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SAIAH SEYYID ALI

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Global Existence of Small Data Solutions to Some Semi-linear σ-Evolution Models

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LOUMI Amine	MCA	Université de Chlef	Président
KAINANE MEZADEK Abdelatif	MCA	Université de Chlef	Rapporteur
KAINANE MEZADEK Mohamed	Professeur	Université de Chlef	Co-Rapporteur
HADJ KADDOUR Tayeb	MCA	Université de Chlef	Examinateur
BENIANI Abderrahmane	Professeur	Université de Ain Témouchent	Examinateur

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Dedication

"And my success is not but through Allah. Upon Him I have relied, and to Him I return." (Surah Hud, 88)

With all my love and respect, I dedicate this doctoral dissertation to my beloved parents, **my father** and **my mother**, the pillars of my life, for their endless support and countless sacrifices.
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اهداء

وَمَا تَوْفِيقِي إِلَّا بِاللَّهِ عَلَيْهِ تَوَكَّلْتُ وَإِلَيْهِ أُنِيبُ (سورة هود، الآية ٨٨)

بكل حب واحترام، أهدي هذه الأطروحة لنيل شهادة الدكتوراه إلى والديّ العزيزين، أبي وأمي، ركيزتيّ في الحياة، على دعمهما اللامتناهي وتضحياتهما التي لا تُحصى.

إلى إخوتي الأعزاء، الذين كانوا وما زالوا مصدر إلهامي، وقوتي، وتشجيعي المستمر.

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*

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*

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CONTENTS

NOTATIONS

- Ω: Open set of $\mathbb{R}^n$.
- X, Y: Banach spaces.
- $\|\cdot\|_X$: Norm in X.
- H: Hilbert space.
- Γ: Euler's Gamma function.
- $L(X,Y)$: The space of linear and continuous applications from X to Y.
- α is Multi-indices
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$ with α is Multi-indices.
- $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n} = \partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2 \cdots \partial^{\alpha_n} x_n$ a differntial operator of ored $|\alpha|$
- $C(\Omega)$: The set of continuous functions $u : \Omega \to \mathbb{R}$.
- $C^k(\Omega)$: The set of k-continuously differentiable functions $u : \Omega \to \mathbb{R}$.
- $C\big([0,T];X\big)$ The set of continuous functions $u : [0,T] \to X$
- $C^k\big([0,T];X\big)$ The set of k-continuously differentiable functions $u : [0,T] \to X$
- J_μ: Bessel functions
- E_α : Mittag-Leffler function
- $E_{\alpha,\beta}$:Generalization of Mittag-Leffler function

- $L^p(\Omega)$: $\{u : \Omega \to \mathbb{R},\ u$ measurable and $\int_\Omega |u(x)|^p\, dx < \infty\}$, for $1 \leq p < \infty$.
- $L^\infty(\Omega)$: $\{u : \Omega \to \mathbb{R},\ u$ measurable and there exist a constant $C \geq 0$ such that $|u(x)| \leq C$ almost everywhere in $\Omega\}$.
- $f \lesssim g$: means that there exists a constant $C \geq 0$ such that $f \leq Cg$, where $f, g \in \mathbb{R}^+$.
- ae :Almost Everywhere.

INTRODUCTION

Fractional differential equations have gained considerable importance due to their wide-ranging applications across various scientific disciplines, including quantum mechanics (Schrödinger equation), solid and fluid mechanics (Navier-Stokes equation), biology (Bertalanffy's equation), electromagnetic fields (Maxwell's equations), and many others. For further details, we refer the reader to [7], [20], [21].

The historical development of weakly coupled semi-linear equations dates back to the late 19th and early 20th centuries, with pioneering contributions from mathematicians such as Henri Poincaré. Their foundational work laid the groundwork for understanding nonlinear dynamics in systems. Subsequent progress in reaction-diffusion equations and the development of advanced mathematical methods have significantly advanced this field.

During the last decade, considerable progress has been achieved in the analysis of semi-linear and quasi-linear $\sigma-$evolution equations with different types of damping. Among the most important classes of models are structurally damped $\sigma-$evolution equations and equations involving double damping mechanisms. These models typically exhibit hybrid parabolic–hyperbolic behavior, leading to delicate mathematical difficulties. In particular, decay rates of solutions depend strongly on the damping structure, the spatial dimension, and the regularity of initial data.

Another important research direction concerns weakly coupled systems of σ-evolution equations. Such systems naturally appear in models describing interacting physical quantities, coupled wave propagation, and multi-component media, see for example, [3, 11, 12, 14, 15, 17, 18]. Compared with single equations, weakly coupled systems present additional mathematical challenges due to nonlinear interactions between components and possible imbalance between nonlinear powers.

Recently, few studies have been conducted regarding the fractional σ-evolution equations, we refer for example to [4], [10] and [11].

The objective of this thesis is to investigate the global existence of solutions to the following weakly coupled system of $k-$semi-linear fractional σ-evolution equations with mass terms and distinct nonlinear memory terms:

$$\begin{cases} \partial_t^{1+\alpha_1} u_1 + (-\Delta)^{\sigma_1} u_1 + m_1^2 u = F_{\mu_1,p_1}(u_k), \\ \partial_t^{1+\alpha_2} u_2 + (-\Delta)^{\sigma_2} u_2 + m_2^2 u_2 = F_{\mu_2,p_2}(u_1), \\ \qquad\qquad . \\ \qquad\qquad . \\ \partial_t^{1+\alpha_k} u_k + (-\Delta)^{\sigma_k} u_k + m_k^2 u_k = F_{\mu_k,p_k}(u_{k-1}) \\ u_\ell(0,x) = u_{0l}(x),\ \partial_t u_\ell(0,x) = 0, \quad \ell = 1, 2, ..., k \end{cases}$$

and to the following weakly coupled system of semi-linear fractional σ-evolution equations without mass terms and with distinct power nonlinearities:

$$\begin{cases} \partial_t^{1+\alpha_1} u + (-\Delta)^{\sigma_1} u = |v|^{p_1},\ u(0,x) = u_0(x)\ u_t(0,x) = 0, \\ \partial_t^{1+\alpha_2} v + (-\Delta)^{\sigma_2} v = |u|^{p_2},\ v(0,x) = v_0(x),\ v_t(0,x) = 0, \end{cases}$$

where $k \geq 2$, and for $l = 1, \ldots, k$, we have $\alpha_l \in (0,1)$, $\mu_l \in (0,1)$, $p_l > 1$, $m_l > 0$, $\sigma_l \geq 1$, and $(t,x) \in [0,\infty) \times \mathbb{R}^n$. The nonlinear memory term is defined as

$$F_{\mu,p}(w)(t,x) := \int_0^t (t-s)^{-\mu} |w(s,x)|^p, ds.$$

The operator $\partial_t^{1+\alpha_l} u$ is understood as $D_t^{\alpha_l}(u_t)$, where $D_t^{\alpha_l}$ denotes the Riemann–Liouville fractional derivative:

$$D_t^{\alpha_l}(f) = \partial_t\big(I_t^{1-\alpha_l} f\big), \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s), ds; ; \text{for } \beta > 0.$$

Here, I_t^β is the fractional integral of order β, and Γ denotes Euler's gamma function.

This thesis is organized into five chapters as follows.

In the first chapter, we discuss fundamental concepts in functional analysis, including the structure of function spaces and the interplay between various norms, along with essential inequalities governing their relationships. Special attention is given to Lebesgue's convergence theorems, which are indispensable when interchanging limits and integrals in mathematical analysis.

We also explore the Fourier transform, its fundamental properties, and its significance in solving partial differential equations (PDEs) and evolution problems. Several special functions—such as modified Bessel functions and Mittag-Leffler functions—that frequently arise in fractional calculus and related applications are revisited. At the end of this chapter, we introduce the concept of fractional derivatives, particularly in the Riemann–Liouville sense, thereby bridging the gap between classical calculus and fractional analysis, and preparing the reader for their application in fractional-order evolution equations and advanced models.

In the second chapter, we focus on the homogeneous problem associated with certain fractional σ-evolution equations, both with and without mass terms. Using modified Bessel functions and Mittag-Leffler functions, we establish polynomial decay estimates in L^p–L^q norms for solutions to the linear fractional evolution equations related to our problem.

The third chapter is devoted to the study of global existence of solutions to weakly coupled systems of $k-$semi-linear fractional σ-evolution equations with mass and different power nonlinearities. Our aim is to show the influence of the parameters and the regularity of the data on the qualitative behavior of solutions. By using some fixed-point theory and L^r–L^q estimates of the linear solution, we prove the global existence of solutions for our problem. In the chapter four, we study of global existence of solutions to weakly coupled systems of semi-linear fractional σ-evolution equations with out mass and different power nonlinearities. We demonstrate that the absence of the mass terms complicates the study of the global existence. Finally, we conclude by presenting auxiliary lemmas and theorems essential for the rigorous justification of our results.

CHAPTER 1

PRELIMINARIES

We begin by recalling the fundamental concepts and select classical results of analysis and fractional calculus that will be employed in the execution of this undertaking.

1.1 The Lebesgue $L^p(\Omega)$ spaces

In this section we discuss an important construction, which is extremely useful in virtually all branches of Analysis.

1.1.1 Definitions and basic properties

Definition 1.1 *(see [1]) Let Ω an open set of $\mathbb{R}^n$. If $f : \Omega \to \mathbb{K}$ is a measurable function, with $1 \leq p < \infty$, then we define*

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p \, d\mu \right)^{\frac{1}{p}},$$

and

$$\|f\|_{L^\infty(\Omega)} = \|f\|_{L^\infty} = esssup_{x\in\Omega}|f(x)| = \inf \{C \mid |f(x)| \leq C \;\; a.e. \;\; on \; \Omega\}.$$

Definition 1.2 *(see [1])*

The space $L^p(\Omega)$ is the set:

$$L^p(\Omega) = \{f : \Omega \to \mathbb{K} \,|\, \|f\|_{L^p(\Omega)} < \infty\} \quad for \;\; p \in [1, \infty[,$$

$$L^\infty(\Omega) = \{f : \Omega \to \mathbb{K} \mid is \; measurable \; and \; there \; exists \; C \geq 0 \; such \; that \; |f(x)| \leq C \;\; a.e. \;\; on \; \Omega\}.$$

Proposition 1 *The space $L^p(\Omega)$ for all $p \geq 1$ is a vector space.*

Hölder's Inequality

Proposition 2 *(see [1]) Let $1 < p < \infty$, we denote by q the conjugate exponent of p, that is $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and we have the following inequality:*

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}}.$$

The previous inequality summarizes the inequality of Cauchy-Schwarz for $p = 2$ as:

$$\int_\Omega |f(x)g(x)|\,dx \leq \left(\int_\Omega |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |g(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Interpolation's inequality

Proposition 3 *Let $u \in L^p \cap L^q$ with $1 \leq p \leq q \leq \infty$. Then $u \in L^r$ for all $r \in [p, q]$ and we have*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^q}^{1-\theta},$$

Where $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$ with $\theta \in [0, 1]$.

Fractional Gagliardo-Nirenberg inequality

Proposition 4 *Let $1 < p, p_0, p_1 < \infty, \sigma > 0$and $s \in [0, \sigma)$,Then ,it holds for all $u \in L^{p_0} \cap H^{\sigma}_{p_1}$*

$$\|u\|_{H^s_{p_1}} \lesssim \|u\|_{L^{p_0}}^{1-\theta} \|u\|_{H^s_{p_1}}^{\theta},$$

where

$$\theta = \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{s}{n}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{\sigma}{n}}.$$

Fractional powers

Proposition 5 *Let $p > 1, p_0, 1 < r < \infty$,and $u \in H^s_r$,where $s \in (\frac{n}{r}, p)$,Let us denote by $F(u)$ one of the functions $|u|^p$,$|u|^{p-1}u$. we have the following hold:*

$$\|F(u)\|_{H^s_r} \lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{H^s_r} \qquad \textit{and} \qquad \|F(u)\|_{H^s_r} \lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{H^s_r}.$$

1.1.2 Lebesgue convergence theorems

In this subsection, we quickly review analysis results that will be used most frequently throughout the rest.

Let (X, μ) be a measure space. Most of the time, X will be an open subset of $\mathbb{R}^n$, and μ the Lebesgue measure, defined on the completion of the Borel sigma-algebra. At the foundation of it all lies the following theorem, which is rarely used in its raw form.

Proposition 6 *(Lebesgue's Monotone Convergence Theorem) Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable functions on X, with values in $\mathbb{R}_+$, this sequence converges pointwise to a measurable function f, and we have:*

$$\int_X f_n d\mu \to \int_X f d\mu \quad in \quad n \to \infty^+.$$

Proposition 7 *(Lebesgue's Dominated Convergence Theorem)*
Let $(f_n)_{n\in\mathbb{N}}$be a sequence of measurable functions on X, with values in $\mathbb{R}$such that:

$$f_n \to f \quad \text{almost everywhere in } X \quad in \quad n \to \infty^+,$$

and there exists a function $g \in L^1(X, d\mu)$ (i.e integrable), such that $|f_n(x)| \leq g(x)$ almost everywhere on X. Then f is integrable and:

$$\int_X |f_n - f| d\mu \to 0 \quad in \quad n \to \infty^+.$$

In particular:

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X \lim_{n\to\infty} f_n d\mu = \int_X f d\mu.$$

The dominated convergence theorem also exists in an $L^p(X, d\mu)$ version.

1.2 Fourier transformation

In this section, we introduce the notion of Fourier transform, which is used to solve our equations by transforming them into the frequency face, where they become easier to analyze and solve.

1.2.1 Fourier transforms in $L^1(\mathbb{R}^n)$

Definition 1.3 *(see [25]) Let be $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is defined by:*

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i(x\cdot\xi)}\, dx, \quad \forall \xi \in \mathbb{R}^n,$$

where $x \cdot \xi = \sum_{i=1}^n x_i \xi_i$.

Example 1 *The Fourier transform of function :*

$$f(x) = \exp(-\alpha|x|) \quad \textit{where } \alpha > 0.$$

We want to compute its Fourier transform :

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\, dt.$$

We separate the integral over the two domains:

$$\mathcal{F}(\omega) = \int_{-\infty}^{0} e^{\alpha x}e^{-i\omega x}\, dt + \int_{0}^{\infty} e^{-\alpha x}e^{-i\omega x}\, dx.$$

We can write it

$$\mathcal{F}(\omega) = \int_{-\infty}^{0} e^{(\alpha - i\omega)x}\, dt + \int_{0}^{\infty} e^{-(\alpha+i\omega)x}\, dx.$$

We compute each integral:

$$\int_{0}^{\infty} e^{-(\alpha+i\omega)x}\, dx = \left[\frac{-e^{-(\alpha+i\omega)x}}{\alpha + i\omega}\right]_0^{\infty} = \frac{1}{\alpha + i\omega}.$$

Adding both results:

$$\mathcal{F}(\omega) = \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}.$$

Definition 1.4 *(see [25]) The convolution product of f and $g \in L^1(\mathbb{R}^n)$ denoted by $f * g$, is the function defined by:*

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\, dy.$$

1.2.2 Properties of the Fourier transform

Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$:

- $\hat{f}$ is a continuous function on $\mathbb{R}^n$, $\lim\limits_{|\xi|\to+\infty} \hat{f}(\xi) = 0$ and we have :

$$\|\hat{f}\|_\infty \leq \|f\|_1.$$

- F is a linear operator. That is for any $f, g \in L^1$ and $a, b \in \mathbb{C}$, we have:

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g).$$

- The Fourier transform of a translated and scaled function is given by:

$$\mathcal{F}(f(bt-a))(\lambda) = \frac{1}{|b|}e^{-i\lambda a/b}\mathcal{F}(f)\left(\frac{\lambda}{b}\right), \quad \forall a \in \mathbb{R}, \forall b \in \mathbb{R}^*.$$

- $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$.
- $\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g)$.

