\sqrt{\mathrm{t}+2}$*, which is non-decreasing. The nonlinear source function* $\mathfrak{f}$ *is defined as:*

$$\mathfrak{f}(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{x}(\mathrm{xt})) = \frac{\sin(\mathrm{t})\mathrm{x}(\frac{9}{10}\mathrm{t}) + [\cos(\mathrm{t}) + \mathrm{t}^2]\mathrm{x}(\mathrm{t})}{e^{2+\mathrm{t}} + 32\pi} + 0.01.$$

It is clear that $\mathfrak{f} \in \mathcal{C}([0, 2] \times \mathbb{R}^2, \mathbb{R})$ *and satisfies the Lipschitz-type inequality:*

$$|\mathfrak{f}(\mathrm{t}, x_1, x_2) - \mathfrak{f}(\mathrm{t}, y_1, y_2)| \leq \frac{1 + \mathrm{t}^2}{e^{2+\mathrm{t}} + 32\pi}(|x_1 - y_1| + |x_2 - y_2|).$$

Thus, assumptions $(Ax_{\boldsymbol{6}})$ *and* $(Ax_{\boldsymbol{7}})$ *are fulfilled with:*

$$l_{\mathfrak{f}}^{*} = 0.18316, \quad L = 0.11989.$$

The function $\mathfrak{g}$ *is given by:*

$$\mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t})) = \frac{1 + \mathrm{t}^{3/2}}{12e^{\mathrm{t}^2+\pi}}\mathrm{x}(\mathrm{t}) + 0.2,$$

which is Lipschitz continuous with constants $l_{\mathfrak{g}}^{*} = 0.11737$ *and* $M = 0.51904$.

Additionally, we calculate:

$$\Lambda_1 = 2.7684 \times 10^{-2}, \quad \Lambda_2 = 8.6624 \times 10^{-2}, \quad \Lambda_3 = 2.2966 \times 10^{-2},$$

implying that:

$$\varepsilon \geq 2.5681 \times 10^{-3}.$$

Let us set $\sigma = 25$, *and observe that:*

$$\Lambda^{*} = 0.95166 < 1.$$

Since all the required hypotheses of Theorem 3.3 are satisfied, it follows that problem (3.16) *admits a unique solution in the interval* $[0, 2]$.

Now, assume $\varrho = \frac{2}{5} > 0$. *According to Theorem 3.4, if* $\mathrm{x} \in \mathcal{C}^3([0, 2], \mathbb{R})$ *satisfies:*

$$\left| {}^{C}D_{0^+}^{0.75,\psi} \left[{}^{C}D_{0^+}^{1.45,\psi} \left(\frac{\mathrm{x}(\mathrm{t})}{\mathfrak{g}(\mathrm{t}, \mathrm{x}(\mathrm{t}))} \right) + \frac{33}{100}\mathrm{x}(\mathrm{t}) \right] - \mathfrak{f}(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{x}(\mathrm{xt})) \right| < \frac{2}{5}, \tag{3.17}$$

then there exists a solution $\mathrm{z} \in \mathcal{C}([0, 2], \mathbb{R})$ *to problem* (3.16) *such that:*

$$\|\mathrm{x} - \mathrm{z}\| \leq \frac{2}{5}\varphi,$$

where

$$\varphi = \frac{\mathbb{S}}{1 - \Lambda^{*}} = 0.33430 > 0.$$

Hence, the fractional boundary value problem (3.16) *is Ulam–Hyers stable on* $[0, 2]$.

Finally, setting $\varrho = 0$ *yields* $\mathfrak{J}(0) = 0$, *which implies that the problem* (3.16) *also enjoys generalized Ulam–Hyers stability.*

3.5 Notes and Remarks

Besides the main problem discussed in this chapter, we have considered a generalized framework of hybrid Langevin fractional pantograph differential equations involving ψ-Caputo fractional derivatives. The analysis focused on investigating the existence, uniqueness, and Ulam stability of the solutions within Banach spaces.

The results were established using Dhage's and Banach's fixed-point theorems, combined with the contraction mapping principle. Several examples were provided to illustrate the applicability and relevance of the theoretical findings.

This work serves as a continuation and generalization of previous studies, particularly those that addressed linear and nonlinear initial and boundary value problems involving various types of fractional derivatives, as discussed in [69,91,92]. Moreover, the stability analysis draws inspiration from the monographs of Abbas et al. [1] and the works [26,30], which extensively develop the Ulam stability concepts for fractional functional equations.

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[4] **H. Bouzid**, A. Benali, A. Salim, L. Tabharit, A Study on Some Classes of Hybrid Langevin Pantograph ψ-Caputo Fractional Coupled Systems, *Pan-Amer. J. Math.* Vol(4), 14, (2025).

chapter

4 Analysis of Existence and Ulam Stability for Hybrid Langevin Pantograph Coupled Systems

4.1 Introduction and Motivations

This chapter focuses on establishing the existence of solutions for a class of coupled hybrid Langevin fractional pantograph differential equations involving Ψ-Caputo fractional derivatives in Banach spaces. The uniqueness of solutions is demonstrated using Banach's fixed-point theorem, while their existence is verified through Dhage's hybrid fixed-point theorem for the sum of three operators. Furthermore, we examine the stability of these solutions in both the Ulam-Hyers sense and its generalized form. To support the theoretical findings, several illustrative examples are provided.

Fractional calculus extends classical differentiation and integration to non-integer orders, offering a powerful mathematical framework. Various definitions of fractional derivatives exist, including the Riemann-Liouville, Caputo, Hilfer, and Hadamard derivatives, among others. For fundamental results in fractional calculus and fractional differential equations, the reader is referred to [1, 2, 9, 24].

Nonlinear coupled systems involving fractional derivatives have gained significant attention in contemporary research due to their broad applications in applied mathematics. As a result, numerous studies and monographs have investigated the existence,

[5]. **H. Bouzid**, A. Benali, A. Salim and I. M. Erhan, Existence and Ulam stability results of hybrid Langevin pantograph ψ-fractional coupled systems. *Filomat*, vol. **39**(10), 3401–3424, **(2025)** .

stability, and uniqueness of solutions for various fractional differential equations and inclusions, utilizing different types of fractional derivatives and boundary conditions.

Additionally, dynamical systems often emerge as special cases of fractional differential equations. The study of stability in differential and integral equations is crucial for practical applications. For fundamental results and recent advancements in Ulam-type stability of differential and integral equations, we refer the reader to [1, 26, 30, 60, 83, 89, 110].

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1,\Psi}\left[{}^{C}D_{0^+}^{\beta_1,\Psi}\left(\frac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right)+\mu_1\mathrm{x(t)}\right]=\mathfrak{f}_1(\mathrm{t},\mathrm{x(t)},\mathrm{y}(\zeta\mathrm{t})), & \mathrm{t}\in J=[0,b], \\ {}^{C}D_{0^+}^{\alpha_2,\Psi}\left[{}^{C}D_{0^+}^{\beta_2,\Psi}\left(\frac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right)+\mu_2\mathrm{y(t)}\right]=\mathfrak{f}_2(\mathrm{t},\mathrm{y(t)},\mathrm{x}(\tilde{\zeta}\mathrm{t})), & \mathrm{t}\in J=[0,b], \end{cases} \tag{4.1}$$

Under the given boundary conditions

$$\begin{cases} \left(\dfrac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right)\Bigg|_{\mathrm{t}=0}=\mathcal{V}_1, \quad \left(\dfrac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right)\Bigg|_{\mathrm{t}=0}=\mathcal{V}_2, \\ \left(\dfrac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right)'\Bigg|_{\mathrm{t}=0}=\left(\dfrac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right)'\Bigg|_{\mathrm{t}=0}=0, \\ \left(\dfrac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right)\Bigg|_{\mathrm{t}=\epsilon_1}=\mathcal{V}_3, \quad \left(\dfrac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right)\Bigg|_{\mathrm{t}=\epsilon_2}=\mathcal{V}_4, \ 0<\epsilon_1,\epsilon_2\leq b, \end{cases} \tag{4.2}$$

where ${}^{C}D_{0^+}^{\alpha_\mathrm{i},\Psi}$, ${}^{C}D_{0^+}^{\beta_\mathrm{i},\Psi}$ are the Ψ-Caputo fractional derivatives of order $\alpha_\mathrm{i}\in(0,1]$, $\beta_\mathrm{i}\in(1,2]$, for $i=1,2$, $\mu_1,\mu_2\in\mathbb{R}\setminus\{0\}$, $0<b$, $\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3,\mathcal{V}_4\in\mathbb{R}$, $(\mathcal{V}_1\neq\mathcal{V}_3$ and $\mathcal{V}_2\neq\mathcal{V}_4)$ and $0<\zeta,\tilde{\zeta}<1$ and $\Psi\in\mathcal{C}^1(J,\mathbb{R}^+)$ be an increasing differentiable function such that $\Psi'(\mathrm{t})\neq 0$, for all $\mathrm{t}\in J$. The given functions $\mathfrak{f}_\mathrm{j}:J\times\mathbb{R}^2\to\mathbb{R}$, $\mathcal{W}_\mathrm{j}:J\times\mathbb{R}\to\mathbb{R}$, and $\mathfrak{g}_\mathrm{j}:J\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ are continuous with $\mathrm{j}=1,2$. It is important to highlight that this study represents the first documented investigation in the literature where a coupled fractional model concurrently examines both Langevin and pantograph systems.

The main contributions of this research are summarized as follows:

- The coupled system (4.1)-(4.2) integrates both Langevin and pantograph components while incorporating diverse boundary conditions, thereby extending existing formulations in the literature.

- The use of function deformation techniques allows the representation of the coupled system (4.1)-(4.2) in a hybrid framework, facilitating the application of Dhage's hybrid fixed-point theorems.
- The introduction of the Ψ-fractional operator unifies multiple fractional derivatives into a single operator, enhancing the integration of classical results and enabling novel applications.
- This work generalizes the results in [72] by investigating a coupled Langevin hybrid fractional system subject to multi-point boundary conditions, where the orders satisfy $\beta_i \in (0, 1]$ and $\alpha_i \in (1, 2]$, for $i = 1, 2$.
- This study provides, for the first time, a comprehensive analysis of existence, uniqueness, and stability properties for the coupled system (4.1)-(4.2).

Now, we consider the following Banach space

$$\Upsilon = \{(x, y) : \ x, y \in \mathcal{C}^3(J, \mathbb{R})\},$$

endowed with the norm

$$\|(x, y)\|_{\Upsilon} = \|x\|_\infty + \|y\|_\infty.$$

4.2 Existence Results

Let $\mathcal{Y} : J \to \mathbb{R}$ be a function satisfying $\mathcal{Y}(\cdot) \in \mathcal{C}(J, \mathbb{R})$, where $\mathcal{G} \in \mathcal{C}^3(J, \mathbb{R} \setminus \{0\})$ and $\mathcal{H} \in \mathcal{C}^3(J, \mathbb{R})$ are continuous functions.

We examine the following linear fractional differential equation, which is associated with equation (4.1).

$$ {}^{C}D^{\alpha,\Psi}_{0^+}\left[{}^{C}D^{\beta,\Psi}_{0^+}\left(\frac{x_(t) - \mathcal{H}(t)}{\mathcal{G}(t)}\right) + \mu x(t)\right] = \mathcal{Y}(t), \quad t \in J, \tag{4.3}$$

where $\alpha \in (0,1]$, $\beta \in (1,2], \mu \in \mathbb{R} - \{0\}$, with according to the boundary conditions

$$\begin{cases} \left(\dfrac{\mathrm{x(t)} - \mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right)\Big|_{\mathrm{t}=0} = c, \\ \left(\dfrac{\mathrm{x(t)} - \mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right)'\Big|_{\mathrm{t}=0} = 0, \\ \left(\dfrac{\mathrm{x(t)} - \mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right)\Big|_{\mathrm{t}=\epsilon} = \ell, \ 0 < \epsilon \leq b \ , \end{cases} \tag{4.4}$$

where $\mathcal{G} \in \mathcal{C}^3(J, \mathbb{R} \setminus \{0\})$, $\mathcal{H} \in \mathcal{C}^3(J, \mathbb{R})$, $c, \ell \in \mathbb{R}$ with $c \neq \ell$, and ϵ is a pre-fixed point satisfying $0 < \epsilon \leq b$. Additionally, $\mathcal{Y} \in \mathcal{C}(J, \mathbb{R})$. The following theorem demonstrates that the problem (4.3)-(4.4) has a unique solution, given by:

$$\begin{aligned} \mathrm{x(t)} :=& \mathcal{G}(\mathrm{t}) \Big[I_{0^+}^{\alpha+\beta,\Psi} \mathcal{Y}(\mathrm{t}) - \mu I_{0^+}^{\beta,\Psi} \mathrm{x(t)} + c - \ell \\ &+ \frac{[\Psi(\mathrm{t}) - \Psi(0)]^\beta}{[\Psi(\epsilon) - \Psi(0)]^\beta} \left(\mu I_{0^+}^{\beta,\Psi} \mathrm{x}(\epsilon) - I_{0^+}^{\alpha+\beta,\Psi} \mathcal{Y}(\epsilon) + \ell - c \right) \Big] + \mathcal{H}(\mathrm{t}). \end{aligned} \tag{4.5}$$

Theorem 4.1. *The function* x *satisfies problem* (4.3)-(4.4) *if and only if it satisfies* (4.5).