Proof see [25]

Proposition 8 *(Energy theorem) Suppose $f(x)$ s real valued function defined over $\mathbb{R}^n$.with Fourier transform given as $\mathcal{F}(\xi)$ we have:*

$$\int_{\mathbb{R}^n} |f(x)|^2 dx = \int_{\mathbb{R}^n} |\mathcal{F}(x)|^2 dx$$

1.2.3 Inverse Fourier transform

(see [25]) Let $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in L^1(\mathbb{R}^n)$. The inverse Fourier transform of a function f is given by the formula:

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \quad \forall x \in \mathbb{R}^n.$$

1.2.4 Derivative of the Fourier transform

Proposition 9 *see [25] Let $f \in L^1(\mathbb{R})$ such that $x \mapsto xf(x) \in L^1(\mathbb{R})$. Then:*

1. *$\hat{f}$ is continuously differentiable.*
2. *$(\mathcal{F}(f))'(\xi) = -i\mathcal{F}(xf)(\xi)$.*

Proposition 10 *(see [25]) Let $f \in L^1(\mathbb{R}^n)$. If $\partial_x^\beta (x^\alpha f(x)) \in L^1(\mathbb{R}^n)$ for any $\alpha, \beta \in \mathbb{N}^n$ and with $|\alpha| \leq K$, then*

$$\xi^\beta \partial_\xi^\alpha \widehat{f}(\xi) = (-i)^{|\alpha|+|\beta|} \mathcal{F}\left(\partial_x^\beta (x^\alpha f(x))\right)(\xi).$$

1.3 Fractional derivatives within the meaning of Riemann-Liouville

The definition of the Riemann[1]-Liouville[2] fractional integral is a generalization of the real order α of the Cauchy formula, established by Augustin-Louis Cauchy $(1789 - 1857)$. This generalization allows for the extension of integration to non-integer orders, thus offering a more general approach for the calculation of integrals.

[1]**Benard Riemann:**(Hanovre 1826 -Italie 1866) fractional integral , is a allemand mathematician . He made a significant contribution to analysis and differential geometry..

[2]**Joseph Liouville:**(Saint Omer 1809-Paris 1882) A French mathematician. Among his most famous works are the discovery of transcendental numbers in 1844, and elliptic integrals.

1.3.1 Gamma Function

The Euler's Gamma function $\Gamma(z)$ is among the fundamental functions within fractional calculus.

Definition 1.5 *(see [1]) The Gamma function, denoted by Γ, is given by:*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \text{for all} \quad z \in \mathbb{C} \text{ with } Re(z) > 0.$$

Which converges in the right half of the complex plane $Re(z) > 0$. Indeed, we have:

$$\begin{aligned} \Gamma(x+iy) &= \int_0^\infty t^{x-1+iy}e^{-t}dt \\ &= \int_0^\infty t^{x-1}e^{iy\log(t)}e^{-t}dt \\ &= \int_0^\infty t^{x-1}[\cos(y\log(t)) + i\sin(y\log(t))]e^{-t}dt. \end{aligned} \tag{1.1}$$

The expression in the square brackets in (1.1) is bounded for all t, convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $x = Re(z) > 1$.

Proposition 11

1- *We have $\Gamma(z)$ is defined and analytic in the region $Re(z) > 0$.*

2- $\Gamma(z+1) = z\Gamma(z)$.

3- $\Gamma(1-z)\Gamma(z) = \dfrac{\pi}{sin(\pi z)}$.

4- $\Gamma(z+1) \approx \sqrt{2\pi}z^{z+\frac{1}{2}}e^{-z}$ *for $|z|$ large, $Re(z) > 0$.*

In particular, $n! \approx \sqrt{2\pi}\, n^{n+1/2}e^{-n}$ (Stirling's formula).

1.3.2 Beta Function

In many cases it is more convenient to use the beta function instead of a certain combination of values of the gamma function.For more details refer to [1]

Definition 1.6 *The beta function, denoted by B, is given by :*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \qquad \forall z, w \in \mathbb{C} \text{ with } Re(z) > 0 \text{ and } Re(w) > 0.$$

(Euler's beta function).

Remark 1 *To establish the relationship between the gamma function and the beta function we will use the Laplace transform. and we will obtain this formula:*

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad \forall z,w\in\mathbb{C} \text{ with } Re(z)>0 \text{ and } Re(w)>0.$$

from which it follows that :

$$B(z,w)=B(w,z).$$

1.3.3 Improved Bessel functions

The theory of Bessel functions is connected with Riccaati's equations, in fact,Bessel functions are defined as solutions of Bessel's equation, Here we need some basic and basic properties of the improved Bessel functions.

Definition 1.7 *The Bessel functions of the first kind J_μ are defined from their power series representation:*

$$J_\mu(s)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\mu+1)}\left(\frac{s}{2}\right)^{2k+\mu},$$

where $\mu>0$. The improved modified Bessel function $\tilde{J}_\mu(s)$ is defined by $\tilde{J}_\mu(s)=\dfrac{J_\mu(s)}{s^\mu}$.

Proposition 12 *Assume that μ is a non-negative integer. We have the following properties:*

1. $sd_s\tilde{J}_\mu(s)=\tilde{J}_{\mu-1}(s)-2\mu\tilde{J}_\mu(s)$.
2. $d_s\tilde{J}_\mu(s)=-s\tilde{J}_{\mu+1}(s)$.
3. $\tilde{J}_{-\frac{1}{2}}(s)=\sqrt{\frac{2}{\pi}}\cos(s)$.
4. *We investigate the asymptotic necessity of estimates involving modified Bessel functions:*

$$|\tilde{J}_\mu(s)|\leqslant Ce^{\pi|\Im\mu|} \; if \; |s|\leqslant 1,$$

and

$$\tilde{J}_\mu(s)=Cs^{-\frac{1}{2}}\cos(s-\frac{\mu}{2}\pi-\frac{\pi}{4})+O(|s|^{-\frac{3}{2}}) \; if \; |s|\geqslant 1.$$

5. $\tilde{J}_{\mu+1}(r|s|)=-\frac{1}{r|s|^2}\partial_r\tilde{J}_\mu(r|x|), \qquad r\neq 0, s\neq 0.$

Proposition 13 *Let $f\in L^p(\mathbb{R}^n)$ a radial function with $p\in[1,2]$, then the reverse of Fourier transformation is also a radial function and it satisfies :*

$$\mathcal{F}^{-1}(f)(x)=\int_0^\infty g(r)r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr, \qquad g(|x|):=f(x). \tag{1.2}$$

Remark 1 *(Equallity)*
for $\mu = 1$,we have this equallity :

$$J_0(x) = \frac{1}{x} J_1(x) + \frac{d}{dx} J_1(x).$$

To prove this equallity we use this resultat from lemma $x d_x \tilde{J}_\mu(x) = \tilde{J}_{\mu-1}(x) - 2\mu \tilde{J}_\mu(x)$,
for $\mu = 1$, we have :

$$\begin{aligned} x d_x \tilde{J}_1(x) &= \tilde{J}_0(x) - 2\tilde{J}_1(x), \\ x d_x \big(\frac{J_1(x)}{x}\big) &= J_0(x, -2\big(\frac{J_1(x)}{x}\big), \\ x\Big(\frac{d_x J_1(x) x - J_1(x)}{x^2}\Big) &= J_0(x) - \frac{2}{x} J_1(x), \\ d_x J_1(x) &= J_0(x) - \frac{1}{x} J_1(x). \end{aligned}$$

then, we get our equallity $J_0(x) = \frac{1}{x} J_1(x) + \frac{d}{dx} J_1(x)$. Also, we can prove the equallity using the definition of Bessel function and some proprities of Gamma function.

Mittag-Leffler function

The Mittag–Leffler function[3] emerges as a natural mathematical tool in the formulation of solutions to fractional integral equations (Saxena et al., 2002). It is particularly important in the investigation of fractional kinetic models, random walks, Lévy flight processes, and anomalous diffusion phenomena. Moreover, the standard and generalized forms of the Mittag–Leffler function bridge the gap between exponential laws and power-type behaviors characteristic of classical kinetic equations and their fractional counterparts (Lang, 1999; Hilfer, 2000; Saxena et al., 2002).

Definition 1.8 *(see [22]) The called funtions of the Mittag-Leffler type,play an important role in the theory of fractional equations(FDEs), First we introduce a two parameter Mittag-Leffler function defined by (1.3):*

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \big(\alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0\big), \tag{1.3}$$

and with $\beta = 1$

$$E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \big(\alpha \in \mathbb{C}, Re(\alpha) > 0\big). \tag{1.4}$$

[3]**Gösta Mittag-Leffler** (1846–1927) was a Swedish mathematician whose work in analytic function theory led to the introduction of the function that now bears his name.

The function $E_\alpha(z)$ was defined and studied by Mittag-Leffler in the year 1903. It is a direct generalization of the exponential function. The function defined by (4.31) *gives a generalization of* (1.4). *This generalization was studied by Wiman in 1905, Agarwal in 1953 and Humbert and Agarwal in 1953, and others.*

Definition 1.9 *The Mittag-Leffler function E_β allows the following implicit definition:*

$$\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}E_\beta(\lambda s^\beta)ds = E_\beta(\lambda t^\beta)-1. \tag{1.5}$$

The Mittag-Leffler function $E_\beta(-t^\beta\langle\xi\rangle_m^2)$ with

$$E_\beta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}, \beta\in\mathbb{C} \text{ avec } Re\beta>0.$$

Basic calculate of Mittag-Leffler Function

As a consequence of the definitions (1.4) and (1.3) the following results hold:

Proposition 14 *Show that the following relations hold :*

i. $E_{\alpha,\beta}(z)=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)}$.

ii. $E_{\alpha,\beta}(z)=\beta E_{\alpha,\beta+1}(z)+\alpha z\,\frac{d}{dz}\,E_{\alpha,\beta+1}(z)$.

iii. $\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha),\quad \big(Re(\beta-m)>0, m\in\mathbb{N}\big)$.

Proof:

i. we have

$$\begin{aligned}E_{\alpha,\beta}(z)&=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}=\sum_{k=-1}^{\infty}\frac{z^{k+1}}{\Gamma(\alpha k+\alpha+\beta)}\\&=zE_{\alpha,\alpha+\beta}(z)+\frac{1}{\Gamma(\beta)},\quad Re(\beta)>0.\end{aligned}$$

ii. we have

$$\begin{aligned}\beta E_{\alpha,\beta+1}(z)+\alpha z\,\frac{d}{dz}\,E_{\alpha,\beta+1}(z)&=\beta E_{\alpha,\beta+1}(z)+\alpha z\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta+1)},\\&=\beta E_{\alpha,\beta+1}(z)+\sum_{k=0}^{\infty}\frac{(\alpha k+\beta-\beta)z^k}{\Gamma(\alpha k+\beta+1)},\\&=E_{\alpha,\beta}(z).\end{aligned}$$

iii.

$$\Big(\frac{d}{dz}\Big)^m\big[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)\big] = \Big(\frac{d}{dz}\Big)^m\sum_{k=0}^{\infty}\frac{z^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)},$$
$$= \sum_{k=0}^{\infty}\frac{z^{\alpha k+\beta-m-1}}{\Gamma(\alpha k+\beta-m)}, \quad Re(\beta)>0.$$

since

$$\Big(\frac{d}{dz}\Big)^m\big(z^{\alpha k+\beta-1}\big) = \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta-m)}z^{\alpha k+\beta-m-1},$$
$$= z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha), \quad \big(Re(\beta-m)>0, m\in\mathbb{N}\big).$$

1.3.4 Riemann-Liouville Integral

Definition 1.10 *(see [19]) Let f be a continuous function on $[a,b]$, $0<\alpha<1$ and $a\le x\le b$. The fractional integral of order α is defined by :*

$$\left(I_a^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt. \tag{1.6}$$

where α is a suitably chosen real (or even complex). Note that formula (1.6) is (at least formally) a generalisation of the n^{th} primitive with a non-integer α order of "primitivation".

Exemples 1.3.1 *Consider the function*

$$f(x) = (x-a)^\beta \in L^1([a,b],\mathbb{R}), \beta\in\mathbb{C},$$

such that $Re(\beta)>0$. Then:

$$I_a^\alpha (x-a)^\beta = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(t-a)^\beta dt. \tag{1.7}$$

To evaluate this integral we set the change, $t=a+(x-a)\tau$.

$$I_a^\alpha (x-a)^\beta = \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^\beta d\tau$$
$$= \frac{(x-a)^{\alpha+\beta}}{\Gamma(\alpha)}B(\beta+1,\alpha).$$
$$I_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(x-a)^{\alpha+\beta} \tag{1.8}$$

after using the Eulerian integral of the first kind (Euler 's beta function). We can see that this is a generalization of the case where $\alpha = 1$

$$\begin{aligned} I_1^\alpha (x-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+2)}(x-a)^{1+\beta} \\ &= \frac{1}{1+\beta}(x-a)^{1+\beta} \end{aligned}$$

because of the relationship: $\Gamma(z+1) = z\Gamma(z)$.