Proof. Assume that x satisfies the problem (4.3)-(4.4) and such that the function $\eta : \mathrm{t} \longrightarrow \left(\frac{\mathrm{x(t)} - \mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})} \right) \in \mathcal{C}^3(J, \mathbb{R})$. We prove that x is a solution to the equation (4.5). Applying the fractional integral $I_{0^+}^{\alpha,\Psi}$ to both sides of (4.3) and using Lemma 1.1, we have

$$^{C}D_{0^+}^{\beta,\Psi} \eta(\mathrm{t}) + \Phi \mathrm{x(t)} = I_{0^+}^{\alpha,\Psi} \mathcal{Y}(\mathrm{t}) + \varsigma_0. \tag{4.6}$$

Now, Applying $I_{0^+}^{\beta,\Psi}$ to both sides of (4.12)

$$\eta(\mathrm{t}) = I_{0^+}^{\alpha+\beta,\Psi} \mathcal{Y}(\mathrm{t}) + \varsigma_0 \frac{[\Psi(\mathrm{t}) - \Psi(0)]^\beta}{\Gamma(\beta+1)} - \mu I_{0^+}^{\beta,\Psi} \mathrm{x(t)} + \varsigma_1 + \varsigma_2 [\Psi(\mathrm{t}) - \Psi(0)]. \tag{4.7}$$

Then,

$$\frac{\mathrm{x(t)} - \mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})} = I_{0^+}^{\alpha+\beta,\Psi} \mathcal{Y}(\mathrm{t}) + \varsigma_0 \frac{[\Psi(\mathrm{t}) - \Psi(0)]^\beta}{\Gamma(\beta+1)} - \mu I_{0^+}^{\beta,\Psi} \mathrm{x(t)} + \varsigma_1 + \varsigma_2 [\Psi(\mathrm{t}) - \Psi(0)],$$

which implies that

$$\begin{aligned} \mathrm{x(t)} =& \mathcal{G}(\mathrm{t}) \Big[I_{0^+}^{\alpha+\beta,\Psi} \mathcal{Y}(\mathrm{t}) + \varsigma_0 \frac{[\Psi(\mathrm{t}) - \Psi(0)]^\beta}{\Gamma(\beta+1)} - \mu I_{0^+}^{\beta,\Psi} \mathrm{x(t)} \\ &+ \varsigma_1 + \varsigma_2 [\Psi(\mathrm{t}) - \Psi(0)] + \mathcal{H}(\mathrm{t}). \end{aligned} \tag{4.8}$$

such as $\varsigma_i \in \mathbb{R}$, with $i = 0, 1, 2$. Next, By the condition $\frac{\mathrm{x}(0)-\mathcal{H}(0)}{\mathcal{G}(0)} = c$ and $\left(\frac{\mathrm{x}(t)-\mathcal{H}(t)}{\mathcal{G}(t)}\right)'\Big|_{t=0} =$ 0 gives

$$\varsigma_1 := c \text{ and } \varsigma_2 := 0. \tag{4.9}$$

On the other hand by $\frac{\mathrm{x}(\epsilon)-\mathcal{H}(\epsilon)}{\mathcal{G}(\epsilon)} = \ell$, we have

$$\varsigma_0 := \frac{\Gamma(\beta+1)}{[\Psi(\epsilon)-\Psi(0)]^\beta}\left(\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\epsilon) - I_{0^+}^{\alpha+\beta,\Psi}\mathcal{Y}(\epsilon) + \ell - c\right). \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8), we obtain (4.5).

Conversely, assume x satisfies the equation (4.5) such that the function $\eta : t \longrightarrow \left(\frac{\mathrm{x}(t)-\mathcal{H}(t)}{\mathcal{G}(t)}\right) \in \mathcal{C}(J, \mathbb{R})$. Applying operator ${}^C D_{0^+}^{\beta,\Psi}$ on both sides of (4.5), and since $\mathcal{G}(\mathrm{t}) \neq 0$ for all $\mathrm{t} \in J$, then, from Lemma 1.1 and Lemma 1.3, we obtain

$${}^C D_{0^+}^{\beta,\Psi}\eta(\mathrm{t}) = I_{0^+}^{\alpha,\Psi}\mathcal{Y}(\mathrm{t}) - \varPhi\mathrm{x}(\mathrm{t}) + \frac{\Gamma(\beta+1)}{[\Psi(\epsilon)-\Psi(0)]^\beta}\left(\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\epsilon) - I_{0^+}^{\alpha+\beta,\Psi}\mathcal{Y}(\epsilon) + \ell - c\right). \tag{4.11}$$

Reapplying, ${}^C D_{0^+}^{\alpha,\Psi}$ to the above equation, we obtain

$${}^C D_{0^+}^{\alpha,\Psi}\left[{}^C D_{0^+}^{\beta,\Psi}\left(\frac{\mathrm{x}(\mathrm{t})-\mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right) + \varPhi\mathrm{x}(\mathrm{t})\right] = \mathcal{Y}(\mathrm{t}), \tag{4.12}$$

Taking the limit $t \to 0$ of (4.5) we obtain

$$\tfrac{\mathrm{x}(0)-\mathcal{H}(0)}{\mathcal{G}(0)} = c, \tag{4.13}$$

Substituting $\mathrm{t} = \epsilon$ into (4.5), we have

$$\frac{\mathrm{x}(\epsilon_1)-\mathcal{H}(\epsilon)}{\mathcal{G}(\epsilon)} = \ell.$$

Now, Applying $D_{0^+}^1$ to both sides of (4.5) gives

$$\begin{aligned}\left(\frac{\mathrm{x}(\mathrm{t})-\mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right)' &= I_{0^+}^{\alpha+\beta-1,\Psi}\mathcal{Y}(\mathrm{t}) - \mu I_{0^+}^{\beta-1,\Psi}\mathrm{x}(\mathrm{t}) \\ &\quad + \frac{\Gamma(\beta+1)}{[\Psi(\epsilon)-\Psi(0)]^\beta}\left(\mu I_{0^+}^{\beta,\Psi}\mathrm{x}(\epsilon) - I_{0^+}^{\alpha+\beta,\Psi}\mathcal{Y}(\epsilon) + \ell - c\right) I_{0^+}^{\beta-1,\Psi}1.\end{aligned} \tag{4.14}$$

Taking the limit $\mathrm{t} \to 0$ of (4.14) we have

$$\left(\tfrac{\mathrm{x}(\mathrm{t})-\mathcal{H}(\mathrm{t})}{\mathcal{G}(\mathrm{t})}\right)'\Big|_{\mathrm{t}=0} = 0. \tag{4.15}$$

This confirms that the boundary conditions given in (4.4) are fulfilled. □

Next, we present the solution for the coupled system (4.1)-(4.2)

Definition 4.1. *A function* $\mathrm{x} \in \mathcal{C}^3(J, \mathbb{R})$ *is said to be a solution of the fractional problem* (4.3)-(4.4) *on* J *if it satisfies both equations* (4.3) *and* (4.4) *on* J.

Lemma 4.1. *Let* $\alpha_{\mathrm{i}} \in (0, 1]$, $\beta_{\mathrm{i}} \in (1, 2]$, *for* $i = 1, 2$, $\mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$, $0 < b$, $0 < \epsilon_1, \epsilon_2 < b$, $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4 \in \mathbb{R}$, $(\mathcal{V}_1 \neq \mathcal{V}_3$ *and* $\mathcal{V}_2 \neq \mathcal{V}_4)$ *and* $0 < \zeta, \tilde{\zeta} < 1$, *let* $\mathfrak{f}_{\mathrm{j}} : J \times \mathbb{R}^2 \to \mathbb{R}$, $\mathcal{W}_{\mathrm{j}} : J \times \mathbb{R} \to \mathbb{R}$, *and* $\mathfrak{g}_{\mathrm{j}} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *are continuous functions with* $\mathrm{j} = 1, 2$. *If the function* $\mathrm{t} \longrightarrow \left(\frac{\mathrm{x(t)} - \mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right) \in \mathcal{C}^3(J, \mathbb{R})$ *and similarly, the function* $\mathrm{t} \longrightarrow \left(\frac{\mathrm{y(t)} - \mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right) \in \mathcal{C}^3(J, \mathbb{R})$, *then* $(\mathrm{x}, \mathrm{y}) \in \mathrm{t}$ *satisfies the coupled system* (4.1)-(4.2) *if and only if* (x, y) *is the fixed point of the operator* $\mathcal{T} : \mathcal{C}(J, \mathbb{R})^2 \longrightarrow \mathcal{C}(J, \mathbb{R})^2$ *defined by*

$$\mathcal{T}(\mathrm{x}, \mathrm{y})(\mathrm{t}) := (\mathcal{T}_1(\mathrm{x}, \mathrm{y})(\mathrm{t}), \mathcal{T}_2(\mathrm{x}, \mathrm{y})(\mathrm{t})),$$

such as

$$\mathcal{T}_1(\mathrm{x}, \mathrm{y})(\mathrm{t}) := \mathfrak{g}_1(\mathrm{t}, \mathrm{x(t)}) \left[\Phi_1(\mathrm{x}, \mathrm{y})(\mathrm{t}) - \tfrac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_1}}{(\Psi(\epsilon_1) - \Psi(0))^{\beta_1}} \Phi_1(\mathrm{x}, \mathrm{y})(\epsilon_1)\right] + \mathcal{W}_1(\mathrm{t}, \mathrm{x(t)}) \quad (4.16)$$

and

$$\mathcal{T}_2(\mathrm{x}, \mathrm{y})(\mathrm{t}) := \mathfrak{g}_2(\mathrm{t}, \mathrm{y(t)}) \left[\Phi_2(\mathrm{x}, \mathrm{y})(\mathrm{t}) - \tfrac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_2}}{(\Psi(\epsilon_2) - \Psi(0))^{\beta_2}} \Phi_2(\mathrm{x}, \mathrm{y})(\epsilon_2)\right] + \mathcal{W}_2(\mathrm{t}, \mathrm{y(t)}), \quad (4.17)$$

where

$$\begin{aligned}\Phi_1(\mathrm{x}, \mathrm{y})(\mathrm{t}) :=& \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\alpha_1 + \beta_1 - 1}}{\Gamma(\alpha + \beta)} \mathfrak{f}_1(s, \mathrm{x}(s), \mathrm{y}(\zeta s)) \mathrm{d}s \\ &- \mu_1 \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\beta_1 - 1}}{\Gamma(\beta_1)} \mathrm{x}(s) \mathrm{d}s + \mathcal{V}_1 - \mathcal{V}_3,\end{aligned}$$

and

$$\begin{aligned}\Phi_2(\mathrm{x}, \mathrm{y})(\mathrm{t}) :=& \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\alpha_2 + \beta_2 - 1}}{\Gamma(\alpha_2 + \beta_2)} \mathfrak{f}_2(s, \mathrm{y}(s), \mathrm{x}(\tilde{\zeta} s)) \mathrm{d}s \\ &- \mu_2 \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t}) - \Psi(s))^{\beta_2 - 1}}{\Gamma(\beta_2)} \mathrm{y}(s) \mathrm{d}s + \mathcal{V}_2 - \mathcal{V}_4.\end{aligned}$$

Given that the functions $\mathfrak{g}_{\mathrm{i}}$ and $\mathcal{W}_{\mathrm{i}}$ are continuous and $\mathfrak{f}_{\mathrm{i}}(\mathrm{t}, \cdot, \cdot) \in \mathcal{C}(J \times \mathbb{R}^2, \mathbb{R})$, for $i = 1, 2$, it follows that $\mathcal{T}(\mathrm{x}, \mathrm{y}) \in \Upsilon$.

The next result relies on the application of the Banach fixed-point theorem. Furthermore, we assume the following conditions for this outcome:

$(Ax_{\mathbf{1}})$ The functions $(\mathfrak{f}_{\mathrm{i}})_{i=1,2} : J \times \mathbb{R}^2 \to \mathbb{R}$, $(\mathcal{W}_{\mathrm{i}})_{i=1,2} : J \times \mathbb{R} \to \mathbb{R}$, and $(\mathfrak{g}_{\mathrm{i}})_{i=1,2} : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ are continuous.

$(Ax_{\mathbf{2}})$ There exist positive functions $l_{\mathfrak{f}_{\mathrm{i}}}, l_{\mathfrak{g}_{\mathrm{i}}}, l_{\mathcal{W}_{\mathrm{i}}} \in \mathcal{C}(J, \mathbb{R}^+)$ such that

$$|\mathfrak{f}_{\mathrm{i}}(\mathrm{t}, \mathrm{x}, \mathrm{y}) - \mathfrak{f}_{\mathrm{i}}(\mathrm{t}, \bar{\mathrm{x}}, \bar{\mathrm{y}})| \leq l_{\mathfrak{f}_{\mathrm{i}}}(\mathrm{t}) \left(|\mathrm{x} - \bar{\mathrm{x}}| + |\mathrm{y} - \bar{\mathrm{y}}|\right),$$
$$|\mathfrak{g}_{\mathrm{i}}(\mathrm{t}, \mathrm{x}) - \mathfrak{g}_{\mathrm{i}}(\mathrm{t}, \bar{\mathrm{x}})| \leq l_{\mathfrak{g}_{\mathrm{i}}}(\mathrm{t})|\mathrm{x} - \bar{\mathrm{x}}|,$$

and

$$|\mathcal{W}_{\mathrm{i}}(\mathrm{t}, \mathrm{x}) - \mathcal{W}_{\mathrm{i}}(\mathrm{t}, \bar{\mathrm{x}})| \leq l_{\mathcal{W}_{\mathrm{i}}}(\mathrm{t})|\mathrm{x} - \bar{\mathrm{x}}|,$$

for $i = 1, 2$, for all $\mathrm{t} \in J$ and $\mathrm{x}, \mathrm{y}, \bar{\mathrm{x}}, \bar{\mathrm{y}} \in \mathbb{R}$,

where

$$l^*_{\mathfrak{f}_{\mathrm{i}}} = \sup_{\mathrm{t} \in J} l_{\mathfrak{f}_{\mathrm{i}}}(\mathrm{t}), l^*_{\mathfrak{g}_{\mathrm{i}}} = \sup_{\mathrm{t} \in J} l_{\mathfrak{g}_{\mathrm{i}}}(\mathrm{t}) \text{ and } l^*_{\mathcal{W}_{\mathrm{i}}} = \sup_{\mathrm{t} \in J} l_{\mathcal{W}_{\mathrm{i}}}(\mathrm{t}),\ i = 1, 2.$$

$(Ax_{\mathbf{3}})$ There exist positive constants $\mathcal{L}_{\mathrm{i}}$, $\mathcal{M}_{\mathrm{i}}$ and $\mathcal{K}_{\mathrm{i}}$, where $|\mathfrak{f}_{\mathrm{i}}(\mathrm{t}, \cdot, \cdot)| < \mathcal{L}_{\mathrm{i}}$, $|\mathfrak{g}_{\mathrm{i}}(\mathrm{t}, \cdot)| < \mathcal{M}_{\mathrm{i}}$, and $|\mathcal{W}_{\mathrm{i}}(\mathrm{t}, \cdot)| < \mathcal{K}_{\mathrm{i}}$, for all $\mathrm{t} \in J$ and $\cdot \in \mathbb{R}$, where $i = 1, 2$.

For the sake of clarity, we denote

$$\begin{cases} \Re_1 = \frac{[(\Psi(b)-\Psi(0))^{\alpha_1+\beta_1}+(\Psi(b)-\Psi(0))^{\beta_1}(\Psi(\epsilon_1)-\Psi(0))^{\alpha_1}]}{\Gamma(\alpha_1+\beta_1+1)}, \\ \Re_2 = \frac{2|\mu_1|(\Psi(b)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}, \\ \Re_3 = (|\mathcal{V}_1| + |\mathcal{V}_3|)\left(\frac{[\Psi(b)-\Psi(0)]^{\beta_1}}{[\Psi(\epsilon_1)-\Psi(0)]^{\beta_1}} + 1\right), \\ \nabla_1 = \frac{[(\Psi(b)-\Psi(0))^{\alpha_2+\beta_2}+(\Psi(b)-\Psi(0))^{\beta_2}(\Psi(\epsilon_2)-\Psi(0))^{\alpha_2}]}{\Gamma(\alpha_2+\beta_2+1)}, \\ \nabla_2 = \frac{2|\mu_2|(\Psi(b)-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}, \\ \nabla_3 = (|\mathcal{V}_2| + |\mathcal{V}_4|)\left(\frac{[\Psi(b)-\Psi(0)]^{\beta_2}}{[\Psi(\epsilon_2)-\Psi(0)]^{\beta_2}} + 1\right), \\ \varpi_1 = (2l^*_{\mathfrak{f}_1}\mathcal{M}_1 + l^*_{\mathfrak{g}_1}\mathcal{L}_1)\Re_1 + (\mathcal{M}_1 + l^*_{\mathfrak{g}_1}\mathrm{t})\Re_2 + l^*_{\mathfrak{g}_1}\Re_3 + l^*_{\mathcal{W}_1}, \\ \varpi_2 = (2l^*_{\mathfrak{f}_2}\mathcal{M}_2 + l^*_{\mathfrak{g}_2}\mathcal{L}_2)\nabla_1 + (\mathcal{M}_2 + l^*_{\mathfrak{g}_2}\mathrm{t})\nabla_2 + l^*_{\mathfrak{g}_2}\nabla_3 + l^*_{\mathcal{W}_2}. \end{cases}$$

Theorem 4.2. *Suppose that $(Ax_{\mathbf{1}})$–$(Ax_{\mathbf{3}})$ holds. If*

$$\sum_{i=1}^{2} \varpi_{\mathrm{i}} < 1, \tag{4.18}$$

then the coupled system (4.1)-(4.2) has a unique solution on $J = (0, b]$.

Proof. Setting

$$\mathrm{t} \geq \left(\frac{\mathcal{M}_1(\Re_3 + \mathcal{L}_1\Re_1) + \mathcal{K}_1}{1 - \mathcal{M}_1\Re_2} + \frac{\mathcal{M}_2(\nabla_3 + \mathcal{L}_2\nabla_1) + \mathcal{K}_2}{1 - \mathcal{M}_2\nabla_2} \right),$$

with $0 \leq \mathcal{M}_1\Re_2, \mathcal{M}_2\nabla_2 < 1$. we show that $\mathcal{T}\mathcal{B}_{\mathrm{r}} \subset \mathcal{B}_{\mathrm{r}}$, where

$$\mathcal{B}_{\mathrm{r}} = \left\{ \mathrm{x}, \mathrm{y} \in \mathcal{C}^3(J; \mathbb{R}) : \|(\mathrm{x}, \mathrm{y})\|_{\Upsilon} \leq \mathrm{r} \right\}.$$

For $(\mathrm{x}, \mathrm{y}) \in \mathcal{B}_{\mathrm{r}}$ and for each $\mathrm{t} \in J$, from the definition of $\mathcal{T}$ and hypothesis $(Ax\mathbf{1})$–$(Ax_{\mathbf{3}})$, we obtain

$$\begin{aligned}
&|\mathcal{T}_1(\mathrm{x}, \mathrm{y})(\mathrm{t})| \\
&\leq \left| \mathfrak{g}_1(\mathrm{t}, \mathrm{x}(\mathrm{t})) \left[\Phi_1(\mathrm{x}, \mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_1}}{(\Psi(\epsilon_1) - \Psi(0))^{\beta_1}} \Phi_1(\mathrm{x}, \mathrm{y})(\epsilon_1) \right] + \mathcal{W}_1(\mathrm{t}, \mathrm{x}(\mathrm{t})) \right| \\
&\leq |\mathfrak{g}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}))| \left[\frac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_1}}{(\Psi(\epsilon_1) - \Psi(0))^{\beta_1}} \right. \\
&\quad \times \left(\frac{\mathcal{L}_1(\Psi(\epsilon_1) - \Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|\mu_1|(\Psi(\epsilon_1) - \Psi(0))^{\beta_1}}{\Gamma(\beta_1 + 1)} \mathrm{r} \right) \\
&\quad \left. + \frac{\mathcal{L}_1(\Psi(\mathrm{t}) - \Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{|\mu_1|(\Psi(\mathrm{t}) - \Psi(0))^{\beta_1}}{\Gamma(\beta_1 + 1)} \mathrm{r} \right] + |\mathcal{W}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}))| \\
&\leq \mathcal{M}_1 \left[\Re_3 + \mathcal{L}_1\Re_1 + \Re_2\mathrm{r} \right] + \mathcal{K}_1 \\
&\leq \mathrm{r}.
\end{aligned}$$

and

$$\begin{aligned}
&|\mathcal{T}_2(\mathrm{x}, \mathrm{y})(\mathrm{t})| \\
&\leq \left| \mathfrak{g}_2(\mathrm{t}, \mathrm{x}(\mathrm{t})) \left[\Phi_2(\mathrm{x}, \mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_1}}{(\Psi(\epsilon_2) - \Psi(0))^{\beta_2}} \Phi_2(\mathrm{x}, \mathrm{y})(\epsilon_2) \right] + \mathcal{W}_2(\mathrm{t}, \mathrm{x}(\mathrm{t})) \right| \\
&\leq |\mathfrak{g}_2(\mathrm{t}, \mathrm{x}(\mathrm{t}))| \left[\frac{(\Psi(\mathrm{t}) - \Psi(0))^{\beta_2}}{(\Psi(\epsilon_2) - \Psi(0))^{\beta_2}} \right. \\
&\quad \times \left(\frac{\mathcal{L}_2(\Psi(\epsilon_2) - \Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{|\mu_2|(\Psi(\epsilon_2) - \Psi(0))^{\beta_2}}{\Gamma(\beta_2 + 1)} \mathrm{r} \right) \\
&\quad \left. + \frac{\mathcal{L}_2(\Psi(\mathrm{t}) - \Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{|\mu_2|(\Psi(\mathrm{t}) - \Psi(0))^{\beta_2}}{\Gamma(\beta_2 + 1)} \mathrm{r} \right] + |\mathcal{W}_2(\mathrm{t}, \mathrm{x}(\mathrm{t}))| \\
&\leq \mathcal{M}_2 \left[\nabla_3 + \mathcal{L}_2\nabla_1 + \nabla_2\mathrm{r} \right] + \mathcal{K}_2 \\
&\leq \mathrm{r}.
\end{aligned}$$

Hence,

$$\|\mathcal{T}(\mathrm{x},\mathrm{y})\|_{\Upsilon} \leq \mathrm{r},$$

which implies that $\mathcal{T}\mathcal{B}_{\mathrm{r}} \subset \mathcal{B}_{\mathrm{r}}$.

Let for $(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}}) \in \mathcal{B}_{\mathrm{r}}$ and for any $\mathrm{t} \in J$. By $(Ax_{\mathbf{2}})$, we get

$$\begin{aligned}
&|\varPhi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \varPhi_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| \\
&\leq \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)} |\mathfrak{f}_1(s,\mathrm{x}(s),\mathrm{y}(\zeta_1 s)) - \mathfrak{f}_1(s,\bar{\mathrm{x}}_1,(s),\bar{\mathrm{y}}(\zeta_1 s))| \mathrm{d}s \\
&\quad + |\mu_1| \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta_1-1}}{\Gamma(\beta_1)} |\mathrm{x}(s)-\bar{\mathrm{x}}(s)| \mathrm{d}s \\
&\leq \left(\frac{2l^*_{\mathfrak{f}_1}(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)} \right) \|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Upsilon},
\end{aligned}$$

and by condition $(Ax_{\mathbf{3}})$, we obtain

$$|\varPhi_1(\mathrm{x},\mathrm{y})(\mathrm{t})| \leq \frac{\mathcal{L}_1(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)} + |\mathcal{V}_1\| + |\mathcal{V}_3|.$$

In the same way, we obtain

$$\begin{aligned}
&|\varPhi_2(\mathrm{x},\mathrm{y})(\mathrm{t}) - \tilde{\varPhi}(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| \\
&\leq \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)} |\mathfrak{f}_2(s,\mathrm{y}(s),\mathrm{x}(\zeta s)) - \mathfrak{f}_2(s,\bar{\mathrm{y}},(s),\bar{\mathrm{x}}(\zeta_2 s))| \mathrm{d}s \\
&\quad + |\mu_2| \int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta_2-1}}{\Gamma(\beta_2)} |\mathrm{y}(s)-\bar{\mathrm{y}}(s)| \mathrm{d}s \\
&\leq \left(\frac{2l^*_{\mathfrak{f}_2}(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_2|(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} \right) \|\mathrm{x}-\bar{\mathrm{x}},\mathrm{y}-\bar{\mathrm{y}}\|_{\Upsilon},
\end{aligned}$$

and by condition $(Ax_{\mathbf{3}})$, we get

$$|\varPhi_2(\mathrm{x},\mathrm{y})(\mathrm{t})| \leq \frac{\mathcal{L}_2(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_2|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{\Gamma(\beta_2+1)} + |\mathcal{V}_2| + |\mathcal{V}_4|.$$

By applying the triangle inequality, we obtain

$$\begin{aligned}
&|\mathcal{T}_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \mathcal{T}_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| \\
&\leq |\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))\varPhi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \mathfrak{g}_1(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))\varPhi_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})|
\end{aligned}$$

$$+\frac{(\Psi(t)-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}|\mathfrak{g}_1(t,x(t))\varPhi_1(x,y)(\epsilon_1)-\mathfrak{g}_1(t,\bar{x}(t))\varPhi_1(\bar{x},\bar{y})(\epsilon_1)|$$
$$+|\mathcal{W}_1(t,x(t))-\mathcal{W}_1(t,\bar{x}(t))|.$$