Theorem 1 *(see [19]) For* $n > 0$*, suppose* $(f_k)_{k=1}^\infty$ *is a uniformly convergent sequence of continuous functions on* $[a,b]$*. Then, we can interchange the fractional integral operator and the limit process almost everywhere:*

$$\left(I_a^n \lim_{k\to\infty} f_k\right)(x) = \left(\lim_{k\to\infty} I_a^n f_k\right)(x).$$

Especially, the sequence of functions $I_a^n(f_k)_{k=1}^\infty$ *is uniformly convergent.*

Preuve . We denote the limit of the sequence (f_k) by f. It is well known that f is continuous. We then find

$$\begin{aligned} |I_a^n f_k(x) - I_a^n f(x)| &\leq \frac{1}{\Gamma(n)}\int_a^x |f_k(t)-f(t)|(x-t)^{n-1}dt \\ &\leq \frac{1}{\Gamma(n)}||f_k - f||_\infty \int_a^x (x-t)^{n-1}dt \\ &= \frac{1}{\Gamma(n+1)}||f_k - f||_\infty (x-a)^n \\ &\leq \frac{1}{\Gamma(n+1)}||f_k - f||_\infty (b-a)^n. \end{aligned} \tag{1.9}$$

Which converges to zero as $(k \longrightarrow \infty)$ uniformly for all $x \in [a,b]$. ∎

Proposition 15 *(see [19]) Let* $f \in C^n([a,b])$*. For* x *fixed, the map* $\alpha \longmapsto (I_a^\alpha f)(x)$ *defined for* $Re(\alpha) > 0$ *is holomorphic and analytically extends to the domain* $Re(\alpha) > -n$*.*

Preuve . Remark that the application in question is well defined and holomorphic for $Re(\alpha) > 0$. Let us show the existence of the analytic continuation. In (1.6) we proceed by an integration by part:

$$(I_a^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)d\left[-\frac{(x-t)^\alpha}{\alpha}\right]dt = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}f(a) + \frac{1}{\Gamma(\alpha+1)}\int_a^x (x-t)^\alpha f'(t)\,dt,$$

$$(I_a^\alpha f)(x) = \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}f(a) + (I_a^{\alpha+1} f')(x). \tag{1.10}$$

It is clear that the right-hand side of the preceding equality is holomorphic in the domain

$Re(\alpha) > -1$ because the terms on the right-hand side are the product of a holomorphic function and a holomorphic function, or the sum of two holomorphic functions. The first term, $\frac{(x-a)^a}{\Gamma(a+1)} f(a)$, is a holomorphic function because it is the product of a holomorphic function (the integral operator) and another holomorphic function (the function $f(a)$). The second term $(I_a^{a+1} f')(x)$, is also a holomorphic function because it is the result of applying the integral operator to the derivative of a holomorphic function. Since the sum of two holomorphic functions is also holomorphic, the right-hand side of the equation is holomorphic in the domain $Re(\alpha) > -1$. This domain is chosen because it ensures that the integral operator $I_a^\alpha f(x)$ is well-defined and holomorphic. When $Re(\alpha) < -1$, the integral operator may not converge, or it may converge to a non-holomorphic function. By restricting the domain to $Re(\alpha) > -1$, we can guarantee that the integral operator produces a holomorphic function, which is essential for the proof of the analytic continuation. Now the final result comes from a simple iteration of (1.10):

$$(I_a^\alpha f)(x) = \sum_{j=0}^{n-1} \frac{(x-a)^{\alpha+j}}{\Gamma(\alpha+1+j)} f^{(j)}(a) + \left(I_a^{\alpha+n} f^{(n)}\right)(x), \tag{1.11}$$

formula that is the expression of the analytical extension. ■

Proposition 16 *(see [19]) Let $f \in C^0([a,b])$. For α and β two complexes numbers such that $Re(\alpha) > 0$ and $Re(\beta) > 0$, we have:*

$$I_a^\alpha (I_a^\beta f) = I_a^{\alpha+\beta} f.$$

And if $Re(\alpha) > 1$, then:

$$\frac{d}{dx} I_a^\alpha f = I_a^{\alpha-1} f.$$

Preuve . Let $f \in C^0([a,b])$ and α, β two valuer complex such as: $Re(\alpha) > 0$ and $Re(\beta) > 0$. For any $x \in [a,b]$, we have :

$$\begin{aligned}
[I_a^\alpha(I_a^\beta f)](x) &= \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}(I_a^\beta f)(t)\,dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_a^t (x-t)^{\alpha-1}(t-h)^{\beta-1}f(h)\,dh\,dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (x-t)^{\alpha-1}(t-h)^{\beta-1}f(h)\chi_{]a,x[}(t)\chi_{]a,t[}(h)\,dh\,dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (x-t)^{\alpha-1}(t-h)^{\beta-1}f(h)\chi_{]a,x[}(t)\chi_{]a,t[}(h)\,dt\,dh \quad (Fubini's\ \ theorem) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b f(h)\int_a^b (x-t)^{\alpha-1}(t-h)^{\beta-1}\chi_{]h,x[}(t)\chi_{]a,x[}(h)\,dt\,dh \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b f(h)\chi_{]a,x[}(h)\int_a^b (x-t)^{\alpha-1}(t-h)^{\beta-1}\chi_{]h,x[}(t)\,dt\,dh \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(h)\int_h^x (x-t)^{\alpha-1}(t-h)^{\beta-1}\,dt\,dh.
\end{aligned}$$

To evaluate this integral we set the change, $t = h + (x-h)\lambda$

$$\begin{aligned}
[I_a^\alpha(I_a^\beta f)](x) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(h)\int_0^1 (x-h)^{\alpha-1+\beta-1}(1-\lambda)^{\alpha-1}\lambda^{\beta-1}(x-h)\,d\lambda\,dh \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-h)^{\alpha+\beta-1}f(h)\int_0^1 (1-\lambda)^{\alpha-1}\lambda^{\beta-1}\,d\lambda\,dh \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-h)^{\alpha+\beta-1}f(h)B(\alpha,\beta)\,dh \\
&= \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-h)^{\alpha+\beta-1}f(h)\,dh \\
&= \frac{1}{\Gamma(\alpha+\beta)}\int_a^x (x-h)^{\alpha+\beta-1}f(h)\,dh \\
&= [I_a^{\alpha+\beta}f](x).
\end{aligned}$$

So

$$[I_a^\alpha(I_a^\beta f)](x) = [I_a^{\alpha+\beta}f](x),\ \forall x \in [a,b].$$

In formula (1.10), which is valid for $Re(\alpha) > -1$, let $\alpha = 0$. We obtain :

$$(I_a^0 f)(x) = f(a) + (I_a^1 f')(x) = f(a) + \int_a^x f'(t)dt = f(x). \tag{1.12}$$

This shows that for the functions C^1 at least we have $I_a^0 f = f$. In fact this identity remains valid for continuous functions. ■

1.3.5 Riemann-Liouville derivative

Definition 1.11 *(see [19]) Let $\alpha \in [m-1, m]$ such that, $m \in \mathbb{N}^\star$.The derivative of order α in the Riemann-Liouville sense is the function defined by:*

$$ {}^{RL}D_a^\alpha f = D^m I_a^{m-\alpha} f, \tag{1.13}$$

and is called the Riemann-Liouville fractional differential operator of order α.

Exemples 1.3.2 *Let f defined by $f(x) = (x-a)^\beta$ with $\beta \in \mathbb{R}$ and $\alpha \in [m-1, m]$ such that, $m \in \mathbb{N}^\star$. Then :*

$$\begin{aligned} {}^{RL}D_a^\alpha (x-a)^\beta &= \left(\frac{d}{dx}\right)^m \left(\frac{\Gamma(\beta+1)}{\Gamma(\beta+1+m-\alpha)}(x-a)^{\beta+m-\alpha}\right) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\frac{\Gamma(\beta+1+m-\alpha)}{\Gamma(\beta+1+m-\alpha)}(x-a)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(x-a)^{\beta-\alpha}, \end{aligned} \tag{1.14}$$

where we used the formula

$$\left(\frac{d}{dx}\right)^p (x-a)^\lambda = \lambda(\lambda-1)\cdots(\lambda-p+1)(x-a)^{\lambda-p}.$$

It is clear that the derivation formula(1.14) is reduced to $\alpha = 1$

$$ {}^{RL}D_a^1 (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta)}(x-a)^{\beta-1} = \beta(x-a)^{\beta-1} = \frac{d}{dx}(x-a)^\beta.$$

In the previous example if we take $\beta = 0$ we obtain the following result :

$$ {}^{RL}D_a^\alpha 1 = \frac{1}{\Gamma(1-\alpha)}(x-a)^{-\alpha}.$$

That is, the Riemann-Liouville derivative of a constant is not zero!.

Lemma 1 *(see [19]) Let $\alpha \in]m-1, m[$ and f a function verifying ${}^{RL}D_a^\alpha f = 0$. Then:*

$$f(x) = \sum_{j=0}^{m-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha-m)}(x-a)^{j+\alpha-m}.$$

Where the c_j are any complexe constants.

Preuve . Let's start from ${}^{RL}D_a^\alpha f = \left(\frac{d}{dx}\right)^m \left[I_a^{m-\alpha} f\right](x) = 0$, so we have first:

$$\left[I_a^{m-\alpha} f\right](x) = \sum_{j=0}^{m-1} c_j (x-a)^j.$$

And by application of I_a^α we get

$$\left[I_a^m f\right](x) = \sum_{j=0}^{m-1} c_j \frac{\Gamma(j+1)}{\Gamma(j+1+\alpha)}(x-a)^{j+\alpha}.$$

■

CHAPTER 2

LINEAR FRACTIONAL σ-EVOLUTION EQUATIONS WITH AND WITH OUT MASS TERMS

In this chapter, we focus on the homogeneous problem to some fractional $\sigma-$evolution equations involving mass terms and with out mass terms. We derive decay estimates for linear Cauchy problem, where the decay behavior is examined in detail as part of a comprehensive analytical study. To achieve polynomial decay in $L^m - L^q$ estimates, we employ modified Bessel functions and Mittag-Lefler functions, which play a central role in capturing the temporal decay properties of the solution to the following problem

$$\begin{cases} \partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = 0, \end{cases} \tag{2.1}$$

where $\alpha \in (0,1)$, $m \geq 0$, $p > 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $\partial_t^{1+\alpha} u = D_t^\alpha(u_t)$ with $D_t^\alpha(f) = \partial_t(I_t^{1-\alpha} f)$ and

$$I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s), \quad \text{for} \quad \beta > 0.$$

$D_t^\alpha(f)$ and $I_t^\beta f$ denote the fractional derivative and the fractional integral of Riemann-Liouville respectively of f on $[0,t]$ and Γ is Euler's Gamma function. By the hypothesis $u_t(0,x) = 0$, so we can write the Cauchy problem (2.1) in the form of an integro-differential equation.

$$\begin{cases} \frac{\partial u}{\partial t} = I_t^\alpha \left(-(-\Delta)^\sigma u - m^2 u \right), \\ u(0,x) = u_0(x). \end{cases} \tag{2.2}$$

A solution to (2.1) is defined as a solution of (2.2). For this reason, we may restrict ourselves in further considerations to the study of (2.2) to obtain results for (2.1).

2.1 Some preliminary

The solution to Cauchy problem (2.2) with parameters $\sigma \geq 1$ and $m > 0$ can be formally expressed via an integral equation formulation, involving a properly defined kernel or memory operator.

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x), \tag{2.3}$$

with

$$G(t,x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} E_{\alpha+1}\left(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\right) d\xi, \tag{2.4}$$

where $\{G^m_{\alpha,\sigma}(t)\}_{t\geq 0}$ denotes the semigroup of operators which is defined via Fourier transform by $(\widehat{G^m_{\alpha,\sigma}(t) * f})(t,\xi) = E_{\alpha+1}\left(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\right) \widehat{f}(\xi) \quad \text{with} \quad \langle\xi\rangle^2_{m,\sigma} = |\xi|^{2\sigma} + m^2.$

Here

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)},\ \beta \in \mathbb{C} \ \ \textit{with} \ \ \mathrm{Re}(\beta) > 0.$$

, where E_β represents the Mittag-Leffler function.

A representation of solutions of the linear integro-differential equation associated to (2.2) or (2.1) with $\sigma \geq 1$ and $m > 0$ is given by

$$u(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x). \tag{2.5}$$

Indeed, we put

$$\begin{aligned} v(t,\xi) &= F_{x\to\xi}(u(t,x))(t,\xi) \\ &= F_{x\to\xi}\left((G^m_{\alpha,\sigma}(t) * u_0)(t,x)\right)(t,\xi) \\ &= E_{\alpha+1}\left(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma}\right) \widehat{u_0}(\xi). \end{aligned}$$

By using (2.2) and (5.13) we have :

$$
\begin{aligned}
&F^{-1}_{\xi\to x}\left\{F_{x\to\xi}\left(\int_0^t I_s^\alpha\left(-(-\Delta)^\sigma u - m^2u\right)(s,x)\,ds\right)(t,\xi)\right\}(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\langle\xi\rangle^2_{m,\sigma}\int_0^t I_s^\alpha(\widehat{G(t,x)}u_0(\xi))\,ds\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}\widehat{G(\tau,x)}u_0(\xi)\,d\tau\,ds\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_0^s (s-\tau)^{\alpha-1}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\,ds\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha)}\int_0^t\int_\tau^t (s-\tau)^{\alpha-1}\,ds\,E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\frac{\langle\xi\rangle^2_{m,\sigma}}{\Gamma(\alpha+1)}\int_0^t (t-\tau)^{\alpha}E_{\alpha+1}(-\tau^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)\,d\tau\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(\Big(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})-1\Big)\widehat{u_0}(\xi)\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\Big(E_{\alpha+1}(-t^{\alpha+1}\langle\xi\rangle^2_{m,\sigma})\widehat{u_0}(\xi)-\widehat{u_0}(\xi)\Big)(t,x)\\
&= F^{-1}_{\xi\to x}\big(\widehat{u}(t,\xi)-\widehat{u_0}(\xi)\big)(t,x) = u(t,x)-u_0(x).
\end{aligned}
$$

Consequently, we arrive at the following conclusion:

$$u = G^m_{\alpha,\sigma}(t) * u_0$$

The presented results provide a formal solution to the stated equations

$$u = u_0(x) + \int_0^t I_s^\alpha\left(-(-\Delta)^\sigma u - m^2u\right)\,ds.$$

2.2 L^p estimates for model oscillating integrals

At first we derive L^p estimates for the model oscillating integral

$$F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}|\xi|^{2\sigma})\big).$$

Proposition 2.2.1 *The following estimate holds in* $\mathbb{R}^n$ *for* $\sigma>0$, $\alpha\geq 0$*:*

$$\|F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}|\xi|^{2\sigma})\big)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}, \tag{2.6}$$

for $p\in[1,\infty]$, $t>0$ *and for all* $n\geq 1$ *satisfying* $n(1-\frac{1}{p})<2\sigma$.