We conclude that

$$\begin{aligned}
&|\mathcal{T}_1(x,y)(t)-\mathcal{T}_1(\bar{x},\bar{y})(t)|\\
&\leq \mathcal{M}_1\left(\frac{2l^*_{\mathfrak{f}_1}(\Psi(t)-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\mu_1|(\Psi(t)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|x-\bar{x},y-\bar{y}\|_\Upsilon\\
&+l^*_{\mathfrak{g}_1}\left(\frac{\mathcal{L}_1(\Psi(t)-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\mu_1|(\Psi(t)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}r\right)\|x-\bar{x},y-\bar{y}\|_\Upsilon\\
&+\mathcal{M}_1(\Psi(t)-\Psi(0))^{\beta_1}\left(\frac{2l^*_{\mathfrak{f}_1}(\Psi(\epsilon_1)-\Psi(0))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\mu_1|}{\Gamma(\beta_1+1)}\right)\|x-\bar{x},y-\bar{y}\|_\Upsilon\\
&+l^*_{\mathfrak{g}_1}(\Psi(t)-\Psi(0))^{\beta_1}\left(\frac{\mathcal{L}_1(\Psi(\epsilon_1)-\Psi(0))^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\mu_1|}{\Gamma(\beta_1+1)}r\right)\|x-\bar{x},y-\bar{y}\|_\Upsilon\\
&+\left[l^*_{\mathfrak{g}_1}(|\mathcal{V}_1|+|\mathcal{V}_3|)\left(\frac{[\Psi(t)-\Psi(0)]^{\beta_1}}{[\Psi(\epsilon_1)-\Psi(0)]^{\beta_1}}+1\right)+l^*_{\mathcal{W}_1}\right]\|x-\bar{x},y-\bar{y}\|_\Upsilon.
\end{aligned}$$

After taking the supremum over J and simplifying, we get

$$\|\mathcal{T}_1(x,y)-\mathcal{T}_1(\bar{x},\bar{y})\|_\infty\leq\varpi_1\|x-\bar{x},y-\bar{y}\|_\Upsilon. \tag{4.19}$$

Similarly, we obtain

$$\|\mathcal{T}_2(x,y)-\mathcal{T}_2(\bar{x},\bar{y})\|_\infty\leq\varpi_2\|x-\bar{x},y-\bar{y}\|_\Upsilon. \tag{4.20}$$

It follows from (4.19) and (4.20) that

$$\|\mathcal{T}(x,y)-\mathcal{T}(\bar{x},\bar{y})\|_\Upsilon\leq(\varpi_1+\varpi_2)\|x-\bar{x},y-\bar{y}\|_\Upsilon.$$

Since $\sum_{i=1}^{2}\varpi_i<1$, $\mathcal{T}$ is a contraction operator. Consequently by utilizing Banach's fixed point theorem, the coupled system (4.1)-(4.2) has a unique solution. In the following result, we establish the existence of solutions for the hybrid coupled Langevin fractional pantograph system (4.1)-(4.2). □

This is achieved by utilizing a theorem derived from Dhage's fixed-point theorem.

Theorem 4.3. *Assume that the following hypotheses hold.*

(Ax_4) *The functions* $\mathfrak{f}_1, \mathfrak{f}_2 : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *are continuous on* J.

(Ax_5) *The functions* $\mathfrak{g}_\mathrm{i} : J \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ *and* $\mathcal{W}_\mathrm{i} : J \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous. Moreover, there exist positive continuous functions* $l_{\mathfrak{g}_\mathrm{i}}, l_{\mathcal{W}_\mathrm{i}} \in \mathcal{C}(J, [0, \infty))$ *such that*

$$|\mathfrak{g}_\mathrm{i}(\mathrm{t}, \mathrm{x}) - \mathfrak{g}_\mathrm{i}(\mathrm{t}, \bar{\mathrm{x}})| \leq l_{\mathfrak{g}_\mathrm{i}}(\mathrm{t})|\mathrm{x} - \bar{\mathrm{x}}|,$$

and

$$|\mathcal{W}_\mathrm{i}(\mathrm{t}, \mathrm{x}) - \mathcal{W}_\mathrm{i}(\mathrm{t}, \bar{\mathrm{x}})| \leq l_{\mathcal{W}_\mathrm{i}}(\mathrm{t})|\mathrm{x} - \bar{\mathrm{x}}|,$$

for $i = 1, 2$, *and for any* $\mathrm{x}, \bar{\mathrm{x}} \in \mathbb{R}$ *and* $t \in J$.

(Ax_6) *There exist functions* $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{y}_1, \mathfrak{y}_2, \mathfrak{y}_3 \in \mathcal{C}(J, \mathbb{R}^+)$ *such that*

$$|\mathfrak{f}_1(\mathrm{t}, \mathrm{x}, \bar{\mathrm{x}})| < \mathfrak{p}_1(\mathrm{t}) + \mathfrak{p}_2(\mathrm{t})|\mathrm{x}| + \mathfrak{p}_3(\mathrm{t})|\bar{\mathrm{x}}|,$$

and

$$|\mathfrak{f}_2(\mathrm{t}, \mathrm{x}, \bar{\mathrm{x}})| < \mathfrak{y}_1(\mathrm{t}) + \mathfrak{y}_2(\mathrm{t})|\mathrm{x}| + \mathfrak{y}_3(\mathrm{t})|\bar{\mathrm{x}}|,$$

for all $\mathrm{t} \in J$ *and* $\mathrm{x}, \bar{\mathrm{x}}, \in \mathbb{R}$.

(Ax_7) *There exists a number* $\gamma > 0$ *such that*

$$\frac{g_1^* \mathcal{N}_\gamma + g_2^* \mathcal{E}_\gamma + \mathcal{W}_1^* + \mathcal{W}_2^*}{1 - [l_{\mathfrak{g}_1}^* \mathcal{N}_\gamma + l_{\mathfrak{g}_2}^* \mathcal{E}_\gamma + l_{\mathcal{W}_1}^* + l_{\mathcal{W}_2}^*]} \leq \gamma,$$

and

$$l_{\mathfrak{g}_1}^* \mathcal{N}_\gamma + l_{\mathfrak{g}_2}^* \mathcal{E}_\gamma + l_{\mathcal{W}_1}^* + l_{\mathcal{W}_2}^* < 1, \tag{4.21}$$

where

$$l_{\mathfrak{g}_\mathrm{j}}^* = \sup_{\mathrm{t} \in J} l_{\mathfrak{g}_\mathrm{j}}(\mathrm{t}), l_{\mathcal{W}_\mathrm{j}}^* = \sup_{\mathrm{t} \in J} l_{\mathcal{W}_\mathrm{j}}(\mathrm{t}), \ \ g_\mathrm{j}^* = \sup_{\mathrm{t} \in J} |\mathfrak{g}_\mathrm{j}(\mathrm{t}, 0)|, \ \ \mathcal{W}_\mathrm{j}^* = \sup_{\mathrm{t} \in J} |\mathcal{W}_\mathrm{j}(\mathrm{t}, 0)|, \ \ \mathrm{j} = 1, 2,$$

$$\mathfrak{p}_\mathrm{i}^* = \sup_{\mathrm{t} \in J} \mathfrak{p}_\mathrm{i}(\mathrm{t}), \ \ \mathfrak{y}_\mathrm{i}^* = \sup_{\mathrm{t} \in J} \mathfrak{y}_\mathrm{i}(\mathrm{t}), i = 1, 2, 3,$$

and

$$\mathcal{N}_\gamma = \mu_1 \mathtt{P}_1^* + [\mu_1(\mathtt{P}_2^* + \mathtt{P}_3^*) + \mu_2]\,\gamma + \mu_3,$$
$$\mathcal{E}_\gamma = \nabla_1 \mathfrak{y}_1^* + [\nabla_1(\mathfrak{y}_2^* + \mathfrak{y}_3^*) + \nabla_2]\,\gamma + \nabla_3.$$

Then the coupled system (4.1)-(4.2) *has at least one mild solution on* J.

Proof. We define a subset $B\gamma$ of $\mathcal{C}^3(J,\mathbb{R})$.

$$B\gamma = \{(\mathrm{x},\mathrm{y}) \in \mathrm{t} \ : \ \|\mathrm{x},\mathrm{y}\|_{\Upsilon} \leq \gamma\}.$$

Next, consider the operator $\mathcal{T}_1$ and $\mathcal{T}_2$ defined in (4.16) and (4.17), respectively. Additionally, introduce the operators $\mathbb{P},\mathbb{Q},\mathbb{U},\mathbb{V} : \mathcal{C}(J,\mathbb{R})^2 \longrightarrow \mathcal{C}(J,\mathbb{R})$ by

$$\begin{cases} \mathbb{P}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t})), \ \mathrm{t} \in J, \\ \mathbb{Q}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t})), \ \mathrm{t} \in J, \\ \mathbb{U}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t})), \ \mathrm{t} \in J, \\ \mathbb{V}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t})), \ \mathrm{t} \in J, \end{cases}$$

and $\mathcal{R},\mathcal{K} : B\gamma \longrightarrow \mathcal{C}(J,\mathbb{R})$ by

$$\begin{cases} \mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \Phi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}\Phi_1(\mathrm{x},\mathrm{y})(\epsilon_1), \ \mathrm{t} \in J, \\ \mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}) = \Phi_2(\mathrm{x},\mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}\Phi_2(\mathrm{x},\mathrm{y})(\epsilon_2), \ \mathrm{t} \in J. \end{cases}$$

Then,

$$\mathcal{T}_1(\mathrm{x},\mathrm{y}) = \mathbb{P}(\mathrm{x},\mathrm{y})\mathcal{R}(\mathrm{x},\mathrm{y}) + \mathbb{Q}(\mathrm{x},\mathrm{y}),$$

and

$$\mathcal{T}_2(\mathrm{x},\mathrm{y}) = \mathbb{U}(\mathrm{x},\mathrm{y})\mathcal{K}(\mathrm{x},\mathrm{y}) + \mathbb{V}(\mathrm{x},\mathrm{y}).$$

Step 1: Firstly, we show that $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{U}$ and $\mathbb{V}$ are Lipschitzian on $\mathcal{C}(J,\mathbb{R})^2$. Let $(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}}) \in \mathcal{C}(J,\mathbb{R})^2$. Then by $(Ax_{\mathbf{5}})$ we have

$$\begin{aligned} |\mathbb{P}(\mathrm{x},\mathrm{y})(\mathrm{t}) - \mathbb{P}(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| &= |\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t})) - \mathfrak{g}_1(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))| \\ &\leq l_{\mathfrak{g}_1}(\mathrm{t})|\mathrm{x}(\mathrm{t}) - \bar{\mathrm{x}}(\mathrm{t})|. \end{aligned}$$

Then for each $\mathrm{t} \in J$ we obtain

$$\|\mathbb{P}(\mathrm{x},\mathrm{y}) - \mathbb{P}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq l^*_{\mathfrak{g}_1}\|(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_{\Upsilon}, \tag{4.22}$$

As before, we have

$$\|\mathbb{U}(\mathrm{x},\mathrm{y}) - \mathbb{U}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq l^*_{\mathfrak{g}_2}\|(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_{\Upsilon}, \tag{4.23}$$

and for each $\mathrm{t} \in J$ we get

$$|\mathbb{Q}(\mathrm{x},\mathrm{y})(\mathrm{t}) - \mathbb{Q}(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| = |\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t})) - \mathcal{W}_1(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}))|$$

$$\leq \; l_{\mathcal{W}_1}(\mathrm{t})|\mathrm{x}(\mathrm{t}) - \bar{\mathrm{x}}(\mathrm{t})|.$$

Then,

$$\|\mathbb{Q}(\mathrm{x},\mathrm{y}) - \mathbb{Q}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq l^*_{\mathcal{W}_1}\|(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Upsilon, \tag{4.24}$$

and

$$\|\mathbb{V}(\mathrm{x},\mathrm{y}) - \mathbb{V}(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\infty \leq l^*_{\mathcal{W}_2}\|(\mathrm{x},\mathrm{y}),(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_\Upsilon. \tag{4.25}$$

Therefore $\mathbb{P}$, $\mathbb{Q}$, $\mathbb{U}$ and $\mathbb{V}$ are Lipschitzian on $\mathcal{C}(J,\mathbb{R})^2$ with Lipschitz constants $l^*_{\mathfrak{g}_\mathrm{i}}$ and $l^*_{\mathcal{W}_\mathrm{i}}$, for $i = 1, 2$.