Proof 1 *In a first step we estimate the following oscillating integrals:*

$$F^{-1}_{\xi\to x}\big(e^{-c_1t|\xi|^{2\rho}}\cos(c_2t|\xi|^{2\rho})\big) \quad \textit{and} \quad F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big),$$

where $c_1 = -\cos(\frac{\pi}{1+\alpha}), c_2 = \sqrt{1-c_1^2}, \rho = \frac{\sigma}{1+\alpha}$ *and* $\tau > 0$. *We prove instead of (2.6) the polynomial type decay estimates*

$$\|F^{-1}_{\xi\to x}\big(e^{-c_1 t|\xi|^{2\rho}}\cos(c_2 t|\xi|^{2\rho})\big)\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}, \tag{2.7}$$

$$\|F^{-1}_{\xi\to x}\big(e^{-\tau t|\xi|^{2\rho}}\big)\|_{L^p} \lesssim (\tau t)^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{2.8}$$

for all $p \in [1, +\infty]$ *and* $t > 0$. *Then, we deduce*

$$\big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big) + \exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\big)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{2.9}$$

for all $p \in [1, +\infty]$ *and* $t > 0$. *It remains to prove that*

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\big\|_{L^p} \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \tag{2.10}$$

for all $p \in [1, +\infty]$ *and* $t > 0$. *Therefore we use the formula*

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma) \sim \int_0^\infty \frac{\exp\big(-t|\xi|^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big)}{s^2 + 2s\cos((1+\alpha)\pi) + 1}\, ds.$$

Taking account of the definition of modified Bessel functions we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\\
&\quad= \int_0^\infty \Big(\int_0^\infty \frac{\exp\big(-tr^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big)}{s^2 + 2s\cos((1+\alpha)\pi) + 1} r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)\, ds\\
&\quad= \int_0^\infty \frac{1}{s^2 + 2s\cos((1+\alpha)\pi) + 1}\Big(\int_0^\infty \exp\big(-tr^{\frac{2\sigma}{1+\alpha}} s^{\frac{1}{1+\alpha}}\big) r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)dr\Big)\, ds\\
&\quad= \int_0^\infty \frac{1}{s^2 + 2s\cos((1+\alpha)\pi) + 1}\Big(F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}} t|\xi|^{\frac{2\sigma}{1+\alpha}}}\big)(x)\Big)\, ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}} t|\xi|^{\frac{2\sigma}{1+\alpha}}}\big)(\cdot)\big\|_{L^p} \lesssim s^{-\frac{n}{2\sigma}(1-\frac{1}{p})} t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}.$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}|\xi|^\sigma)\big)\big\|_{L^p}\\
&\qquad\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})} \int_0^\infty \frac{s^{-\frac{n}{2\sigma}(1-\frac{1}{p})}}{s^2 + 2s\cos((1+\alpha)\pi) + 1}\, ds \lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(1-\frac{1}{p})}
\end{aligned}$$

if $n(1-\frac{1}{p}) < 2\sigma$.

Now let us turn to L^p estimates for the model oscillating integral

$$F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big).$$

At the first glance one might expect an exponential type decay estimate. We are able to prove a potential type decay estimate only.

Proposition 17 *The following estimate holds in $\mathbb{R}^n$ for $\sigma > 0$, $\alpha \geq 0$, $m > 0$ and for all $n \geq 1$:*

$$\|F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)\|_{L^p} \lesssim (1+t)^{-(1+\alpha)} \tag{2.11}$$

for $p \in [1,\infty]$ and $t \geq 0$.

Proof 2 *In a first step we estimate the following oscillating integrals:*

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big) \quad \textit{and} \quad F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big),$$

*where $c = -\cos(\frac{\pi}{1+\alpha}), \kappa = \frac{1}{1+\alpha} \in (\frac{1}{2},1)$ and $\tau > 0$. We prove instead of (**??**) the exponential type decay estimate*

$$\big\|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)\big\|_{L^p} + \big\|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)\big\|_{L^p} \lesssim e^{-Ct} \tag{2.12}$$

with a suitable positive $C = C(m,\alpha)$, for $p \in [1,\infty]$ and $t \geq 0$. By using modified Bessel functions we have for $n = 3$

$$\begin{aligned}
F^{-1}_{\xi\to x}\Big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\Big)(x) &= \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^2\tilde{J}_{\frac{1}{2}}(r|x|)\,dr \\
&= -\frac{1}{|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr \\
&= -\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r\partial_r(\cos(r|x|))\,dr.
\end{aligned}$$

Using twice integration by parts we obtain

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x) \\
&= -\tfrac{\sqrt{2}}{\sqrt{\pi}|x|^4}\int_0^\infty \Big(t\big(h_1(r)r^{2\sigma-2}\langle r\rangle^4_{m,\sigma} + h_2(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma} + h_3(r)r^{6\sigma-2}\big)\langle r\rangle^{2\kappa-6}_{m,\sigma} \\
&\quad + t^2\big(h_4(r)r^{4\sigma-2}\langle r\rangle^2_{m,\sigma} + h_5(r)r^{6\sigma-2}\big)\langle r\rangle^{4\kappa-6}_{m,\sigma} + t^3h_6(r)r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big) \\
&\quad \times e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr,
\end{aligned} \tag{2.13}$$

where $h_i(r) = a_i\cos(g(r)) + b_i\sin(g(r))$, $i = 1,\cdots,6$, $g(r) = t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}$ and a_i, b_i, $i = 1,\cdots,6$, are constants which depend on α and σ only.
To estimate (2.13) we use the inequality

$$\langle r\rangle^{2\kappa}_{m,\sigma} \geq 2^{\kappa-1}\langle r\rangle^{2\kappa}_{2^{-1/2}m,\sigma} + 2^{-1}m^{2\kappa}. \tag{2.14}$$

Then, we get

$$\big|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x)\big| \lesssim \frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.$$

For the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)$ *we have*

$$\begin{aligned}F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(x) &= \int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r^2 \tilde{J}_{\frac{1}{2}}(r|x|)\,dr \\ &= -\frac{1}{|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r\partial_r \tilde{J}_{-\frac{1}{2}}(r|x|)\,dr = -\frac{\sqrt{2}}{\sqrt{\pi}|x|^2}\int_0^\infty e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}} r\partial_r(\cos(r|x|))\,dr.\end{aligned}$$

Using twice integration by parts we obtain

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(x) \\ &\quad= -\frac{\sqrt{2}}{\sqrt{\pi}|x|^4}\int_0^\infty \Big(-2\sigma(4\sigma^2-1)\kappa\tau t r^{2\sigma-2}\langle r\rangle^{2\kappa-2}_{m,\sigma} - 24\sigma^3\kappa(\kappa-1)\tau t r^{4\sigma-2}\langle r\rangle^{2\kappa-4}_{m,\sigma} \\ &\qquad - 8\sigma^3\kappa(\kappa-1)(\kappa-2)\tau t r^{6\sigma-2}\langle r\rangle^{2\kappa-6}_{m,\sigma} + 24\sigma^3\kappa^2\tau^2 t^2 r^{4\sigma-2}\langle r\rangle^{4\kappa-4}_{m,\sigma} \\ &\qquad + 8\sigma^3\kappa^2(\kappa-1)\tau^2 t^2 r^{6\sigma-2}\langle r\rangle^{4\kappa-6}_{m,\sigma} - 8\sigma^3\kappa^3\tau^3 t^3 r^{6\sigma-2}\langle r\rangle^{6\kappa-6}_{m,\sigma}\Big) e^{-\tau t\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(r|x|)\,dr.\end{aligned}$$

This leads to the estimate

$$\big|F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(x)\big| \lesssim \frac{e^{-\frac{\tau}{2}tm^{2\kappa}}}{\langle x\rangle^4_m}.$$

Summarizing all estimates we proved the statement (2.12) in the case $n=3$.
Now, let us study the case n *odd and* $n\geq 4$. *Then we carry out* $\frac{n+1}{2}$ *steps of partial integration. We obtain after* $\frac{n-1}{2}$ *steps and by applying the rules*

$$\tilde{J}_{\mu+1}(r|x|) = -\frac{1}{r|x|^2}\partial_r\tilde{J}_\mu(r|x|)$$

for real non-negative μ *the relation*

$$\begin{aligned}&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x) = \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr \\ &\quad= (-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\tilde{J}_{-\frac{1}{2}}(r|x|)\,dr \\ &\quad= (-1)^{\frac{n-1}{2}}\frac{1}{|x|^{n-1}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\cos(r|x|)\,dr \\ &\quad= (-1)^{\frac{n+1}{2}}\frac{1}{|x|^{n+1}}\sqrt{\frac{2}{\pi}}\int_0^\infty \Big(\frac{\partial^2}{\partial r^2}\Big)\Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-1}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\cos(r|x|)\,dr.\end{aligned}$$

All integrals have the form

$$\begin{aligned}&\int_0^\infty \langle r\rangle^\rho_{m,\sigma} r^\delta e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\cos(r|x|)\,dr \\ &\qquad \text{or} \int_0^\infty \langle r\rangle^\rho_{m,\sigma} r^\delta e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\sin(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\cos(r|x|)\,dr,\end{aligned}$$

where ρ is a negative integer depending on κ and n and δ is a non negative real depending on σ and n. For this reason we conclude the estimate

$$\big|F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x)\big| \lesssim \frac{e^{-\frac{c}{2}tm^{2\kappa}}}{\langle x\rangle^{n+1}_m}.$$

In the same way we obtain the same estimate for $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(x)$. All together implies the statement (2.12) for odd $n \geq 4$.
For $n = 2$ we have

$$F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x) = \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}) r\tilde{J}_0(r|x|)\,dr.$$

From the relation $J_0(s) = \frac{1}{s}J_1(s) + \frac{d}{ds}J_1(s)$ it follows that

$$\tilde{J}_0(r|x|) = 2\tilde{J}_1(r|x|) + r\partial_r\tilde{J}_1(r|x|) = \frac{1}{r}\partial_r(r^2\tilde{J}_1(r|x|)).$$

Then, we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x)\\
&= -\int_0^\infty 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\tilde{J}_1(r|x|)\,dr\\
&= -\int_0^{\frac{1}{|x|}} 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\tilde{J}_1(r|x|)\,dr\\
&\quad -\int_{\frac{1}{|x|}}^\infty 2\kappa\sigma t r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\tilde{J}_1(r|x|)\,dr.
\end{aligned}$$

Using the boundedness of $\tilde{J}_1(s)$ for $s \in [0,1]$ the first integral can be estimated by

$$e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(4\sigma+2\kappa-2)}.$$

Remark that $4\sigma+2\kappa-2 > 2$. To estimate the second integral we apply the following asymptotic formula for $\tilde{J}_1(s)$ for $s \geq 1$:

$$\tilde{J}_1(s) = cs^{-\frac{3}{2}}\cos\Big(s-\frac{3}{4}\pi\Big) + O(|s|^{-\frac{5}{2}}).$$

Consequently, the integral can be estimated as follows:

$$\int_{\frac{1}{|x|}}^\infty r^{4\sigma-1}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}O((r|x|)^{-\frac{5}{2}})\,dr \lesssim |x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}.$$

It remains to estimate

$$\begin{aligned}
&\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr,\\
&\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^\infty r^{\frac{8\sigma-5}{2}}\langle r\rangle^{2\kappa-2}_{m,\sigma}e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos\Big(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\sin(r|x|)\,dr.
\end{aligned}$$

We explain only the first integral because the second one can be treated in the same way. We write the first integral as follows:

$$\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr$$
$$=\frac{1}{|x|^{\frac{3}{2}}}\int_{\frac{1}{|x|}}^{1} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr \qquad (2.15)$$
$$+\frac{1}{|x|^{\frac{3}{2}}}\int_{1}^{\infty} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\cos(r|x|)\,dr. \qquad (2.16)$$

The integral in (2.15) is equal to

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1} r^{\frac{8\sigma-5}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\partial_r(\sin(r|x|))\,dr. \qquad (2.17)$$

After partial integration and by using (4) the limit terms can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$. The new integral is equal to

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{\frac{1}{|x|}}^{1}\Big(c_1 r^{\frac{8\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+c_2 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+c_3 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\sin\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)$$
$$+c_4 r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{4\kappa-4}\cos\Big(t\langle r\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2}+\frac{\pi}{1+\alpha}\Big)\Big)e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\sin(r|x|)\,dr.$$

It can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$, too. After integration by parts the integral in (2.16) can be estimated by

$$\frac{1}{|x|^{\frac{5}{2}}}\int_{1}^{\infty} r^{\frac{12\sigma-7}{2}}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-ct\langle r\rangle_{m,\sigma}^{2\kappa}}\,dr.$$

The latter integral can be estimated by $|x|^{-\frac{5}{2}}e^{-\frac{c}{2}tm^{2\kappa}}$. Finally, we have for the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle_{m,\sigma}^{2\kappa}}\big)$

$$F^{-1}(e^{-\tau t\langle\xi\rangle_{m,\sigma}^{2\kappa}})(x)=\int_0^{\infty}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}r\tilde{J}_0(r|x|)\,dr$$
$$=\int_0^{\infty}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}\partial_r(r^2\tilde{J}_1(r|x|))\,dr=\int_0^{\infty}2\sigma\kappa\tau t r^{2\sigma+1}\langle r\rangle_{m,\sigma}^{2\kappa-2}e^{-\tau t\langle r\rangle_{m,\sigma}^{2\kappa}}\tilde{J}_1(r|x|)\,dr.$$

Then, we derive the same estimates as we did before for estimating the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle_{m,\sigma}^{2\kappa}}\cos(t\langle\xi\rangle_{m,\sigma}^{2\kappa}\sqrt{1-c^2})\big)(x)$. Summarizing all estimates yields the statement (2.12) for $n=2$.

Now for the case of even $n \geq 4$ we carry out $\frac{n}{2}-1$ steps of partial integration. In this way we obtain

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(e^{-ct\langle\xi\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle\xi\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})\big)(x)\\
&= \int_0^\infty e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\tilde{J}_0(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n-2}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)\frac{1}{r}\partial_r(r^2\tilde{J}_1(r|x|))\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\\
&= \frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}} \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr \qquad (2.18)\\
&\quad+\frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr. \qquad (2.19)
\end{aligned}$$

For the integral in (4.15) we are able to derive the following estimate:

$$\begin{aligned}
&\Big|\frac{1}{|x|^{n-2}}\int_0^{\frac{1}{|x|}} \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|\\
&\quad\lesssim \frac{1}{|x|^{n-2}}e^{-\frac{c}{2}tm^{2\kappa}}\int_0^{\frac{1}{|x|}} r^{2\sigma+1}\langle r\rangle^{2\kappa-2}_{m,\sigma}\,dr \lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+2\kappa+2\sigma-2)}.
\end{aligned}$$

For the integral in (4.17) we follow the same arguments to obtain the following estimate:

$$\begin{aligned}
&\Big|\frac{1}{|x|^{n-2}}\int_{\frac{1}{|x|}}^\infty \Big(\frac{\partial}{\partial r}\frac{1}{r}\Big)^{\frac{n}{2}}\Big(e^{-ct\langle r\rangle^{2\kappa}_{m,\sigma}}\cos(t\langle r\rangle^{2\kappa}_{m,\sigma}\sqrt{1-c^2})r^{n-1}\Big)r^2\tilde{J}_1(r|x|)\,dr\Big|\\
&\quad\lesssim e^{-\frac{c}{2}tm^{2\kappa}}\langle x\rangle_m^{-(n+\frac{1}{2})}.
\end{aligned}$$

In the same way we can estimate the oscillating integral $F^{-1}_{\xi\to x}\big(e^{-\tau t\langle\xi\rangle^{2\kappa}_{m,\sigma}}\big)(x)$. All together implies the statement (2.12) for even $n \geq 4$. To complete the proof it remains to show

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p} \lesssim (1+t)^{-(1+\alpha)}$$

for $p \in [1,\infty]$ and $t \geq 0$. Therefore we use the formula

$$l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma}) \sim \int_0^\infty \frac{\exp\big(-t\langle\xi\rangle^{\frac{2}{1+\alpha}}_{m,\sigma}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.$$

Taking account of the definition of modified Bessel functions we get

$$\begin{aligned}
&F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\\
&\quad=\int_0^\infty\Big(\int_0^\infty\frac{\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)}{s^2+2s\cos((1+\alpha)\pi)+1}r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&\quad=\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\Big(\int_0^\infty\exp\big(-t\langle r\rangle_m^{\frac{2}{1+\alpha}}s^{\frac{1}{1+\alpha}}\big)r^{n-1}\tilde{J}_{\frac{n}{2}-1}(r|x|)\,dr\Big)\,ds\\
&\quad=\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\Big(F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(x)\Big)\,ds.
\end{aligned}$$

The estimate

$$\big\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(\cdot)\big\|_{L^p}\lesssim e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}$$

implies

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\\
&\quad\lesssim\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\|F^{-1}_{\xi\to x}\big(e^{-s^{\frac{1}{1+\alpha}}t\langle\xi\rangle_{m,\sigma}^{\frac{2}{1+\alpha}}}\big)(\cdot)\|_{L^p}\,ds\\
&\quad\lesssim\int_0^\infty\frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds.
\end{aligned}$$

For $t\in(0,1]$ we may conclude

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\lesssim\int_0^\infty\frac{1}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds\lesssim 1.$$

For $t\geq 1$ we have

$$\begin{aligned}
\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}&\lesssim\int_0^\infty\frac{e^{-\frac{1}{2}s^{\frac{1}{1+\alpha}}tm^{\frac{2}{1+\alpha}}}}{s^2+2s\cos((1+\alpha)\pi)+1}\,ds\\
&\lesssim\int_0^\infty\exp\big(-\tilde{C}_1ts^{\frac{1}{1+\alpha}}\big)\,ds.
\end{aligned}$$

After the change of variables $\tau:=ts^{\frac{1}{1+\alpha}}$ it follows

$$\begin{aligned}
\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}&\lesssim\int_0^\infty\exp\big(-\tilde{C}_1ts^{\frac{1}{1+\alpha}}\big)ds\\
&\lesssim t^{-(1+\alpha)}\int_0^\infty\tau^\alpha\exp\big(-\tilde{C}_1\tau\big)d\tau\lesssim t^{-(1+\alpha)}.
\end{aligned}$$

We deduce for all $p\in[1,\infty]$ the estimate

$$\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\lesssim(1+t)^{-(1+\alpha)}\quad\textit{for all}\quad t\geq 0.$$

Summarizing all the estimates we may conclude

$$\begin{aligned}
&\big\|F^{-1}_{\xi\to x}\big(E_{1+\alpha}(-t^{1+\alpha}\langle\xi\rangle^2_{m,\sigma})\big)\big\|_{L^p}\\
&\quad\lesssim \big\|F^{-1}_{\xi\to x}\big(\exp\big(a_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}+\big\|F^{-1}_{\xi\to x}\big(\exp\big(b_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\\
&\qquad+\big\|F^{-1}_{\xi\to x}\big(l_{1+\alpha}(t^{\frac{1+\alpha}{2}}\langle\xi\rangle_{m,\sigma})\big)\big\|_{L^p}\\
&\quad\lesssim e^{-Ct}+(1+t)^{-(1+\alpha)}\lesssim(1+t)^{-(1+\alpha)}.
\end{aligned}$$

This completes the proof.

2.3 $L^m - L^q$ estimates for solutions to linear Cauchy problems

2.3.1 Linear Cauchy problems without any mass term

Proposition 18 *Let $u_0 \in L^m(\mathbb{R}^n)$, $n \geq 1$, $m \geq 1$ and $\alpha \in (0,1)$. Then the solution of the linear Cauchy problem*

$$\begin{aligned}
&\partial_t^{1+\alpha}u+(-\Delta)^\sigma u=0,\\
&u(x,0)=u_0(x),\quad u_t(0,x)=0
\end{aligned}\tag{2.20}$$

satisfies the following $L^m - L^q$ estimates:

$$\|u(t,\cdot)\|_{L^q}\lesssim t^{-\frac{n(1+\alpha)}{2\sigma}(\frac{1}{m}-\frac{1}{q})}\|u_0\|_{L^m}\tag{2.21}$$

for all $1 \leq m \leq q \leq \infty$ provided that $n(\frac{1}{m}-\frac{1}{q}) < 2\sigma$.

Proof 3 *The inequality (2.21) follows from Young's inequality and Proposition 2.2.1.*

Proposition 19 *Let $u_0 \in L^r(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ and $\alpha \in (0,1)$. Then the solution of the linear Cauchy problem*

$$\begin{aligned}
&\partial_t^{1+\alpha}u+(-\Delta)^\sigma u=0,\\
&u(x,0)=u_0(x),\quad u_t(0,x)=0
\end{aligned}\tag{2.22}$$

satisfies the following estimate for any fixed $\delta > 0$ small:

$$\|u(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\beta^{r,\delta}_{\alpha,q,\sigma}}(\|u_0\|_{L^r}+\|u_0\|_{L^q})\quad\text{for all}\quad q\in[r,\infty],\tag{2.23}$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma}:=\min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big);1-\delta\Big\}.$$

Proof 4 *For $t \in (0,1]$ we set $m = q$ in (2.21) to get the $L^q - L^q$ estimate*

$$\|u(t,\cdot)\|_{L^q} \lesssim \|u_0\|_{L^q}.$$

For $t \geq 1$ we choose $m = r$ in (2.21) if $n(\frac{1}{r} - \frac{1}{q}) < 2\sigma$. Otherwise, in (2.21) the parameter m is chosen as the solution to

$$\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{m} - \frac{1}{q}\Big) = 1 - \delta$$

with a fixed sufficiently small positive δ. In this way, we may conclude the $L^r - L^q$ estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\beta^{r,\delta}_{\alpha,q,\sigma}} \|u_0\|_{L^r}.$$

Gluing both estimates together we derive the desired estimate (2.23).

2.3.2 Linear Cauchy problems with a mass term

Proposition 2.3.1 *Let $w_0 \in L^r(\mathbb{R}^n)$, $n \geq 1$, $r \geq 1$ and $\alpha \in (0,1)$. Then the solution of the linear Cauchy problem*

$$\begin{aligned} &\partial_t^{1+\alpha} w + (-\Delta)^\sigma w + m^2 w = 0, \\ &w(x,0) = w_0(x), \quad w_t(0,x) = 0 \end{aligned} \tag{2.24}$$

satisfies the following $L^r - L^q$ estimates:

$$\|w(t,\cdot)\|_{L^q} \lesssim (1+t)^{-(1+\alpha)} \|w_0\|_{L^r} \tag{2.25}$$

for all $1 \leq r \leq q \leq \infty$.

Proof: The estimate (2.25) is a direct consequence of the formula (2.11) and Young's inequality.

CHAPTER 3

WEAKLY COUPLED SYSTEMS OF $k-$ SEMI-LINEAR FRACTIONAL σ-EVOLUTION EQUATIONS WITH MASS AND DIFFERENT NONLINEAR MEMORY TERMS

This chapter is devoted to the the weakly coupled system of k semi-linear fractional $\sigma-$evolution equations with mass and different memory terms. We are interested to the global existence of small data solutions to the following Cauchy problem, namely

$$\begin{cases} \partial_t^{1+\alpha_1} u_1 + (-\Delta)^{\sigma_1} u_\ell + m_1^2 u = F_{\mu_1,p_1}(u_k), \\ \partial_t^{1+\alpha_2} u_2 + (-\Delta)^{\sigma_2} u_2 + m_2^2 u_2 = F_{\mu_2,p_2}(u_1), \\ \quad\quad\quad . \\ \quad\quad\quad . \\ \partial_t^{1+\alpha_k} u_k + (-\Delta)^{\sigma_k} u_k + m_k^2 u_k = F_{\mu_k,p_k}(u_{k-1}) \\ u_\ell(0,x) = u_{0l}(x),\ \partial_t u_\ell(0,x) = 0, \quad \ell = 1,2,...,k \end{cases} \tag{3.1}$$

where $k \geq 2$, for $l = 1, \cdots, k$, $\alpha_l \in (0,1)$, $\mu_l \in (0,1)$, $p_l > 1$, $m_l > 0$, $\sigma_l \geq 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$,

$$F_{\mu,p}(w)(t,x) := \int\limits_0^t (t-s)^{-\mu} |w(s,x)|^p ds, \tag{3.2}$$

$\partial_t^{1+\alpha_l} u = D_t^{\alpha_l}(u_t)$ with

$$D_t^{\alpha_l}(f) = \partial_t (I_t^{1-\alpha_l} f) \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s)\, ds \ \text{ for } \ \beta > 0.$$

$D_t^\alpha(f)$ and $I_t^\beta f$ are defined as above.

Our main results on global (in time) existence of small data Sobolev solutions are given in the next section.

3.1 Main results

Theorem 1 *(No loss of decay) (No loss of decay) Let us assume* $0 < \alpha_\ell < 1$, $\alpha_\ell < \mu_\ell < 1$, $\sigma_\ell \geq 1$, $r_\ell \geq 1$ *and* $m_\ell > 0$ *for all* $\ell = 1, ..., k$. *Assume that for all* $\epsilon > 0$

$$p_1 > \max\Big\{\frac{r_k}{r_1} - \epsilon, \frac{1}{\mu_k - \alpha_k}\Big\},$$
$$p_\ell > \max\Big\{\frac{r_{\ell-1}}{r_\ell} - \epsilon, \frac{1}{\mu_{\ell-1} - \alpha_{\ell-1}}\Big\}, \quad \textit{for all} \quad \ell = 2, ..., k-1.$$

Then there exists a positive constant ε *such that for any data* $(u_{01}, .., u_{0k}) \in \mathcal{A}_k := \prod_{\ell=1}^{k} L^{r_\ell}(\mathbb{R}^n)$ *with* $\|(u_{01}, .., u_{0k})\|_{\mathcal{A}_k} \leq \varepsilon$, *we have a uniquely determined global (in time) Sobolev solution*

$$u \in \prod_{\ell=1}^{k} C\big([0,\infty), L^{r_\ell}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (3.1). *Moreover, the solution satisfies for all* $t \geq 0$ *and* $l = 1, ..., k$ *the decay estimates*

$$\|u_\ell(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_\ell - \mu_\ell} \|u_{0l}\|_{L^{r_\ell}} \quad \textit{for all} \quad q \in [r_\ell, \infty].$$

Theorem 2 *(Loss of decay) Let us assume* $0 < \alpha_\ell < 1$, $\alpha_\ell < \mu_\ell < 1$, $\sigma_\ell \geq 1$, $r_\ell \geq 1$ *and* $m_\ell > 0$ *for all* $\ell = 1, ..., k$. *Assume that for all* $\epsilon > 0$

$$\max\Big\{1, \frac{\alpha_1 - \mu_1 + 1}{\mu_k - \alpha_k}, \frac{r_k}{r_1} - \epsilon\Big\} < p_1 < \frac{1}{\mu_k - \alpha_k},$$

,

$$\max\Big\{1, \frac{\alpha_2 - \mu_2 + 1}{\mu_1 - \alpha_1 - \gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}, \frac{r_1}{r_2} - \epsilon\Big\} < p_2 < \frac{1}{\mu_1 - \alpha_1 - \gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}$$

and for $l = 3, \cdots, k-1$

$$p_\ell < \frac{1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma^{(\mu_k, ..., \mu_{\ell-2})}_{(\alpha_k, ..., \alpha_{\ell-2})}(p_1, ..., p_{\ell-1})}$$

$$p_l > \max\Big\{1, \frac{\alpha_\ell - \mu_\ell + 1}{\mu_{\ell-1} - \alpha_{\ell-1} - \gamma^{(\mu_k, ..., \mu_{\ell-2})}_{(\alpha_k, ..., \alpha_{\ell-2})}(p_1, ..., p_{\ell-1})}, \frac{r_{\ell-1}}{r_\ell} - \epsilon\Big\}$$

and

$$p_k > \max\Big\{\frac{r_k}{r_{k-1}} - \epsilon, \frac{1}{\mu_{k-1} - \alpha_{k-1} - \gamma^{(\mu_k, ..., \mu_{k-2})}_{(\alpha_k, ..., \alpha_{k-2})}(p_1, ..., p_{k-1})}\Big\},$$

where, for $l = 3, \cdots, k-1$

$$\begin{cases} \gamma^{(\mu_k)}_{(\alpha_k)}(p_1) = 1 - p_1(\mu_k - \alpha_k) \\ \gamma^{(\mu_k,\mu_1)}_{(\alpha_k,\alpha_1)}(p_1,p_2) = 1 - p_2(\mu_1 - \alpha_1) + p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1) \\ \gamma^{(\mu_k,\ldots,\mu_{\ell-1})}_{(\alpha_k,\ldots,\alpha_{\ell-1})}(p_1,...,p_\ell) = 1 - p_\ell(\mu_\ell - \alpha_\ell) + p_\ell\gamma^{(\mu_k,\ldots,\mu_{\ell-2})}_{(\alpha_k,\ldots,\alpha_{\ell-2})}(p_1,...,p_{\ell-1}). \end{cases} \tag{3.3}$$

Then there exists a positive constant ε *such that for any data* $(u_{01},..,u_{0k}) \in \mathcal{A}_k := \prod_{\ell=1}^k L^{r_\ell}(\mathbb{R}^n)$ *with* $\|(u_{01},..,u_{0k})\|_{\mathcal{A}_k} \le \varepsilon$*, we have a uniquely determined global (in time) Sobolev solution*

$$u \in \prod_{\ell=1}^{k} C\big([0,\infty), L^{r_\ell}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (3.1)*. Moreover, the solution satisfies for all* $t \ge 0$ *and* $l = 2, \cdots, k-1$*, the decay estimate*

$$\begin{aligned} &\|u_1(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_1-\mu_1+\gamma^{\mu_k}_{\alpha_k}(p_1)}\|u_{01}\|_{L^{r_1}} \quad \textit{for all} \quad q \in [r_1,\infty], \\ &\|u_\ell(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_\ell-\mu_\ell+\gamma^{(\mu_k,\ldots,\mu_{\ell-1})}_{(\alpha_k,\ldots,\alpha_{\ell-1})}(p_1,\ldots,p_\ell)}\|u_{0\ell}\|_{L^{r_\ell}} \quad \textit{for all} \quad q \in [r_\ell,\infty], \\ &\|u_k(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_k-\mu_k}\|u_{0k}\|_{L^{r_k}} \quad \textit{for all} \quad q \in [r_k,\infty]. \end{aligned}$$

We suppose $r_1 = r_2 = 1$ in the following result.