Step 2: We demonstrate that the operators $\mathcal{R}$ and $\mathcal{K}$ are completely continuous on $B\gamma$. To achieve this, we first establish that the operators $\mathcal{R}$ and $\mathcal{K}$ are continuous on $\mathcal{C}(J,\mathbb{R})$. Let $\{\mathrm{x}_n,\mathrm{y}_n\}_{n\in\mathbb{N}}$ be a sequence in $B\gamma$ that converges to a point $(\mathrm{x},\mathrm{y}) \in B\gamma$. Applying the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{R}(\mathrm{x}_n,\mathrm{y}_n)(\mathrm{t}) \\
&= \lim_{n\to+\infty}\left\{\Phi_1(\mathrm{x}_n,\mathrm{y}_n)(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}\Phi_1(\mathrm{x}_n,\mathrm{y}_n)(\epsilon_1)\right\} \\
&= \lim_{n\to+\infty}\left\{\int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\mathfrak{f}_1(s,\mathrm{x}_n(s),\mathrm{y}_n(\zeta_1 s))\mathrm{d}s\right. \\
&\quad - \mu_1\int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}\mathrm{x}_n(s)\mathrm{d}s + \mathcal{V}_1 - \mathcal{V}_3 \\
&\quad - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}\left(\int_0^{\epsilon_1} \frac{\Psi'(s)(\Psi(\epsilon_1)-\Psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}\mathfrak{f}_1(s,\mathrm{x}_n(s),\mathrm{y}_n(\zeta_1 s))\mathrm{d}s\right. \\
&\quad \left.\left. -\mathfrak{f}_1\int_0^{\epsilon_1} \frac{\Psi'(s)(\Psi(\epsilon_1)-\Psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}\mathrm{x}_n(s)\mathrm{d}s + \mathcal{V}_1 - \mathcal{V}_3\right)\right\} \\
&= \mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}).
\end{aligned}$$

Hence,

$$\|\mathcal{R}(\mathrm{x}_n,\mathrm{y}_n) - \mathcal{R}(\mathrm{x},\mathrm{y})\|_\infty \longrightarrow 0, \text{ as } n \to \infty,$$

for all $\mathrm{t} \in J$. Similarly, we also have

$$\begin{aligned}
&\lim_{n\to+\infty} \mathcal{K}(\mathrm{x}_n,\mathrm{y}_n)(\mathrm{t}) \\
&= \lim_{n\to+\infty}\left\{\Phi_2(\mathrm{x}_n,\mathrm{y}_n)(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}\Phi_2(\mathrm{x}_n,\mathrm{y}_n)(\epsilon_2)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to+\infty}\Bigg\{\int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}\mathfrak{f}_2(s,\mathrm{y}_n(s),\mathrm{x}_n(\zeta s))\mathrm{d}s \\
&\quad -\mu_2\int_0^{\mathrm{t}} \frac{\Psi'(s)(\Psi(\mathrm{t})-\Psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}\mathrm{y}_n(s)\mathrm{d}s+\mathcal{V}_2-\mathcal{V}_4 \\
&\quad -\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}\Bigg(\int_0^{\epsilon_2} \frac{\Psi'(s)(\Psi(\epsilon_2)-\Psi(s))^{\alpha_2+\beta_2-1}}{\Gamma(\alpha_2+\beta_2)}\mathfrak{f}_2(s,\mathrm{y}_n(s),\mathrm{x}_n(\zeta s))\mathrm{d}s \\
&\quad -\mu_2\int_0^{\epsilon_2} \frac{\Psi'(s)(\Psi(\epsilon_2)-\Psi(s))^{\beta_2-1}}{\Gamma(\beta_2)}\mathrm{y}_n(s)\mathrm{d}s+\mathcal{V}_2-\mathcal{V}_4\Bigg)\Bigg\} \\
&= \mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}).
\end{aligned}
$$

Hence,

$$\|\mathcal{K}(\mathrm{x}_n,\mathrm{y}_n)-\mathcal{K}(\mathrm{x},\mathrm{y})\|_\infty \longrightarrow 0, \text{ as } n\to\infty,$$

for all $\mathrm{t}\in J$. This shows that $\mathcal{R}$ and $\mathcal{K}$ are continuous operators on $B\gamma$.

Next, we prove that the sets $\mathcal{R}(B\gamma)$ and $\mathcal{K}(B\gamma)$ are uniformly bounded in $B\gamma$. For any $(\mathrm{x},\mathrm{y})\in B\gamma$ and $\mathrm{t}\in J$, we have

$$|\mathit{\Phi}_1(\mathrm{x},\mathrm{y})(\mathrm{t})| \le \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}\left(\mathtt{P}_1^*+(\mathtt{P}_2^*+\mathtt{P}_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)}+\frac{|\mu_1|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}\gamma}{\Gamma(\beta_1+1)}+|\mathcal{V}_1|+|\mathcal{V}_3|,$$

and

$$|\mathit{\Phi}_2(\mathrm{x},\mathrm{y})(\mathrm{t})| \le \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}\left(\mathfrak{y}_1^*+(\mathfrak{y}_2^*+\mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)}+\frac{|\mu_2|(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}\gamma}{\Gamma(\beta_2+1)}+|\mathcal{V}_2|+|\mathcal{V}_4|,$$

Therefore,

$$
\begin{aligned}
|\mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t})| &\le |\mathit{\Phi}_1(\mathrm{x},\mathrm{y})(\mathrm{t})|+\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}|\mathit{\Phi}_1(\mathrm{x},\mathrm{y})(\epsilon_1)| \\
&\le \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}\left(\mathtt{P}_1^*+(\mathtt{P}_2^*+\mathtt{P}_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)}+\frac{2|\mu_1|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}\gamma}{\Gamma(\beta_1+1)} \\
&\quad +\frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}(\Psi(\epsilon_1)-\Psi(0))^{\alpha_1}\left(\mathtt{P}_1^*+(\mathtt{P}_2^*+\mathtt{P}_3^*)\gamma\right)}{\Gamma(\alpha_1+\beta_1+1)} \\
&\quad +(|\mathcal{V}_1|+|\mathcal{V}_3|)\left(\frac{[\Psi(b)-\Psi(0)]^{\beta_1}}{[\Psi(\epsilon_1)-\Psi(0)]^{\beta_1}}+1\right) \\
&\le \mathcal{N}_\gamma.
\end{aligned} \tag{4.26}
$$

Then, we obtain

$$\|\mathcal{R}(\mathrm{x},\mathrm{y})\|_\infty \le \infty,$$

for all $(\mathrm{x},\mathrm{y}) \in B\gamma$. We also have

$$
\begin{aligned}
&|\mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t})| \\
&\leq |\Phi_2(\mathrm{x},\mathrm{y})(\mathrm{t})| + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}|\Phi_2(\mathrm{x},\mathrm{y})(\epsilon_2)| \\
&\leq \frac{(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}\left(\mathfrak{y}_1^* + (\mathfrak{y}_2^*+\mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)} + \frac{2|\mu_2|(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}\gamma}{\Gamma(\beta_2+1)} \\
&\quad + \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}(\Psi(\epsilon_2)-\Psi(0))^{\alpha_2}\left(\mathfrak{y}_1^* + (\mathfrak{y}_2^*+\mathfrak{y}_3^*)\gamma\right)}{\Gamma(\alpha_2+\beta_2+1)} \\
&\quad + (|\mathcal{V}_2|+|\mathcal{V}_4|)\left(\frac{[\Psi(b)-\Psi(0)]^{\beta_2}}{[\Psi(\epsilon_2)-\Psi(0)]^{\beta_2}}+1\right) \qquad (4.27) \\
&\leq \mathcal{E}_\gamma.
\end{aligned}
$$

Then, we get

$$\|\mathcal{K}(\mathrm{x},\mathrm{y})\| \leq \infty, \ \forall(\mathrm{x},\mathrm{y}) \in B\gamma.$$

This prove that the sets $\mathcal{R}(B\gamma)$ and $\mathcal{K}(B\gamma)$ are uniformly bounded in $B\gamma$.

On the other hand, we demonstrate that $\mathcal{R}(B\gamma)$ and $\mathcal{K}(B\gamma)$ are equicontinuous sets in $B\gamma$. We take $\mathrm{t}_1,\mathrm{t}_2 \in [0,b]$ with $\mathrm{t}_1 < \mathrm{t}_2$ and $(\mathrm{x},\mathrm{y}) \in B\gamma$. Then,

$$
\begin{aligned}
&|\mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_1)| \\
&\leq \int_0^{\mathrm{t}_1} \frac{\Psi'(s)[(\Psi(\mathrm{t}_2)-\Psi(s))^{\alpha_1+\beta_1-1} - (\Psi(\mathrm{t}_1)-\Psi(s))^{\alpha_1+\beta_1-1}]}{\Gamma(\alpha_1+\beta_1)}|\mathfrak{f}_1(s,\mathrm{x}(s),\mathrm{y}(\zeta s))|\mathrm{d}s \\
&\quad + \int_{\mathrm{t}_1}^{\mathrm{t}_2} \frac{\Psi'(s)(\Psi(\mathrm{t}_2)-\Psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|\mathfrak{f}_1(s,\mathrm{x}(s),\mathrm{y}(\zeta s))|\mathrm{d}s \\
&\quad + \int_0^{\mathrm{t}_1} \frac{|\mu_1|\Psi'(s)[(\Psi(\mathrm{t}_2)-\Psi(s))^{\beta_1-1} - (\Psi(\mathrm{t}_1)-\Psi(s))^{\beta_1-1}]}{\Gamma(\beta_1)}|\mathrm{x}(s)|\mathrm{d}s \\
&\quad + \int_{\mathrm{t}_1}^{\mathrm{t}_2} \frac{|\mu_1|\Psi'(s)(\Psi(\mathrm{t}_2)-\Psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|\mathrm{x}(s)|\mathrm{d}s \\
&\quad + \frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_1} - (\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_1}]}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}} \\
&\quad \times \left(\int_0^{\epsilon_1} \frac{\Psi'(s)(\Psi(\epsilon_1)-\Psi(s))^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}|\mathfrak{f}_1(s,\mathrm{x}(s),\mathrm{y}(\zeta s))|\mathrm{d}s\right. \\
&\quad \left. +|\mu_1|\int_0^{\epsilon_1} \frac{\Psi'(s)(\Psi(\epsilon_1)-\Psi(s))^{\beta_1-1}}{\Gamma(\beta_1)}|\mathrm{x}(s)|\mathrm{d}s + |\mathcal{V}_1| + |\mathcal{V}_3|\right).
\end{aligned}
$$

Therefore,

$$|\mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_1)|$$

$$\leq |\varPhi_1(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \varPhi_1(\mathrm{x},\mathrm{y})(\mathrm{t}_1)| + \tfrac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_1}-(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_1}]}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}|\varPhi_1(\mathrm{x},\mathrm{y})(\epsilon_1)|$$
$$\leq \frac{[(\Psi(\mathrm{t}_1)-\Psi(0))^{\alpha_1+\beta_1}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\alpha_1+\beta_1}]}{\Gamma(\alpha_1+\beta_1+1)}[\mathrm{P}_1^* + (\mathrm{P}_2^* + \mathfrak{p}_3^*)\gamma]$$
$$+ \left(\frac{(\Psi(\mathrm{t}_2)-\Psi(\mathrm{t}_1))^{\beta_1}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_1}+(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)|\mathfrak{f}_1|\gamma$$
$$+ [(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_1}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_1}]\left(\tfrac{[\Psi(\epsilon_1)-\Psi(0)]^{\alpha_1}}{\Gamma(\alpha_1+\beta_1+1)} + \tfrac{|\mu_1|}{\Gamma(\beta_1+1)} + |\mathcal{V}_1| + |\mathcal{V}_3|\right).$$

Since,

$$|\mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \mathcal{R}(\mathrm{x},\mathrm{y})(\mathrm{t}_1)| \longrightarrow 0, \text{ as } \mathrm{t}_1 \to \mathrm{t}_2,$$

In addition, we have

$$|\mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}_1)|$$
$$\leq |\varPhi_2(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \varPhi_2(\mathrm{x},\mathrm{y})(\mathrm{t}_1)| + \frac{[(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_2}-(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_2}]}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}|\varPhi_2(\mathrm{x},\mathrm{y})(\epsilon_2)|$$
$$\leq \frac{[(\Psi(\mathrm{t}_1)-\Psi(0))^{\alpha_2+\beta_2}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\alpha_2+\beta_2}]}{\Gamma(\alpha_2+\beta_2+1)}[\mathrm{P}_1^* + (\mathrm{P}_2^* + \mathfrak{p}_3^*)\gamma]$$
$$+ \left(\frac{(\Psi(\mathrm{t}_2)-\Psi(\mathrm{t}_1))^{\beta_2}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_2}+(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right)|\mathfrak{f}_2|\gamma$$
$$+ [(\Psi(\mathrm{t}_1)-\Psi(0))^{\beta_2}-(\Psi(\mathrm{t}_2)-\Psi(0))^{\beta_2}]\left(\frac{[\Psi(\epsilon_2)-\Psi(0)]^{\alpha_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_1|}{\Gamma(\beta_2+1)} + |\mathcal{V}_2| + |\mathcal{V}_4|\right).$$

Consequently,

$$|\mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}_2) - \mathcal{K}(\mathrm{x},\mathrm{y})(\mathrm{t}_1)| \longrightarrow 0, \text{ as } \mathrm{t}_1 \to \mathrm{t}_2,$$

Thus, $\mathcal{R}(B\gamma)$ and $\mathcal{K}(B\gamma)$ are equicontinuous on J. Hence, by the Arzelà-Ascoli theorem, $\mathcal{R}$ and $\mathcal{K}$ are completely continuous on $B\gamma$.