Theorem 3 *(Loss of decay) Let us assume* $0 < \alpha_\ell < 1$*,* $\alpha_\ell < \mu_\ell < 1$*,* $\sigma_\ell \ge 1$*,* $m_\ell > 0$ *for all* $\ell = 1, ..., k$*. Assume that for all* $\epsilon > 0$

$$p_1 = \frac{1}{\mu_k - \alpha_k}.$$

$$p_\ell = \frac{1}{\mu_{\ell-1} - \alpha_{\ell-1}}. \quad \ell = 2,...,k-1$$

$$p_k > \frac{1}{\mu_{k-1} - \alpha_{k-1} - \epsilon\gamma(p_{k-1})},$$

where

$$\begin{cases} \gamma(p_1) = 1 \\ \gamma(p_l) = 1 + p_l\gamma(p_{l-1}), \textit{ for } l = 2, \cdots, k-1. \end{cases} \tag{3.4}$$

Then there exists a positive constant ε *such that for any data*

$$(u_{01},..,u_{0k}) \in \mathcal{A}_k = \prod_{\ell=1}^{k} L^1(\mathbb{R}^n) \quad \textit{with} \quad \|(u_{01},..,u_{0k})\|_{\mathcal{A}_k} \le \varepsilon$$

we have a uniquely determined global (in time) Sobolev solution

$$u \in \prod_{\ell=1}^{k} C\big([0,\infty), L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (3.1). *Moreover, the solution satisfies, for* $l = 1, \cdots, k-1$ *and for all* $t \geq 0$*, the decay estimate*

$$\|u_\ell(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_\ell - \mu_\ell}\big(\ln(e+t)\big)^{\gamma(p_l)}\|u_{0l}\|_{L^1} \quad \textit{for all} \quad q \in [1,\infty],$$
$$\|u_k(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_k - \mu_k}\|u_{0k}\|_{L^1} \quad \textit{for all} \quad q \in [1,\infty].$$

Remark 1 *The nonlinear term* $F_{\mu,p}(t,w)$ *in (3.2) may be written as*

$$F_{\mu,p}(t,w) = \Gamma(1-\mu) I_t^{1-\mu}(|w|^p)$$

where Γ *is the Euler Gamma function, and* $I_t^{1-\mu}(|w|^p)$ *is the fractional Riemann-Liouville integral of* $|w|^p$ *in* $[0,t]$*. Therefore, it is reasonable to expect relations with the case of a powers nonlinearities introduced in [11] as* μ_l *tends to* 1*, for all* $l = 1, \cdots, k$ *and* $k = 2$*.*

3.2 Treatment of the weakly coupled linear systems

We will apply the decay estimates for solutions to

$$\begin{cases} \partial_t^{1+\alpha_1} u_1 + (-\Delta)^{\sigma_1} u_1 + m_1^2 u = 0, \\ \partial_t^{1+\alpha_2} u_2 + (-\Delta)^{\sigma_2} u_2 + m_2^2 u_2 = 0, \\ \qquad\qquad . \\ \qquad\qquad . \\ \partial_t^{1+\alpha_k} u_k + (-\Delta)^{\sigma_k} u_k + m_k^2 u_k = 0, \\ u_\ell(0,x) = u_{\ell,0}(x),\ \partial_t u_\ell(0,x) = 0, \quad \ell = 1,2,...,k \end{cases} \tag{3.5}$$

to prove the global (in time) existence of small data Sobolev solutions to the weakly coupled systems of semi-linear models (4.14). We write their solutions in the following form:

$$u_l^{ln}(t,x) := G_{\alpha_l,\sigma_l}^{m_l}(t,x) *_{(x)} u_{0l}(x), \text{ for all } l = 1, \cdots, k. \tag{3.6}$$

Proposition 20 *Let* $u_{0l} \in L^{r_l}$ *with* $r_l \geq 1$ *for all* $l = 1, \cdots, k$ *and . Then the solution of the linear Cauchy problem* (4.14) *satisfies the following* $L^{r_l} - L^q$ *estimates :*

$$\big\|u_l^{ln}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-(1+\alpha_l)}\|u_{0l}\|_{L^{r_l}} \textit{ for all } q \in [r_l,\infty].$$

Applying Duhamel's principle gives the formal integral formulation of solutions to (4.1) as follows

$$\begin{aligned} u_1(t,x) &:= u_1^{ln}(t,x) + \textstyle\int_0^t G_{\alpha_1,\sigma_1}^{m_1}(t-\tau,\cdot) *_{(x)} F_{\mu_1,p_1}(u_k)d\tau = (u_1^{ln} + u_1^{nl})(t,x), \\ u_l(t,x) &:= u_l^{ln}(t,x) + \textstyle\int_0^t G_{\alpha_l,\sigma_l}^{m_l}(t-\tau,\cdot) *_{(x)} F_{\mu_l,p_l}(u_{l-1})d\tau = (u_l^{ln} + u_l^{nl})(t,x). \end{aligned} \tag{3.7}$$

for all $l = 2, \cdots, k$.

3.3 Proof of main results

Before showing our results we recall the following lemma from [2].

Lemma 2 *Suppose that $\theta \in [0,1), a \geq 0$ and $b \geq 0$. Then there exists a constant $C = C(a,b,\theta) > 0$ such that for all $t > 0$ the following estimate holds:*

$$\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau \leq \begin{cases} C(1+t)^{-\min\{a+\theta,b\}} & if \quad \max\{a+\theta,b\} > 1, \\ C(1+t)^{-\min\{a+\theta,b\}}\ln(e+t) & if \quad \max\{a+\theta,b\} = 1, \\ C(1+t)^{1-a-\theta-b} & if \quad \max\{a+\theta,b\} < 1. \end{cases} \tag{3.8}$$

3.3.1 Proof of Theorem 4.2

we introduce for all $T > 0$ the following spaces $X^k(T)$:

$$X^k(T) := \prod_{\ell=1}^{k} C\big([0,T], L^{r_\ell}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

endowed with the corresponding norms as follows:

$$\|u\|_{X^k(T)} := \|(u_1,u_2,...,u_k)\|_{X^k(T)} := \sup_{0\leqslant t\leqslant T}\Big\{\sum_{\ell=1}^{k} M_\ell(t,u_l),\Big\},$$

where

$$M_\ell(t,u_l) = 1+t)^{\mu_\ell-\alpha_\ell}\big(\|u_\ell(t,\cdot)\|_{L^{r_\ell}} + \|u_\ell(t,\cdot)\|_{L^\infty},$$

and the operator N by

$$N \; : \; u = (u_1,u_2,...,u_k) \in X^k(T) \to N(u) = N(u)(t,x) := u^{ln}(t,x) + u^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X^k(T)$. We will show that the operator N fulfills the following two inequalities:

$$\|N(u)\|_{X^k(T)} \lesssim \|(u_{01},u_{02},...u_{0k})\|_{\mathcal{A}_k} + \sum_{\ell=1}^{\ell=k}\|u\|^{p_\ell}_{X^k(T)}, \tag{3.9}$$

$$\|N(u)-N(\overline{u})\|_{X^k(T)} \lesssim \|u-\overline{u}\|_{X^k(T)}\sum_{\ell=1}^{\ell=k}\Big(\|u\|^{p_\ell-1}_{X^k(T)} + \|\overline{u}\|^{p_l-1}_{X^k(T)}\Big) \tag{3.10}$$

Then, we can conclude global existence results of small data solution by applying Banach's fixed point theorem. At first,Using the definition of the norm in $X^k(T)$ and the Propositions 21,we heve:

$$\|u^{ln}\|_{X^k(T)} \lesssim \|(u_{01},u_{02},...,u_{0k})\|_{\mathcal{A}_k}.$$

Hence, we continus to complete the proof of (4.19) it is reasonable to we shall show the following inequality

$$\|u^{nl}\|_{X^k(T)} \lesssim \sum_{\ell=1}^{\ell=k} \|u\|^{p_\ell}_{X^k(T)}.$$

If $u := (u_1, u_2, \cdots, u_k) \in X^k(T)$, at the first stage by interpolation we derive for $l = 1, \cdots, k$

$$\|u_\ell(t,\cdot)\|_{L^q} \lesssim (1+t)^{(\alpha_\ell-\mu_\ell)}\|u\|_{X^k(T)} \quad \text{for all} \quad q \in [r_\ell, \infty].$$

On the other hand, we get

$$\begin{aligned}\|u_1^{nl}(t,\cdot)\Big\|_{L^q} &\lesssim \Big\| \int_0^t G^{m_1}_{\alpha_1,\sigma_1}(t-\tau,\cdot) *_{(x)} I^{\alpha_1}_s(F_{\mu_1,p_1}(u_k))d\tau\|_{L^q} \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} \int_0^s (s-\rho)^{-\mu_1}\|u_k(\rho,\cdot)\|^{p_1}_{L^{p_1 q}}\, d\rho\, ds\, d\tau \\ &\lesssim \|u\|^{p_1}_{X^k(T)} I_1(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_1 q \in [r_k,\infty],\end{aligned}$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} \int_0^s (s-\rho)^{-\mu_1}(1+\rho)^{-p_1(\mu_k-\alpha_k)}\, d\rho\, ds\, d\tau. \tag{3.11}$$

We are interested to estimate the right-hand side of (5.9). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_1}(1+\rho)^{-p_1(\mu_k-\alpha_k)}\, d\rho.$$

Thanks to Lemma 2 we obtain $\omega(s) \lesssim (1+s)^{-\mu_1}$, if we assume that $p_1 > \frac{1}{\mu_k-\alpha_k}$. On other hands, the conditions $q \in [r_1,\infty]$ and $p_1 q \in [r_k,\infty]$ implies that $p_1 \geq \frac{r_k}{r_1}$.
Once more we apply Lemma 2 to obtain

$$\begin{aligned} I_1(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-\mu_1}\, ds\, d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)}(1+\tau)^{\alpha_1-\mu_1}\, d\tau \\ &\lesssim (1+t)^{\alpha_1-\mu_1}.\end{aligned}$$

For $l = 2, \cdots, k$ and $q \in [r_l, \infty]$ In the same manner, one also concludes that,

$$\|u_\ell^{nl}(t,\cdot)\|_{L^q} \lesssim \|u\|^{p_\ell}_{X^k(T)} I_\ell(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_\ell q \in [r_{\ell-1},\infty],$$

Analogously, we arrive at

$$I_\ell(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \int_0^s (s-\rho)^{-\mu_\ell}(1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})}\, d\rho\, ds\, d\tau. \tag{3.12}$$

On the other hand, we have We are interested to estimate the right-hand side of (3.12). For this we need the Lemma 2. We put

$$\omega(s)=\int_0^s (s-\rho)^{-\mu_\ell}(1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})}\,d\rho.$$

Thanks to Lemma 2 we obtain $\omega(s)\lesssim(1+s)^{-\mu_\ell}$, if we assume that $p_\ell>\frac{1}{\mu_{\ell-1}-\alpha_{\ell-1}}$. On other hands, the conditions $q\in[r_\ell,\infty]$ and $p_\ell q\in[r_{\ell-1},\infty]$ implies that $p_\ell\geq\frac{r_\ell-1}{r_\ell}$.
Once more we apply Lemma 2 to obtain

$$\begin{aligned}
I_\ell(t)&\lesssim\int_0^t(1+t-\tau)^{-(1+\alpha_\ell)}\int_0^\tau(\tau-s)^{\alpha_\ell-1}(1+s)^{-\mu_\ell}\,ds\,d\tau\\
&\lesssim\int_0^t(1+t-\tau)^{-(1+\alpha_\ell)}(1+s)^{\alpha_\ell-\mu_\ell}\,d\tau\\
&\lesssim(1+t)^{\alpha_\ell-\mu_\ell}.
\end{aligned}$$

To prove (4.27) we assume that u and $\tilde{u}$ are two vector-function belonging to $X^k(T)$. Then, we get

$$\begin{aligned}
&N(u)-N(\tilde{u})\\
&=\Big(\int_0^t G^{m_1}_{\alpha_1,\sigma_1}(t-s)\star I^{\alpha_1}_s\Big(\int_0^s(s-\rho)^{-\mu_1}\Big(|u_k(\rho,\cdot)|^{p_1}-|\tilde{u_k}(\rho,\cdot)|^{p_1}\Big)\,d\rho\Big)(t,s,x)\,ds,\cdots,\\
&\int_0^t G^{m_k}_{\alpha_k,\sigma_k}(t-s)\star I^{\alpha_k}_s\Big(\int_0^s(s-\rho)^{-\mu_k}\Big(|u_{k-1}(\rho,\cdot)|^{p_k}-|\tilde{u_{k-1}}(\rho,\cdot)|^{p_k}\Big)\,d\rho\Big)(t,s,x)\,ds\Big).
\end{aligned}$$

We estimate, for $q\in[r_1,\infty]$

$$\begin{aligned}
&\Big\|\int_0^t G^{m_1}_{\alpha_1,\sigma_1}(t-s)\star I^{\alpha_1}_s\Big(\int_0^s(s-\rho)^{-\mu_1}\Big(|u_k(\rho,\cdot)|^{p_1}-|\tilde{u_k}(\rho,\cdot)|^{p_1}\Big)\,d\rho\Big)(t,s,\cdot)\,ds\Big\|_{L^q}\\
&\lesssim\int_0^t(1+t-\tau)^{-(1+\alpha_1)}\int_0^\tau(\tau-s)^{\alpha_1-1}\int_0^s(s-\rho)^{-\mu_1}\Big\||u_k(\rho,\cdot)|^{p_1}-|\tilde{u_k}(\rho,\cdot)|^{p_1}\Big\|_{L^q}d\rho\,ds d\tau.
\end{aligned}$$

Using Hölder's inequality implies the estimate

$$\begin{aligned}
&\Big\||u_k(s,\cdot)|^{p_1}-|\tilde{u_k}(s,\cdot)|^{p_1}\Big\|_{L^q}\\
&\qquad\lesssim\big\|u_k(s,\cdot)-\tilde{u_k}(s,\cdot)\big\|_{L^{qp_1}}\big(\|u_k(s,\cdot)\|^{p_1-1}_{L^{qp_1}}+\|\tilde{u_k}(s,\cdot)\|^{p_1-1}_{L^{qp_1}}\big).
\end{aligned}$$

By using the definition of the norm of the solution space $X^k(T)$ we obtain for $p_1\geq\frac{r_k}{r_1}$ and $0\leqslant s\leqslant t$ the following estimates:

$$\begin{aligned}
&\big\|u_k(s,\cdot)-\tilde{u_k}(s,\cdot)\big\|_{L^{qp_1}}\lesssim(1+s)^{\alpha_k-\mu_k}M_2(s,u_k-\tilde{u_k}),\\
&\big\|u_k(s,\cdot)\|^{p_1-1}_{L^{qp_1}}\lesssim(1+s)^{(p_1-1)(\alpha_k-\mu_k)}\|u\|^{p_1-1}_{X^k(T)},\\
&\big\|\tilde{u_k}(s,\cdot)\|^{p_1-1}_{L^{qp_1}}\lesssim(1+s)^{(p_1-1)(\alpha_k-\mu_k)}\|u\|^{p_1-1}_{X^k(T)}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}&\big\||u_k(s,\cdot)|^{p_1}-|\tilde{u_k}(s,\cdot)|^{p_1}\big\|_{L^q}\\&\qquad\lesssim (1+s)^{-p_1(\mu_k-\alpha_k)}M_k(s,u_k-\tilde{u_k})\Big(\|u\|^{p_1-1}_{X^k(T)}+\|\tilde{u}\|^{p_1-1}_{X^k(T)}\Big).\end{aligned}$$

By the same argument, we obtain for $p_\ell \geq \frac{r_{\ell-1}}{r_\ell}$ $l=2,...,k.$ and $0\leqslant s\leqslant t$ the following estimate

$$\begin{aligned}&\big\||u_\ell(s,\cdot)|^{p_\ell}-|\tilde{u_\ell}(s,\cdot)|^{p_\ell}\big\|_{L^q}\\&\lesssim (1+s)^{p_\ell(\alpha_\ell-\mu_\ell)}M_1(s,u_{\ell-1}-\tilde{u_{\ell-1}})\Big(\|(u)\|^{p_\ell-1}_{X^k(T)}+\|\tilde{u}\|^{p_\ell-1}_{X^k(T)}\Big).\end{aligned}$$

So, for $p_1>\frac{1}{\mu_k-\alpha_k}$ and $p_l>\frac{1}{\mu_{l-1}-\alpha_{l-1}}$ for all $l=2,\cdots,k$, we obtain the desired estimate (4.20).