Step 3: we show that hypothesis (3) of Theorem 1.4 is satisfied. For $(\mathrm{x},\mathrm{y}) \in \mathcal{C}^3(J,\mathbb{R})$ and $(\bar{\mathrm{x}},\bar{\mathrm{y}}) \in B\gamma$, where $\mathrm{x} = \mathbb{P}(\mathrm{x},\mathrm{y})\mathcal{R}(\bar{\mathrm{x}},\bar{\mathrm{y}}) + \mathbb{Q}(\mathrm{x},\mathrm{y})$ and $\mathrm{y} = \mathbb{U}(\mathrm{x},\mathrm{y})\mathcal{K}(\bar{\mathrm{x}},\bar{\mathrm{y}}) + \mathbb{V}(\mathrm{x},\mathrm{y})$, we get

$$|\mathrm{x}(\mathrm{t})| = |\mathbb{P}(\mathrm{x},\mathrm{y})(\mathrm{t})\mathcal{R}(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t}) + \mathbb{Q}(\mathrm{x},\mathrm{y})(\mathrm{t})|$$
$$\leq [|\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t})) + \mathfrak{g}_1(\mathrm{t},0)| + |\mathfrak{g}_1(\mathrm{t},0)|]\mathcal{N}_\gamma + |\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t})) + \mathcal{W}_1(\mathrm{t},0)| + |\mathcal{W}_1(\mathrm{t},0)|$$
$$\leq [l_{\mathfrak{g}_1}^* \|\mathrm{x}\| + g_1^*]\mathcal{N}_\gamma + l_{\mathcal{W}_1}^* \|x\| + \mathcal{W}_1^*,$$

and

$$|\mathrm{y}(\mathrm{t})| = |\mathbb{U}(\mathrm{x},\mathrm{y})(\mathrm{t})\mathcal{K}(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t}) + \mathcal{V}(\mathrm{x},\mathrm{y})(\mathrm{t})|$$

$$\leq [|\mathfrak{g}_2(t,y(t)) + \mathfrak{g}_2(t,0)| + |\mathfrak{g}_2(t,0)|]\mathcal{E}_\gamma + |\mathcal{W}_2(t,y(t)) + \mathcal{W}_2(t,0)| + |\mathcal{W}_2(t,0)|$$
$$\leq [l^*_{\mathfrak{g}_2}\|y\| + g^*_2]\mathcal{E}_\gamma + l^*_{\mathcal{W}_2}\|y\| + \mathcal{W}^*_2.$$

which implies that

$$\|(x,y)\|_\Upsilon \leq \frac{g^*_1\mathcal{N}_\gamma + g^*_2\mathcal{E}_\gamma + \mathcal{W}^*_1 + \mathcal{W}^*_2}{1 - [l^*_{\mathfrak{g}_1}\mathcal{N}_\gamma + l^*_{\mathfrak{g}_2}\mathcal{E}_\gamma + l^*_{\mathcal{W}_1} + l^*_{\mathcal{W}_2}]} \leq \gamma.$$

This shows that condition (3) of Theorem 1.4 is satisfied.

Step 4: Finally, we have

$$\rho = \|(\mathcal{R}(B\gamma), \mathcal{K}(B\gamma))\|_\Upsilon = \sup\{\|\mathcal{R}(x,y)\|_\infty + \|\mathcal{K}(x,y)\|_\infty : (x,y) \in B\gamma\} \leq \mathcal{N}_\gamma + \mathcal{E}_\gamma.$$

Fromabove estimate we obtain

$$l^*_{\mathfrak{g}_1}\mathcal{N}_\gamma + l^*_{\mathfrak{g}_2}\mathcal{E}_\gamma + l^*_{\mathcal{W}_1} + l^*_{\mathcal{W}_2} < 1.$$

Hence, all the conditions of Theorem 1.4 are satisfied, and therefore, the operator equation $x = \mathbb{P}(x,y)\mathcal{R}(x,y) + \mathbb{Q}(x,y)$ and $y = \mathbb{U}(x,y)\mathcal{K}(x,y) + \mathbb{V}(x,y)$ has a solution in $B\gamma$. Consequently, the coupled system (4.1)-(4.2) has a solution on J. □

4.3 Ulam-Type Stability

This section discusses the Ulam stability of the coupled system (4.1)-(4.2). Let $(x,y) \in \mathcal{C}^3(J,\mathbb{R})$ and $\varrho_1, \varrho_2 > 0$. We consider the following inequalities:

$$\begin{cases} \left|{}^C D^{\alpha_1,\Psi}_{0^+}\left[{}^C D^{\beta_1,\Psi}_{0^+}\left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right) + \mu_1 x(t)\right] - \mathfrak{f}_1(t,x(t),y(\zeta t))\right| < \varrho, & t \in J, \\ \left|{}^C D^{\alpha_2,\Psi}_{0^+}\left[{}^C D^{\beta_2,\Psi}_{0^+}\left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right) + \mu_2 y(t)\right] - \mathfrak{f}_2(t,y(t),x(\tilde{\zeta} t))\right| < \varrho, & t \in J, \end{cases} \tag{4.28}$$

Definition 4.2 ([26,30]). *The coupled system (4.1)-(4.2) is Ulam-Hyers stable if there exists a real number $\aleph = \aleph_1 + \aleph_2 > 0$ with $\aleph_1, \aleph_2 > 0$ such that for each $\varrho > 0$ and for each solution $(x,y) \in \Upsilon$ to the previous inequality (4.28), there exists a solution $(\bar{x},\bar{y}) \in \Upsilon$ of the coupled system (4.1)-(4.2) with*

$$\|(x,y) - (\bar{x},\bar{y})\|_\Upsilon \leq \varrho\aleph.$$

Definition 4.3 ([26,30]). *The coupled system (4.1)-(4.2) is generalized Ulam-Hyers stable if there exists $\mathcal{J} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, such that $\mathcal{J}(0) = 0$ for any $\varrho > 0$, and for each solution $(\mathrm{x}, \mathrm{y}) \in \Upsilon$ to the inequality (4.28), there exists a solution $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \Upsilon$ of the coupled system (4.1)-(4.2) with*

$$\|(\mathrm{x}, \mathrm{y}) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\Upsilon} \leq \mathcal{J}(\varrho).$$

Remark 4.1. *It is clear that:*

1. *Definition 4.2⇒ Definition 4.3.*

A function $(\mathrm{x}, \mathrm{y}) \in \Upsilon$ is a solution of the inequality (4.28) if and only if there exist a function $\Omega_{\mathrm{i}} \in \mathcal{C}(J, \mathbb{R})$such that for all $\mathrm{i} = 1, 2$.

i. $|\Omega_i(\mathrm{t})| \leq \varrho$, *for all* $\mathrm{t} \in J$.

ii. ${}^{C}D_{0^+}^{\alpha_1,\psi}\left[{}^{C}D_{0^+}^{\beta_1,\psi}\left(\frac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right) + \mu_1\mathrm{x(t)}\right] - \mathfrak{f}_1(\mathrm{t}, \mathrm{x(t)}, \mathrm{y}(\zeta\mathrm{t})) = \Omega_1(\mathrm{t})$, *for all* $\mathrm{t} \in J$.

iii. ${}^{C}D_{0^+}^{\alpha_2,\psi}\left[{}^{C}D_{0^+}^{\beta_2,\psi}\left(\frac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right) + \mu_2\mathrm{y(t)}\right] - \mathfrak{f}_2(\mathrm{t}, \mathrm{y(t)}, \mathrm{x}(\tilde{\zeta}\mathrm{t})) = \Omega_2(\mathrm{t})$, *for all* $\mathrm{t} \in J$.

We now present the Ulam stability of the solution to problem (4.1)-(4.2).

Theorem 4.4. *Assume that (Ax_1)-(Ax_3) are satisfied. then the coupled system (4.1)-(4.2) is Ulam-Hyers stable under the condition (4.18).*

Proof. Assume $\varrho > 0$ and $(\mathrm{x}, \mathrm{y}) \in \mathcal{B}_{\mathrm{r}}$ is a function that fulfills the inequality (4.28), and let $(\bar{\mathrm{x}}, \bar{\mathrm{y}}) \in \mathcal{B}_{\mathrm{r}}$ is the sole solution of the coupled system (4.1)-(4.2). Since $(\mathrm{x}, \mathrm{y}) \in \mathcal{B}_{\mathrm{r}}$ is a function satisfes the inequality (4.28). It follows from Remark 4.1 that

$$\begin{cases} {}^{C}D_{0^+}^{\alpha_1,\Psi}\left[{}^{C}D_{0^+}^{\beta_1,\Psi}\left(\frac{\mathrm{x(t)}-\mathcal{W}_1(\mathrm{t,x(t)})}{\mathfrak{g}_1(\mathrm{t,x(t)})}\right) + \mu_1\mathrm{x(t)}\right] - \mathfrak{f}_1(\mathrm{t}, \mathrm{x(t)}, \mathrm{y}(\zeta\mathrm{t})) = \Omega_1(\mathrm{t}), & \mathrm{t} \in J, \\ {}^{C}D_{0^+}^{\alpha_2,\Psi}\left[{}^{C}D_{0^+}^{\beta_2,\Psi}\left(\frac{\mathrm{y(t)}-\mathcal{W}_2(\mathrm{t,y(t)})}{\mathfrak{g}_2(\mathrm{t,y(t)})}\right) + \mu_2\mathrm{y(t)}\right] - \mathfrak{f}_2(\mathrm{t}, \mathrm{y(t)}, \mathrm{x}(\tilde{\zeta}\mathrm{t})) = \Omega_2(\mathrm{t}), & \mathrm{t} \in J, \end{cases}$$

under the given boundary conditions

$$\begin{cases} \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_1(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_1(\mathrm{t}, \mathrm{x(t)})}\right)\Bigg|_{\mathrm{t}=0} = \mathcal{V}_1, \quad \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_2(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_2(\mathrm{t}, \mathrm{x(t)})}\right)\Bigg|_{\mathrm{t}=0} = \mathcal{V}_2, \\ \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_1(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_1(\mathrm{t}, \mathrm{x(t)})}\right)'\Bigg|_{\mathrm{t}=0} = \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_2(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_2(\mathrm{t}, \mathrm{x(t)})}\right)'\Bigg|_{\mathrm{t}=0} = 0, \\ \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_1(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_1(\mathrm{t}, \mathrm{x(t)})}\right)\Bigg|_{\mathrm{t}=\epsilon_1} = \mathcal{V}_3, \quad \left(\dfrac{\mathrm{x(t)} - \mathcal{W}_2(\mathrm{t}, \mathrm{x(t)})}{\mathfrak{g}_2(\mathrm{t}, \mathrm{x(t)})}\right)\Bigg|_{\mathrm{t}=\epsilon_2} = \mathcal{V}_4, \ 0 < \epsilon_1, \epsilon_2 \leq b, \end{cases}$$

Using Lemma 4.1 once more, we have

$$\mathrm{x}(\mathrm{t}) := \mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))\left[\Phi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}}\Phi_1(\mathrm{x},\mathrm{y})(\epsilon_1)\right] + \mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t})),$$

and

$$\mathrm{y}(\mathrm{t}) := \mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))\left[\Phi_2(\mathrm{x},\mathrm{y})(\mathrm{t}) - \frac{(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{(\Psi(\epsilon_2)-\Psi(0))^{\beta_2}}\Phi_2(\mathrm{x},\mathrm{y})(\epsilon_2)\right] + \mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t})),$$

where

$$\Phi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) := I_{0^+}^{\alpha_1+\beta_1,\Psi}[\mathfrak{f}_1(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\zeta\mathrm{t})) + \Omega_1(\mathrm{t})] - \mu_1 I_{0^+}^{\alpha_1,\Psi}\mathrm{x}(\mathrm{t}) + \mathcal{V}_1 - \mathcal{V}_3,$$

and

$$\Phi_2(\mathrm{x},\mathrm{y})(\mathrm{t}) := I_{0^+}^{\alpha_2+\beta_2,\Psi}[\mathfrak{f}_2(\mathrm{t},\mathrm{y}(\mathrm{t}),\mathrm{x}(\tilde{\zeta}\mathrm{t})) + \Omega_2(\mathrm{t})] - \mu_2 I_{0^+}^{\alpha_2,\Psi}\mathrm{y}(\mathrm{t}) + \mathcal{V}_2 - \mathcal{V}_4.$$

Moreover, using part (i) of Remark 4.1 and $(Ax_{\mathbf{2}})$, we can obtain the following formula for each $\mathrm{t} \in J$.

$$\begin{aligned}
&|\Phi_1(\mathrm{x},\mathrm{y})(\mathrm{t}) - \Phi_1(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| \\
&\leq I_{0^+}^{\alpha_1+\beta_1,\Psi}|\mathfrak{f}_1(\mathrm{t},\mathrm{x}(\mathrm{t}),\mathrm{y}(\zeta\mathrm{t})) - \mathfrak{f}_1(\mathrm{t},\bar{\mathrm{x}}(\mathrm{t}),\bar{\mathrm{y}},(\zeta\mathrm{t}))| \\
&\quad + \mu_1 I_{0^+}^{\alpha_1,\Psi}|\mathrm{x}(\mathrm{t}) - \bar{\mathrm{x}}(\mathrm{t})| + I_{0^+}^{\alpha_1+\beta_1,\Psi}|\Omega_1(\mathrm{t})| \\
&\leq \left(\frac{2l^*_{\mathfrak{f}_1}(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1|(\Psi(\mathrm{t})-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)}\right)\|(\mathrm{x},\mathrm{y})-(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_{\Upsilon} \\
&\quad + \frac{\varrho(\Psi(\mathrm{t})-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)},
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
&|\Phi_2(\mathrm{x},\mathrm{y})(\mathrm{t}) - \Phi_2(\bar{\mathrm{x}},\bar{\mathrm{y}})(\mathrm{t})| \\
&\leq \left(\frac{2l^*_{\mathfrak{f}_2}(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_2|(\Psi(\mathrm{t})-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)}\right)\|(\mathrm{x},\mathrm{y})-(\bar{\mathrm{x}},\bar{\mathrm{y}})\|_{\Upsilon} \\
&\quad + \frac{\varrho(\Psi(\mathrm{t})-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)},
\end{aligned} \tag{4.30}$$

in addition to,

$$\begin{cases} |\varPhi_1(x,y)(t)| & \le \frac{(\mathcal{L}_1+\varrho)(\Psi(t)-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1|(\Psi(t)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)} r + |\mathcal{V}_1| + |\mathcal{V}_3|, \\ |\varPhi_1(\bar{x},\bar{y})(t)| & \le \frac{\mathcal{L}_1(\Psi(t)-\Psi(0))^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{|\mu_1|(\Psi(t)-\Psi(0))^{\beta_1}}{\Gamma(\beta_1+1)} r + |\mathcal{V}_1| + |\mathcal{V}_3|, \\ |\varPhi_2(x,y)(t)| & \le \frac{(\mathcal{L}_2+\varrho)(\Psi(t)-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_2|(\Psi(t)-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} r + |\mathcal{V}_2| + |\mathcal{V}_4|, \\ |\varPhi_2(\bar{x},\bar{y})(t)| & \le \frac{\mathcal{L}_2(\Psi(t)-\Psi(0))^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{|\mu_2|(\Psi(t)-\Psi(0))^{\beta_2}}{\Gamma(\beta_2+1)} r + |\mathcal{V}_2| + |\mathcal{V}_4|, \end{cases} \tag{4.31}$$

Applying the triangle inequality (4.29)-(4.31), we have

$$\begin{aligned} &|x(t) - \bar{x}(t)| \\ &\le |\mathfrak{g}_1(t, x(t))\varPhi_1(x,y)(t) - \mathfrak{g}_1(t, \bar{x}_1(t))\varPhi_1(\bar{x},\bar{y})(t)| \\ &\quad + \frac{(\Psi(t)-\Psi(0))^{\beta_1}}{(\Psi(\epsilon_1)-\Psi(0))^{\beta_1}} |\mathfrak{g}_1(t, x(t))\varPhi_1(x,y)(\epsilon_1) - \mathfrak{g}_1(t, \bar{x}(t))\varPhi_1(\bar{x},\bar{y})(\epsilon_1)| \\ &\quad + |\mathcal{W}_1(t, x(t)) - \mathcal{W}_1(t, \bar{x}(t))|, \end{aligned}$$

we obtain

$$\|x - \bar{x}\|_\infty \le \varpi_1 \|(x,y) - (\bar{x},\bar{y})\|_\Upsilon + \mathcal{M}_1 \mu_1 \varrho. \tag{4.32}$$

On the other hand, we have

$$\|y - \bar{y}\|_\infty \le \varpi_2 \|(x,y) - (\bar{x},\bar{y})\|_\Upsilon + \mathcal{M}_2 \nabla_1 \varrho. \tag{4.33}$$

Combining the two last inequalities (4.32) and (4.33), we get

$$\|(x,y) - (\bar{x},\bar{y})\|_\Upsilon \le \left[1 - \sum_{i=1}^{2} \varpi_i\right]^{-1} (\mathcal{M}_1\mu_1 + \mathcal{M}_2\nabla_2)\, \varrho. \tag{4.34}$$

Let is put $\aleph = \left[1 - \sum_{i=1}^{2} \varpi_i\right]^{-1} (\mathcal{M}_1\mu_1 + \mathcal{M}_2\nabla_2)$. Taking into account $\sum_{i=1}^{2} \varpi_i < 1$, we notice that $\mu > 0$. Thus, we have

$$\|(x,y) - (\bar{x},\bar{y})\|_\Upsilon \le \varrho\aleph.$$

As a result, the coupled system (4.1)-(4.2) exhibits stability in the Ulam-Hyers sense. This concludes the proof based on the definition of Ulam-Hyers stability. □

Theorem 4.5. *Suppose the conditions of Theorem 4.2 hold. If there exists $\mathcal{J} \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, such that $\mathcal{J}(0) = 0$ with $\varrho > 0$. Therefore, the coupled system (4.1)-(4.2) is generalized Ulam-Hyers stable.*

Proof. For $\mathcal{J}(\varrho) = \mathscr{A}\varrho$; $\mathcal{J}(0) = 0$. We prove that the solution to the coupled system (4.1)-(4.2) is also generalized Ulam-Hyers stable. □

4.4 An Example

Example 4.1. *Consider the coupled system*

$$\begin{cases} {}^{C}D_{0^+}^{0.45,\sqrt{t+1}}\left[{}^{C}D_{0^+}^{1.65,\sqrt{t+1}}\left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)+\frac{25}{100}x(t)\right]=\mathfrak{f}_1(t,x(t),y(\frac{1}{7}t)), \quad t\in J,\\ {}^{C}D_{0^+}^{0.70,\sqrt{t+1}}\left[{}^{C}D_{0^+}^{1.75,\sqrt{t+1}}\left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)+2y(t)\right]=\mathfrak{f}_2(t,y(t),x(\frac{4}{10}t)), \quad t\in J, \end{cases} \tag{4.35}$$

under the boundary conditions

$$\begin{cases} \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)(t=0)=\frac{15}{100}, \quad \left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)(t=0)=\frac{30}{100},\\ \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)'\Big|_{t=0}=\left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)'\Big|_{t=0}=0,\\ \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)(\frac{7}{6})=\frac{-15}{100}, \quad \left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)(\frac{6}{5})=\frac{-30}{100}, 0<\frac{7}{6},\frac{6}{5}<2 , \end{cases} \tag{4.36}$$

where $J=(0,2]$, $b=2$ *and*

$$\begin{cases} \mathfrak{g}_1(t,x(t))=\frac{(t-\frac{1}{2})(|x(t)|+1)}{\pi e^{-t+3}}+10^{-1}, \; t\in J, \; x\in\mathcal{C}(J,\mathbb{R}),\\ \mathfrak{g}_2(t,y(t))=\frac{\cos(t)(|y(t)|+0.01)}{\pi^2+100t^2}+10^{-2}, \; t\in J, \; y\in\mathcal{C}(J,\mathbb{R}),\\ \mathcal{W}_1(t,x(t))=\frac{\sqrt{t-\frac{1}{2}x(t)}}{12\pi e\sqrt{6-t}}+\frac{1}{e^{2+t}+32\pi}, \; t\in J, \; x\in\mathcal{C}(J,\mathbb{R}),\\ \mathcal{W}_2(t,y(t))=\frac{\cos(2t)\;y(t)}{12\pi(t+1)}+\frac{e^2}{32\pi}, \; t\in J, \; y\in\mathcal{C}(J,\mathbb{R}),\\ \mathfrak{f}_1(t,x(t),y(\frac{t}{7}))=\frac{\sqrt{t-\frac{1}{2}}\left(\sin(t)y(\frac{1}{7}t)+\cos(t)x(t)-1\right)}{55e^{-t+3}(1+|x(t)|\sqrt{t-\frac{1}{2}})}, \; t\in J, \; x,y\in\mathcal{C}(J,\mathbb{R}),\\ \mathfrak{f}_2(t,y(t),x(\frac{4t}{10}))=\frac{\sin(t)\left(\sin(t)x(\frac{4}{10}t)+y(t)-0.02\right)}{(55+e^{-t+3})(1+|y(t)|)}, \; t\in J, \; x,y\in\mathcal{C}(J,\mathbb{R}). \end{cases}$$

Clearly, the continuous functions $\mathfrak{f}_i\in\mathcal{C}(J\times\mathbb{R}^2,\mathbb{R})$, $\mathcal{W}_i\in\mathcal{C}(J\times\mathbb{R},\mathbb{R})$ *and* $\mathfrak{g}_i\in\mathcal{C}(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, *for* $i=1,2$. *Hence the condition* (Ax_1) *is satisfied.*

For each $\mathrm{x}, \bar{\mathrm{x}}, \mathrm{y}, \bar{\mathrm{y}}, \in \mathbb{R}$ *and* $\mathrm{t} \in J$, *we have*

$$\begin{cases} |\mathfrak{f}_1(\mathrm{t},\mathrm{x},\mathrm{y}) - \mathfrak{f}_1(\mathrm{t},\bar{\mathrm{x}},\bar{\mathrm{y}})| \leq \frac{\sqrt{\mathrm{t}-\frac{1}{2}}}{55e^{-\mathrm{t}+3}} \left(|\mathrm{x}-\bar{\mathrm{x}}| + |\ \mathrm{y}-\bar{\mathrm{y}}|\right), \\ |\mathfrak{f}_2(\mathrm{t},\mathrm{x},\mathrm{y}) - \mathfrak{f}_2(\mathrm{t},\bar{\mathrm{x}},\bar{\mathrm{y}})| \leq \frac{1}{55+e^{-\mathrm{t}+3}} \left(|\mathrm{x}-\bar{\mathrm{x}}| + |\ \mathrm{y}-\bar{\mathrm{y}}|\right), \\ |\mathfrak{g}_1(\mathrm{t},\mathrm{x}) - \mathfrak{g}_1(\mathrm{t},\bar{\mathrm{x}})| \leq \frac{(\mathrm{t}-\frac{1}{2})}{\pi e^{-\mathrm{t}+3}} \left|\mathrm{x}-\bar{\mathrm{x}},\right|, \\ |\mathfrak{g}_2(\mathrm{t},\mathrm{y}) - \mathfrak{g}_2(\mathrm{t},\bar{\mathrm{y}})| \leq \frac{1}{\pi^2+100t^2} \left|\mathrm{y}-\bar{\mathrm{y}},\right|, \\ |\mathcal{W}_1(\mathrm{t},\mathrm{x}) - \mathcal{W}_1(\mathrm{t},\bar{\mathrm{x}})| \leq \frac{\sqrt{\mathrm{t}-\frac{1}{2}}}{12\pi e\sqrt{6-\mathrm{t}}} \left|\mathrm{x}-\bar{\mathrm{x}},\right|, \\ |\mathcal{W}_2(\mathrm{t},\mathrm{y}) - \mathcal{W}_2(\mathrm{t},\bar{\mathrm{y}})| \leq \frac{1}{12\pi(\mathrm{t}+1)} \left|\mathrm{y}-\bar{\mathrm{y}},\right|. \end{cases}$$

Hence condition $(Ax_{\mathbf{2}})$ *is satisfied with*

$$l_{\mathfrak{f}_1}(\mathrm{t}) = \frac{\sqrt{\mathrm{t}-\frac{3}{2}}}{55e^{-\mathrm{t}+3}}, \quad l_{\mathfrak{f}_2}(\mathrm{t}) = \frac{1}{55+e^{-\mathrm{t}+3}},$$
$$l_{\mathfrak{g}_1}(\mathrm{t}) = \frac{(\mathrm{t}-\frac{1}{2})}{\pi e^{-\mathrm{t}+3}}, \quad l_{\mathfrak{g}_2}(\mathrm{t}) = \frac{1}{\pi^2+100t^2},$$
$$l_{\mathcal{W}_1}(\mathrm{t}) = \frac{\sqrt{\mathrm{t}-\frac{1}{2}}}{12\pi e\sqrt{6-\mathrm{t}}}, \; l_{\mathcal{W}_2}(\mathrm{t}) = \frac{1}{12\pi(\mathrm{t}+1)},$$

so we have

$$l^*_{\mathfrak{f}_1} = \frac{\sqrt{3}}{55e\sqrt{2}}, \quad l^*_{\mathfrak{f}_1} = \frac{1}{55+e^2},$$
$$l^*_{\mathfrak{g}_1} = \frac{3}{2\pi e}, \quad l^2_{\mathfrak{g}_1} = \frac{1}{\pi^2+100},$$
$$l^*_{\mathcal{W}_1} = \frac{\sqrt{3}}{24\pi e\sqrt{2}}, \quad l^*_{\mathcal{W}_2} = \frac{1}{24\pi}.$$

and so the condition $(Ax_{\mathbf{3}})$ *is satisfied with*

$$\mathcal{M}_1 = \frac{1}{55e}, \quad \mathcal{L}_1 = \frac{3}{2\pi e}, \; \mathcal{K}_1 = \frac{7621}{500000} \; \mathcal{M}_2 = \frac{1}{55}, \quad \mathcal{L}_2 = \frac{1}{100}, \; \mathcal{K}_2 = 8.6763 \times 10^{-2}.$$

On the other hand, we have

$$\Re_1 = 0.629, \quad \Re_2 = 0.26959, \quad \Re_3 = 0.88371,$$
$$\nabla_1 = 0.25870, \quad \nabla_2 = 1.4409, \quad \nabla_3 = 1.8182 \times 10^{-2},$$

Therefore,

$$\mathrm{r} \geq 0.11142.$$

So, let's assume that

$$\mathrm{r} = 1.$$

We can show that:

$$\sum_{i=1}^{2} \varpi_{\mathrm{i}} = 0.28275 < 1.$$

Thus, all the conditions of Theorem 4.2 are satisfied. Hence, our coupled system (4.35)-(4.36) has a unique solution on $[0, 2]$.