Remark 2 *All estimates (4.19) and (4.20) are uniformly with respect to* $T\in(0,\infty)$ *if* $p_1>\max\{\frac{r_2}{r_1}-\epsilon;\frac{1}{\mu_2-\alpha_2}\}$ *and* $p_2>\max\{\frac{r_1}{r_2}-\epsilon;\frac{1}{\mu_1-\alpha_1}\}$*, for all* $\epsilon>0$.

From (4.19) it follows that N maps $X^k(T)$ into itself for all T and for small data. By standard contraction arguments, the estimates (4.19) and (4.20) lead to the existence of unique solution to $u=N(u)$ and, consequently, to (3.1), that is, the solution of (3.1) satisfies the desired decay estimate. Since all constants are independent of T, after letting T tend to ∞ we obtain a global (in time) existence result for small data solutions to (3.1). This completes the proof.

3.3.2 Proof of Theorem 4.4

We introduce for all $T>0$ the space $X^k(T)$ as follows:

$$X^k(T):=\prod_{\ell=1}^{k}C\big([0,T],L^{r_\ell}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\begin{aligned}\|u\|_{X^k(T)}:=\sup_{0\leqslant t\leqslant T}\Big\{&(1+t)^{-\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}M_1(t,u_1)+\sum_{l=2}^{k-1}(1+t)^{-\gamma^{(\mu_k,\dots,\mu_{\ell-1})}_{(\alpha_k,\dots,\alpha_{\ell-1})}(p_1,\dots,p_\ell)}M_\ell(t,u_l)\\&+M_k(t,u_k)\Big\},\end{aligned}$$

where

$$M_k(t,u_k) := (1+t)^{\mu_k-\alpha_k}\big(\|u_k(t,\cdot)\|_{L^{r_k}} + \|u_k(t,\cdot)\|_{L^\infty},$$
$$M_\ell(t,u_l) = (1+t)^{\mu_\ell-\alpha_\ell}\big(\|u_\ell(t,\cdot)\|_{L^{r_\ell}} + \|u_\ell(t,\cdot)\|_{L^\infty},$$

$$.\begin{cases} \gamma^{(\mu_k)}_{(\alpha_k)}(p_1) = 1-p_1(\mu_k-\alpha_k) \\ \gamma^{(\mu_k,\mu_1)}_{(\alpha_k,\alpha_1)}(p_1,p_2) = 1-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1) \\ \gamma^{(\mu_k,\mu_1,\mu_2)}_{(\alpha_k,\alpha_1,\alpha_2)}(p_1,p_2,p_3) = 1-p_3(\mu_2-\alpha_2)+p_3\gamma^{(\mu_k,\mu_1)}_{(\alpha_k,\alpha_1)}(p_1,p_2) \\ \quad . \\ \quad . \\ \quad . \\ \gamma^{(\mu_k,\ldots,\mu_{\ell-1})}_{(\alpha_k,\ldots,\alpha_{\ell-1})}(p_1,\ldots,p_\ell) = 1-p_\ell(\mu_{l-1}-\alpha_{l-1})+p_\ell\gamma^{(\mu_k,\ldots,\mu_{\ell-2})}_{(\alpha_k,\ldots,\alpha_{\ell-2})}(p_1,\ldots,p_{\ell-1}), \end{cases}$$

for $l=3,\cdots,k-1$. and the operator N by

$$N\ :\ u=(u_1,u_2,\ldots,u_k)\in X^k(T)\to N(u)=N(u)(t,x):=u^{ln}(t,x)+u^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (3.1) as a fixed point of the operator N. We will prove that the operator N satisfies for $u=(u_1,u_2,\ldots,u_k)$; $\overline{u}=(\overline{u}_1,\overline{u}_2,\ldots,\overline{u}_k)$ in $X^k(T)$, the following two inequalities:

$$\|N(u)\|_{X^k(T)} \lesssim \|(u_{01},u_{02},\ldots u_{0k})\|_{\mathcal{A}_k} + \sum_{\ell=1}^{\ell=k}\|u\|^{p_\ell}_{X^k(T)}, \tag{3.13}$$

$$\|N(u)-N(\overline{u})\|_{X^k(T)} \lesssim \|u-\overline{u}\|_{X^k(T)}\sum_{\ell=1}^{\ell=k}\Big(\|u\|^{p_\ell-1}_{X^k(T)} + \|\overline{u}\|^{p_l-1}_{X^k(T)}\Big) \tag{3.14}$$

Using the definition of the norm in $X^k(T)$ and the Propositions 21, we may conclude:

$$\|u^{ln}\|_{X^k(T)} \lesssim \|(u_{01},u_{02},\ldots,u_{0k})\|_{\mathcal{A}_k}.$$

Hence, in order to complete the proof of (4.19) it is reasonable to we shall show the following inequality

$$\|u^{nl}\|_{X^k(T)} \lesssim \sum_{\ell=1}^{\ell=k}\|u\|^{p_\ell}_{X^k(T)}.$$

If $u:=(u_1,u_2,\cdots,u_k)\in X^k(T)$, then by interpolation we derive for $l=2,\cdots,k-1$

$$\|u_1(t,\cdot)\|_{L^q} \lesssim (1+t)^{(\alpha_1-\mu_1)+\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}\|u\|_{X^k(T)} \quad \text{for all} \quad q\in[r_1,\infty]$$
$$\|u_\ell(t,\cdot)\|_{L^q} \lesssim (1+t)^{(\alpha_\ell-\mu_\ell)+\gamma^{(\mu_k,\ldots,\mu_{\ell-1})}_{(\alpha_k,\ldots,\alpha_{\ell-1})}(p_1,\ldots,p_\ell)}\|u\|_{X^k(T)} \quad \text{for all} \quad q\in[r_\ell,\infty],$$
$$\|u_k(t,\cdot)\|_{L^q} \lesssim\ (1+t)^{\alpha_k-\mu_k}\|u\|_{X^k(T)} \quad \text{for all} \quad q\in[r_k,\infty].$$

On the other hand, for $q \in [r_1, \infty]$, we have

$$\|u_1^{nl}(t,\cdot)\|_{L^q} \lesssim \|u\|_{X(T)}^{p_1} I_1(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_1 q \in [r_k, \infty],$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau - s)^{\alpha_1 - 1} \int_0^s (s-\rho)^{-\mu_1} (1+\rho)^{-p_1(\mu_k - \alpha_k)}\, d\rho\, ds\, d\tau. \tag{3.15}$$

We are interested to estimate the right-hand side of (3.15). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_1} (1+\rho)^{-p_1(\mu_k-\alpha_k)}\, d\rho.$$

Thanks to Lemma 2 we obtain $\omega(s) \lesssim (1+s)^{1-\mu_1-p_1(\mu_k-\alpha_k)}$, if we assume that $p_1 < \frac{1}{\mu_k - \alpha_k}$.
Once more we apply Lemma 2 to obtain

$$\begin{aligned}
I_1(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} (1+s)^{1-\mu_1-p_1(\mu_k-\alpha_k)}\, ds\, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)} (1+\tau)^{1+\alpha_1-\mu_1-p_1(\mu_k-\alpha_k)}\, d\tau \\
&\lesssim (1+t)^{1+\alpha_1-\mu_1-p_1(\mu_k-\alpha_k)} \\
&\lesssim (1+t)^{\alpha_1-\mu_1+\gamma_{\alpha_k}^{\mu_k}(p_1)}.
\end{aligned}$$

On other hands, the conditions $q \in [r_1, \infty]$ and $p_1 q \in [r_k, \infty]$ implies that $p_1 \geq \frac{r_k}{r_1}$.
For $l = 2$ and $q \in [r_2, \infty]$, we have

$$\begin{aligned}
\|u_2^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \||u_1(s,\cdot)|^{p_2}\|_{L^q}\, ds\, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \int_0^s (s-\rho)^{-\mu_2} \|u_1(\rho,\cdot)\|_{L^{p_2 q}}^{p_2}\, d\rho\, ds\, d\tau \\
&\lesssim \|u\|_{X(T)}^{p_2} I_2(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_2 q \in [r_1, \infty],
\end{aligned}$$

where

$$\begin{aligned}
I_2(t) = &\int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} \\
&\times \int_0^s (s-\rho)^{-\mu_2} (1+\rho)^{-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)}\, d\rho\, ds\, d\tau.
\end{aligned} \tag{3.16}$$

On the other hand, we have We are interested to estimate the right-hand side of (3.16). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_2} (1+\rho)^{-p_2(\mu_1-\alpha_1)+p_2\gamma_{(\alpha_k)}^{(\mu_k)}(p_1)}\, d\rho.$$

Thanks to Lemma 2 we obtain

$$\omega(s) \lesssim (1+s)^{1-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)},$$

if we assume that $p_2 < \frac{1}{\mu_1-\alpha_1-\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}$ and $p_1 > \frac{1+\alpha_1-\mu_1}{\mu_k-\alpha_k}$.
Once more we apply Lemma 2 to obtain

$$\begin{aligned}
I_2(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} \int_0^\tau (\tau-s)^{\alpha_2-1} (1+s)^{1-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}\, ds\, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_2)} (1+\tau)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)}\, d\tau \\
&\lesssim (1+t)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)} \\
&\lesssim (1+t)^{1+\alpha_2-\mu_2-p_2(\mu_1-\alpha_1)+p_2\gamma^{(\mu_k)}_{(\alpha_k)}(p_1)} \\
&\lesssim (1+t)^{\alpha_2-\mu_2+\gamma^{(\mu_k,\mu_1)}_{(\alpha_k,\alpha_1)}(p_1,p_2)}.
\end{aligned}$$

On other hands, the conditions $q \in [r_2, \infty]$ and $p_2 q \in [r_1, \infty]$ implies that $p_2 \geq \frac{r_1}{r_2}$.
For $l = 3, \cdots, k-1$ and $q \in [r_l, \infty]$, we have

$$\begin{aligned}
\|u^{nl}_\ell(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \||u_{\ell-1}(s,\cdot)|^{p_\ell}\|_{L^q}\, ds\, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \int_0^s (s-\rho)^{-\mu_\ell} \|u_{\ell-1}(\rho,\cdot)\|^{p_\ell}_{L^{p_\ell q}}\, d\rho\, ds\, d\tau \\
&\lesssim \|u\|^{p_\ell}_{X(T)} I_\ell(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_\ell q \in [r_{\ell-1}, \infty],
\end{aligned}$$

where

$$\begin{aligned}
I_\ell(t) = &\int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \\
&\times \int_0^s (s-\rho)^{-\mu_\ell} (1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})}\, d\rho\, ds\, d\tau. \qquad (3.17)
\end{aligned}$$

On the other hand, we have We are interested to estimate the right-hand side of (3.17). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_\ell} (1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})}\, d\rho.$$

Thanks to Lemma 2 we obtain

$$\omega(s) \lesssim (1+s)^{1-\mu_\ell-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})},$$

if we assume that

$$p_\ell < \frac{1}{(\mu_{\ell-1}-\alpha_{\ell-1}) - \gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})}$$

and

$$p_{l-1} > \frac{1+\alpha_{l-1}-\mu_{l-1}}{\mu_{\ell-2}-\alpha_{\ell-2}-\gamma^{(\mu_k,\dots,\mu_{\ell-3})}_{(\alpha_k,\dots,\alpha_{\ell-3})}(p_1,\dots,p_{\ell-2})}$$

On other hands, the conditions $q \in [r_\ell, \infty]$ and $p_\ell q \in [r_{\ell-1}, \infty]$ implies that $p_\ell \geq \frac{r_\ell - 1}{r_\ell}$. Once more we apply Lemma 2 to obtain

$$\begin{aligned} I_\ell(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1}(1+s)^{1-\mu_\ell-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})}\,ds\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)}(1+s)^{\alpha_\ell-\mu_\ell+1-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})}\,d\tau \\ &\lesssim (1+t)^{\alpha_\ell-\mu_\ell+1-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})+p_\ell\gamma^{(\mu_k,\dots,\mu_{\ell-2})}_{(\alpha_k,\dots,\alpha_{\ell-2})}(p_1,\dots,p_{\ell-1})} \\ &\lesssim (1+t)^{\alpha_\ell-\mu_\ell+\gamma^{(\mu_k,\dots,\mu_{\ell-1})}_{(\alpha_k,\dots,\alpha_{\ell-1})}(p_1,\dots,p_\ell)} \end{aligned}$$

Finally, for $q \in [r_k, \infty]$, we have

$$\begin{aligned} \|u_k^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \||u_{k-1}(s,\cdot)|^{p_k}\|_{L^q}\,ds\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \int_0^s (s-\rho)^{-\mu_k}\|u_{k-1}(\rho,\cdot)\|^{p_k}_{L^{p_k q}}\,d\rho\,ds\,d\tau \\ &\lesssim \|u\|^{p_k}_{X(T)} I_k(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_k q \in [r_{k-1}, \infty], \end{aligned}$$

where

$$\begin{aligned} I_k(t) = &\int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \\ &\times \int_0^s (s-\rho)^{-\mu_k}(1+\rho)^{-p_k\left(\mu_{k-1}-\alpha_{k-1}-\gamma^{(\mu_1,\dots,\mu_{k-2})}_{(\alpha_1,\dots,\alpha_{k-2})}(p_1,\dots,p_{k-1})\right)}\,d\rho\,ds\,d\tau. \end{aligned} \tag{3.18}$$

We are interested to estimate the right-hand side of (3.18). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_k}(1+\rho)^{-p_k\left(\mu_{k-1}-\alpha_{k-1}-\gamma^{(\mu_1,\dots,\mu_{k-2})}_{(\alpha_1,\dots,\alpha_{k-1})}(p_1,\dots,p_{k-1})\right)}\,d\rho.$$

Thanks to Lemma 2 we obtain $\omega(s) \lesssim (1+s)^{-\mu_k}$, if we assume that

$$p_k > \frac{1}{\mu_{k-1}-\alpha_{k-1}-\gamma^{(\mu_k,\dots,\mu_{k-2})}_{(\alpha_k,\dots,\alpha_{k-2})}(p_1,\dots,p_{k-1})}$$

and

$$\mu_{k-1}-\alpha_{k-1}-\gamma^{(\mu_k,\dots,\mu_{k-2})}_{(\alpha_k,\dots,\alpha_{k-2})}(p_1,\dots,p_{k-1}) > 0$$

wich equivalent to

$$p_{k-1} > \frac{\alpha_{k-1}-\mu_{k-1}+1}{(\mu_{k-2}-\alpha_{k-2})-\gamma^{(\mu_k,\dots,\mu_{k-3})}_{\alpha_k,\dots,\alpha_{k-3}}(p_1,\dots,p_{k-2})}.$$

Once more we apply Lemma 2 to obtain

$$\begin{aligned} I_k(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1}(1+s)^{-\mu_k}\, d\rho\, ds\, d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)}(1+\tau)^{\alpha_k-\mu_k}\, d\tau \\ &\lesssim (1+t)^{\alpha_k-\mu_k} \end{aligned}$$

The proof of (4.27) is similar to the proof of (4.20) of Theorem 4.2. This completes the proof.