Let $\varrho = \frac{2}{3} > 0$, *as illustrated Theorem 4.4 and by (4.28). If* $\mathrm{x}, \mathrm{y} \in \mathcal{C}([0, 2], \mathbb{R})$ *complies with*

$$|{}^{C}D_{0^+}^{0.45,\sqrt{\mathrm{t}}}\left[{}^{C}D_{0^+}^{1.65,\sqrt{\mathrm{t}}}\left(\frac{\mathrm{x}(\mathrm{t}) - \mathcal{W}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}))}\right) + \tfrac{25}{100}\mathrm{x}(\mathrm{t})\right] - \mathfrak{f}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{y}(\tfrac{1}{7}\mathrm{t}))| < \frac{2}{3},$$

and

$$|{}^{C}D_{0^+}^{0.70,\sqrt{\mathrm{t}}}\left[{}^{C}D_{0^+}^{1.75,\sqrt{\mathrm{t}}}\left(\frac{\mathrm{y}(\mathrm{t}) - \mathcal{W}_2(\mathrm{t}, \mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t}, \mathrm{y}(\mathrm{t}))}\right) + 2\mathrm{y}(\mathrm{t})\right] - \mathfrak{f}_2(\mathrm{t}, \mathrm{y}(\mathrm{t}), \mathrm{x}(\tfrac{4}{10}\mathrm{t}))| < \frac{2}{3},$$

there exists a solution $\bar{\mathrm{x}}, \bar{\mathrm{y}}, \in \mathcal{C}([0, 2], \mathbb{R})$ *of the coupled system (4.35)-(4.36) with*

$$\|(\mathrm{x}, \mathrm{y}) - (\bar{\mathrm{x}}, \bar{\mathrm{y}})\|_{\mathrm{t}} \leq \frac{2}{3}\aleph,$$

which

$$\aleph = \left[1 - \sum_{i=1}^{2} \varpi_{\mathrm{i}}\right]^{-1} (\mathcal{M}_1\Re_1 + \mathcal{M}_2\nabla_2) = 2.0383 > 0.$$

Consequently, the coupled system (4.35)-(4.36) is Ulam-Hyers stable on $[0, 2]$. *Finally, we assume that* $\varrho = 0$, *we obtain* $\mathcal{J}(0) = 0$. *Hence, the problem (4.35)-(4.36) is generalized Ulam-Hyers stable.*

Example 4.2. *Consider the coupled system*

$$\begin{cases} {}^{C}D_{0^+}^{0.45,\frac{t^2}{2}+\mathrm{t}}\left[{}^{C}D_{0^+}^{1.65,\frac{t^2}{2}+\mathrm{t}}\left(\frac{\mathrm{x}(\mathrm{t})-\mathcal{W}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}{\mathfrak{g}_1(\mathrm{t},\mathrm{x}(\mathrm{t}))}\right) + \frac{25}{100}\mathrm{x}(\mathrm{t})\right] = \mathfrak{f}_1(\mathrm{t}, \mathrm{x}(\mathrm{t}), \mathrm{y}(\frac{1}{5}\mathrm{t})), & \mathrm{t} \in J, \\ {}^{C}D_{0^+}^{0.75,\frac{t^2}{2}+\mathrm{t}}\left[{}^{C}D_{0^+}^{1.25,\frac{t^2}{2}+\mathrm{t}}\left(\frac{\mathrm{x}(\mathrm{t})-\mathcal{W}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}{\mathfrak{g}_2(\mathrm{t},\mathrm{y}(\mathrm{t}))}\right) + \frac{4}{100}\mathrm{y}(\mathrm{t})\right] = \mathfrak{f}_2(\mathrm{t}, \mathrm{y}(\mathrm{t}), \mathrm{x}(\frac{1}{5}\mathrm{t})), & \mathrm{t} \in J, \end{cases} \tag{4.37}$$

under the boundary conditions

$$\begin{cases} \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)\Big|_{t=0} = \frac{2}{10}, \quad \left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)\Big|_{t=0} = \frac{35}{100}, \\ \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)'\Big|_{t=0} = \left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)'\Big|_{t=0} = 0, \\ \left(\frac{x(t)-\mathcal{W}_1(t,x(t))}{\mathfrak{g}_1(t,x(t))}\right)\Big|_{t=\frac{3}{2}} = \frac{-1}{10}, \quad \left(\frac{y(t)-\mathcal{W}_2(t,y(t))}{\mathfrak{g}_2(t,y(t))}\right)\Big|_{t=\frac{5}{4}} = \frac{-25}{100}, 0 < \frac{3}{2}, \frac{5}{4} \leq 2, \end{cases} \tag{4.38}$$

where $J = (0,2]$, $b = 2$ *and*

$$\begin{cases} \mathfrak{g}_1(t,x(t)) = \frac{\sqrt{(t-\frac{1}{2})}}{\pi e^{-t+3}}(|\cos(t)||x(t)|) + 0.02, \ t \in J, \ x \in \mathcal{C}(J,\mathbb{R}), \\ \mathfrak{g}_2(t,y(t)) = \frac{|\sin(t)|}{24^t}(|y(t)| + |\cos(t)|) + 1, \ t \in J, \ y \in \mathcal{C}(J,\mathbb{R}), \\ \mathcal{W}_1(t,x(t)) = \frac{\sqrt{t-1}|\sin(t)|x(t)}{33\pi e} + \frac{16\pi}{35e^{2+t}}, \ t \in J, \ x \in \mathcal{C}(J,\mathbb{R}), \\ \mathcal{W}_2(t,y(t)) = \frac{|\cos^2(t)|}{73e^{2+t}}(|y(t)| + |\tan^{-1}(t)| + 2\pi), \ t \in J, \ y \in \mathcal{C}(J,\mathbb{R}), \\ \mathfrak{f}_1(t,x(t),y(\frac{t}{5})) = \frac{\sqrt{t-\frac{1}{2}}|\sin(t)|\left(\frac{1}{x(\frac{1}{5}t)+\cos(t)x(t)+1}\right)}{70e^{-t+3}(2+|x(t)|+|x(\frac{t}{5})|)}, \ t \in J, \ x,y \in \mathcal{C}(J,\mathbb{R}), \\ \mathfrak{f}_2(t,y(t),x(\frac{t}{5})) = \frac{|\sin^2(t)|}{35e^{t+3}}\left(\frac{\cos(t)x(t)}{\pi+|x(t)|} + \frac{x(\frac{1}{5}t)}{2\pi+|x(\frac{1}{5}t)|}\right) + \frac{e^{\pi}}{35t^2}, \ t \in J, \ x,y \in \mathcal{C}(J,\mathbb{R}). \end{cases}$$

Clearly, the continuous function $\mathfrak{f}_1, \mathfrak{f}_2 \in \mathcal{C}(J \times \mathbb{R}^2, \mathbb{R})$. *Hence the condition* (Ax_4) *is satisfied.*

For each $x, \bar{x}, y, \bar{y}, \in \mathbb{R}$ *and* $t \in J$, *we have*

$$\begin{cases} |\mathfrak{g}_1(t,x) - \mathfrak{g}_1(t,\bar{x})| \leq \frac{\sqrt{(t-\frac{1}{2})}}{\pi e^{-t+3}}|x-\bar{x},|, \\ |\mathfrak{g}_2(t,y) - \mathfrak{g}_2(t,\bar{y})| \leq \frac{|\sin(t)|}{24^t}|y-\bar{y},|, \\ |\mathcal{W}_1(t,x) - \mathcal{W}_1(t,\bar{x})| \leq \frac{\sqrt{t-1}}{33\pi e}|x-\bar{x},|, \\ |\mathcal{W}_2(t,y) - \mathcal{W}_2(t,\bar{y})| \leq \frac{|\cos^2(t)|}{73e^{2+t}}|y-\bar{y},|. \end{cases}$$

Hence condition (Ax_5) *is satisfied with*

$$\begin{cases} l_{\mathfrak{g}_1}(t) = \frac{\sqrt{(t-\frac{1}{2})}}{\pi e^{-t+3}}, \\ l_{\mathcal{W}_1}(t) = \frac{\sqrt{t-1}}{33\pi e}, \\ l_{\mathfrak{g}_2}(t) = \frac{|\sin(t)|}{24^t}, \\ l_{\mathcal{W}_1}(t) = \frac{|\cos^2(t)|}{73e^{2+t}}, \end{cases}$$

so we have

$$l^*_{\mathfrak{g}_1} = \frac{\sqrt{3}}{\pi e\sqrt{2}},\ l^*_{\mathcal{W}_1} = \frac{1}{33\pi e},\ l^*_{\mathfrak{g}_2} = \frac{1}{24},$$
$$l^*_{\mathcal{W}_2} = \frac{1}{73e^3}.$$

Let $\mathrm{x}, \mathrm{y} \in \mathbb{R}$, *then*

$$|\mathfrak{f}_1(\mathrm{t}, \mathrm{x}, \mathrm{y})| \leq \frac{\sqrt{\mathrm{t} - \frac{1}{2}}}{70e^{-\mathrm{t}+3}} (|\mathrm{y}| + |\mathrm{x}| + 1),\ \mathrm{t} \in J,$$
$$|\mathfrak{f}_2(\mathrm{t}, \mathrm{x}, \mathrm{y})| \leq \frac{|\sin^2(\mathrm{t})|}{35e^{\mathrm{t}+3}} (|\mathrm{x}| + |\mathrm{x}|) + |\tan^{-1}(\mathrm{t})| + e^{\pi},\ \mathrm{t} \in J,$$

and so the condition $(Ax_{\boldsymbol{6}})$ *is satisfied with*

$$\mathfrak{p}_1(\mathrm{t}) = \mathfrak{p}_2(\mathrm{t}) = \mathfrak{p}_3(\mathrm{t}) = \frac{\sqrt{\mathrm{t} - \frac{1}{2}}}{70e^{-\mathrm{t}+3}},$$
$$\mathfrak{p}^*_1 = \mathfrak{p}^*_2 = \mathfrak{p}^*_3 = \frac{\sqrt{3}}{70e\sqrt{2}},$$
$$\mathfrak{y}_1(\mathrm{t}) = \frac{e^{\pi}}{35t^2}, \quad \mathfrak{y}_2(\mathrm{t}) = \mathfrak{y}_3(\mathrm{t}) = \frac{|\sin^2(\mathrm{t})|}{35e^{\mathrm{t}+3}},$$
$$\mathfrak{y}^*_1 = \frac{e^{\pi}}{35}, \quad \mathfrak{y}^*_2 = \mathfrak{y}^*_3 = \frac{1}{35e^4}.$$

Setting

$$\mathfrak{g}^*_1 = \frac{\sqrt{6}}{\pi e}, \quad \mathcal{W}^*_1 = \frac{16\pi}{35e^3},$$
$$\mathfrak{g}^*_2 = \frac{1}{12}, \quad \mathcal{W}^*_2 = \frac{5\pi}{70e^4}.$$

In addition, condition $(Ax_{\boldsymbol{7}})$ *of Theorem 4.3 are satisfied if we take*

$$7.2494 \leq \gamma \leq 983.11$$

By Theorem 4.3, our coupled system (4.37)-(4.38) *has at least one solution in* J.

4.5 Notes and Remarks

In addition to the primary system studied in this chapter, we have also considered a particular fractional problem that may be viewed as a special case of the general hybrid Langevin fractional pantograph system discussed herein. This specific formulation has been treated separately in a related publication, and it inherits its structure from the main system while introducing additional simplifications that allow for focused analysis under more restrictive assumptions.

The problem is formulated within the framework of Banach spaces and involves the generalized ψ-Caputo fractional derivative, which extends classical operators to better capture memory effects and hereditary behavior. The theoretical analysis encompasses the existence, uniqueness, and Ulam-Hyers-type stability of solutions. The uniqueness results are derived via Banach's fixed-point theorem, while existence is established through Dhage's hybrid fixed-point theorem involving the sum of three operators. Stability results are presented in the sense of Ulam-Hyers and its generalized form, ensuring the robustness of solutions under perturbations.

This particular problem serves as a concrete application of the general theory developed in this chapter. It illustrates how the proposed framework can be adapted to simplified settings, providing insight into its versatility and broader applicability. The findings are supported by illustrative examples that validate the theoretical results and highlight their effectiveness in concrete scenarios.

[6] **H. Bouzid**, A. Benali, A. Salim and A. Jehad, Qualitative Results on Hybrid Langevin–Pantograph ψ-Fractional Differential Equations with Multiple Point Boundary Conditions, (**submitted**).

Conclusion and Perspectives

In this thesis, we have established several results concerning the existence, uniqueness, and Ulam stability of solutions for boundary value and nonlocal problems associated with differential equations involving generalized Caputo-type fractional derivatives. Furthermore, we have investigated nonlocal and boundary value problems for classes of nonlinear Langevin fractional pantograph equations and hybrid Langevin fractional pantograph systems formulated using generalized Caputo fractional derivatives.

The methods employed are primarily based on a variety of fixed point theorems, including those of Krasnoselskii, Dhage, Schaefer, as well as the Banach contraction principle. Additionally, the same types of problems and systems have been extended to the framework of Banach spaces, where the existence results are obtained by applying Darbo's fixed point theorem in conjunction with the technique of the measure of noncompactness.

Given the recent development of proportional fractional and integral derivatives, future research directions include studying nonlinear fractional differential equations involving generalized proportional fractional derivatives. In recent years, various researchers have proposed different forms of integrals and conformable derivatives to recover certain analytical properties not satisfied by classical fractional derivatives. Building on these ideas, modified conformable or proportional fractional derivatives have been introduced, leading to new types of nonlocal fractional operators with kernels involving exponential functions. For further reading on these developments, we refer to [32–35, 35] and the references therein.

From a broader perspective, it would be natural to expand the current findings by considering fractional differential inclusions, more complex hybrid equations, and nonlinear coupled systems. Furthermore, extending the problems studied from Banach spaces to Fréchet spaces would provide an even richer theoretical framework, possibly using alternative fixed-point methods and new compactness conditions adapted to more general settings. Future investigations may also incorporate delay and advance arguments, study random or stochastic fractional problems, and explore generalizations involving recently introduced generalized Mittag-Leffler functions.

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