3.3.3 Proof of Theorem 4.6

We introduce for all $T>0$ the space $X^k(T)$ as follows:

$$X^k(T) := \prod_{\ell=1}^{k} C\big([0,T], L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|u\|_{X^k(T)} := \sup_{0\leqslant t\leqslant T}\Big\{\sum_{l=1}^{k-1}(1+t)^{-\gamma(p_l)}M_l(t,u_l)+M_k(t,u_k)\Big\},$$

where, for $l=1,\cdots,k$

$$M_\ell(t,u_l) = (1+t)^{\mu_\ell-\alpha_\ell}\big(\|u_\ell(t,\cdot)\|_{L^1}+\|u_\ell(t,\cdot)\|_{L^\infty},$$

$$\begin{cases} \gamma(p_1)=1 \\ \gamma(p_l)=1+p_l\gamma(p_{\ell-1}), \text{ for } l=2,\cdots,k-1. \end{cases}$$

The operator N is defined by

$$N\ :\ u=(u_1,u_2,...,u_k)\in X^k(T)\to N(u)=N(u)(t,x):=u^{ln}(t,x)+u^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (3.1) as a fixed point of the operator N. We will prove that the operator N satisfies for $u=(u_1,u_2,...,u_k)$; $\overline{u}=(\overline{u}_1,\overline{u}_2,...,\overline{u}_k)$ in $X^k(T)$, the following two inequalities:

$$\|N(u)\|_{X^k(T)} \lesssim \|(u_{01},u_{02},...u_{0k})\|_{\mathcal{A}_k} + \sum_{\ell=1}^{\ell=k}\|u\|^{p_\ell}_{X^k(T)}, \tag{3.19}$$

$$\|N(u)-N(\overline{u})\|_{X^k(T)} \lesssim \|u-\overline{u}\|_{X^k(T)}\sum_{\ell=1}^{\ell=k}\Big(\|u\|^{p_\ell-1}_{X^k(T)}+\|\overline{u}\|^{p_l-1}_{X^k(T)}\Big) \tag{3.20}$$

Using the definition of the norm in $X^k(T)$ and the Propositions 21, we may conclude:

$$\|u^{ln}\|_{X^k(T)} \lesssim \|(u_{01}, u_{02}, ..., u_{0k})\|_{\mathcal{A}_k}.$$

Hence, in order to complete the proof of (4.41) it is reasonable to we shall show the following inequality

$$\|u^{nl}\|_{X^k(T)} \lesssim \sum_{\ell=1}^{\ell=k} \|u\|^{p_\ell}_{X^k(T)}.$$

If $u := (u_1, u_2, \cdots, u_k) \in X^k(T)$, then by interpolation we derive for $l = 1, \cdots, k-1$

$$\begin{aligned}
\|u_\ell(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(\alpha_\ell-\mu_\ell)+\gamma(p_\ell)}\|u\|_{X^k(T)} \quad \text{for all} \quad q \in [1,\infty],\\
\|u_k(t,\cdot)\|_{L^q} &\lesssim \ (1+t)^{\alpha_k-\mu_k}\|u\|_{X^k(T)} \quad \text{for all} \quad q \in [1,\infty].
\end{aligned}$$

On the other hand, for $q \in [1,\infty]$, we have

$$\|u_1^{nl}(t,\cdot)\|_{L^q} \lesssim \|u\|^{p_1}_{X(T)} I_1(t) \quad \text{for all} \quad t \in [0,T],$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1} \int_0^s (s-\rho)^{-\mu_1}(1+\rho)^{-p_1(\mu_k-\alpha_k)}\, d\rho\, ds\, d\tau. \tag{3.21}$$

We are interested to estimate the right-hand side of (3.21). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_1}(1+\rho)^{-p_1(\mu_k-\alpha_k)}\, d\rho.$$

Thanks to Lemma 2 we obtain $\omega(s) \lesssim (1+s)^{-\mu_1}\ln(1+s)$, if we assume that $p_1 = \frac{1}{\mu_k-\alpha_k}$.
Once more we apply Lemma 2 to obtain

$$\begin{aligned}
I_1(t) &\lesssim \ln(e+t)\int_0^t (1+t-\tau)^{-(1+\alpha_1)} \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-\mu_1}\, ds\, d\tau\\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_1)}(1+\tau)^{\alpha_1-\mu_1}\, d\tau\\
&\lesssim (1+t)^{\alpha_1-\mu_1}\ln(e+t)\\
&\lesssim (1+t)^{\alpha_1-\mu_1}\big(\ln(e+t)\big)^{\gamma(p_1)}.
\end{aligned}$$

For $l = 2, \cdots, k-1$ and $q \in [1,\infty]$, we have

$$\begin{aligned}
\|u_\ell^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \||u_{\ell-1}(s,\cdot)|^{p_\ell}\|_{L^q}\, ds\, d\tau\\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \int_0^s (s-\rho)^{-\mu_\ell}\|u_{\ell-1}(\rho,\cdot)\|^{p_\ell}_{L^{p_\ell q}}\, d\rho\, ds\, d\tau\\
&\lesssim \|u\|^{p_\ell}_{X(T)} I_\ell(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_\ell q \in [r_{\ell-1},\infty],
\end{aligned}$$

where

$$I_\ell(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \times \int_0^s (s-\rho)^{-\mu_\ell}(1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})}\big(\ln(1+\rho)\big)^{p_\ell\gamma(p_{\ell-1})}\, d\rho\, ds\, d\tau. \tag{3.22}$$

Remark that

$$I_\ell(t) \lesssim \big(\ln(e+t)\big)^{p_\ell\gamma(p_{\ell-1})} \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1} \times \int_0^s (s-\rho)^{-\mu_\ell}(1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})}\, d\rho\, ds\, d\tau. \tag{3.23}$$

On the other hand, we have We are interested to estimate the right-hand side of (3.23). For this we need the Lemma 2. We put

$$\omega(s) = \int_0^s (s-\rho)^{-\mu_\ell}(1+\rho)^{-p_\ell(\mu_{\ell-1}-\alpha_{\ell-1})}\, d\rho.$$

Thanks to Lemma 2 we obtain

$$\omega(s) \lesssim (1+s)^{-\mu_\ell},$$

if we assume that

$$p_\ell = \frac{1}{\mu_{\ell-1}-\alpha_{\ell-1}}.$$

Once more we apply Lemma 2 to obtain

$$\begin{aligned} I_\ell(t) &\lesssim \big(\ln(e+t)\big)^{p_\ell\gamma(p_{\ell-1})} \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)} \int_0^\tau (\tau-s)^{\alpha_\ell-1}(1+s)^{-\mu_\ell}\, ds\, d\tau \\ &\lesssim \big(\ln(e+t)\big)^{p_\ell\gamma(p_{\ell-1})} \int_0^t (1+t-\tau)^{-(1+\alpha_\ell)}(1+\tau)^{\alpha_\ell-\mu_\ell}\, d\tau \\ &\lesssim (1+t)^{\alpha_\ell-\mu_\ell}\big(\ln(e+t)\big)^{p_\ell\gamma(p_{\ell-1})} \end{aligned}$$

Finally, for $q \in [1,\infty]$, we have

$$\begin{aligned} \|u_k^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \||u_{k-1}(s,\cdot)|^{p_k}\|_{L^q}\, ds\, d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \int_0^s (s-\rho)^{-\mu_k}\|u_{k-1}(\rho,\cdot)\|^{p_k}_{L^{p_k q}}\, d\rho\, ds\, d\tau \\ &\lesssim \|u\|^{p_k}_{X(T)} I_k(t) \quad \text{for all} \quad t\in[0,T] \end{aligned}$$

where

$$I_k(t) = \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \times \int_0^s (s-\rho)^{-\mu_k}(1+\rho)^{-p_k(\mu_{k-1}-\alpha_{k-1})}\big(\ln(1+\rho)\big)^{p_k\gamma(p_{k-1})}\, d\rho\, ds\, d\tau. \tag{3.24}$$

We are interested to estimate the right-hand side of (3.25). Let $\epsilon > 0$ small enough and we use the fact that $\ln(e+t) \lesssim (1+t)^{\epsilon}$ to obtain

$$
\begin{aligned}
I_k(t) \lesssim & \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1} \\
& \times \int_0^s (s-\rho)^{-\mu_k} (1+\rho)^{-p_k(\mu_{k-1}-\alpha_{k-1}-\epsilon\gamma(p_{k-1}))}\, d\rho\, ds\, d\tau. \qquad (3.25)
\end{aligned}
$$

For this we need the Lemma 2. We put

$$
\omega(s) = \int_0^s (s-\rho)^{-\mu_k} (1+\rho)^{-p_k(\mu_{k-1}-\alpha_{k-1}-\epsilon\gamma(p_{k-1}))}\, d\rho
$$

Thanks to Lemma 2 we obtain $\omega(s) \lesssim (1+s)^{-\mu_k}$, if we assume that

$$
p_k > \frac{1}{\mu_{k-1}-\alpha_{k-1}-\epsilon\gamma(p_{k-1})}.
$$

Once more we apply Lemma 2 to obtain

$$
\begin{aligned}
I_k(t) &\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} \int_0^\tau (\tau-s)^{\alpha_k-1}(1+s)^{-\mu_k}\, d\rho\, ds\, d\tau \\
&\lesssim \int_0^t (1+t-\tau)^{-(1+\alpha_k)} (1+\tau)^{\alpha_k-\mu_k}\, d\tau \\
&\lesssim (1+t)^{\alpha_k-\mu_k}.
\end{aligned}
$$

The proof of (4.42) is similar to the proof of (4.20) of Theorem 4.2. This completes the proof.

CHAPTER 4

WEAKLY COUPLED SYSTEMS OF SEMI-LINEAR FRACTIONAL $\sigma-$EVOLUTION EQUATIONS WITH OUT MASS AND DIFFERENT POWER NONLINEARITIES

The main purpose of this chapter is to study global (in time) existence of small data Sobolev solutions to the Cauchy problem for the following weakly coupled systems of semi-linear fractional $\sigma-$evolution equations with different power nonlinearities, namely

$$\begin{cases} \partial_t^{1+\alpha_1}u + (-\Delta)^{\sigma_1}u = |v|^{p_1},\ u(0,x) = u_0(x)\ u_t(0,x) = 0, \\ \partial_t^{1+\alpha_2}v + (-\Delta)^{\sigma_2}v = |u|^{p_2},\ v(0,x) = v_0(x),\ v_t(0,x) = 0, \end{cases} \tag{4.1}$$

where for $i = 1, 2$, $\alpha_i \in (0, 1)$, $\sigma_i \geq 1$, $(t, x) \in [0, \infty) \times \mathbb{R}^n$, $\partial_t^{1+\alpha_i}u = D_t^{\alpha_i}(u_t)$ with

$$D_t^{\alpha_i}(f) = \partial_t(I_t^{1-\alpha_i}f) \quad \text{and} \quad I_t^\beta f = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)\,ds \ \text{ for } \ \beta > 0.$$

The notations $D_t^\alpha(f)$ and $I_t^\beta f$ are defined as above.

By using $L^r - L^q$ estimates of Sobolev solutions to related linear models with vanishing right-hand side , We explain connections between the fractional orders α_1, α_2 and the exponents (p_1, p_2) in (4.1), which allow to prove the global (in time) existence of small-data Sobolev solutions by applying the fixed-point argument.

4.1 Main results

4.1.1 The case $n \geq \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$

Theorem 4.1 *Let us assume $0 < \alpha_1, \alpha_2 < 1$, $\sigma_1, \sigma_2 \geq 1$ and $r_1, r_2 \geq 1$. We assume that $n \geq \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$. Moreover, for all $\epsilon > 0$ the exponents p_1 and p_2 satisfy the conditions*

$$p_1 > p^{r_1,r_2}_{\alpha_2,\sigma_2}(n,\epsilon) := \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\alpha_2}, \frac{r_2}{r_1} - \epsilon\Big\}$$

and

$$p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n,\epsilon) := \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\alpha_1}, \frac{r_1}{r_2} - \epsilon\Big\},$$

where

$$p^r_{\alpha,\sigma}(n) := 1 + \frac{(n(r-1) + 2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha) + 2\sigma r}.$$

Then there exists a positive constant ε such that for any data

$$(u_0, v_0) \in \mathcal{A}^{r_2}_{r_1} := L^{r_1}(\mathbb{R}^n) \times \big(L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} \leq \varepsilon$, we have a uniquely determined global (in time) Sobolev solution

$$(u,v) \in C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). Moreover, the solution satisfies the following decay estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1} + \alpha_1}\|u_0\|_{L^{r_1}} \text{ for all } q \in [r_1,\infty], \tag{4.2}$$

$$\|v(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2} + \alpha_2}\|v_0\|_{L^{r_2}} \text{ for all } q \in [r_2,\infty], \tag{4.3}$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big); 1-\delta\Big\}.$$

The constant C is independent of u_0 and v_0.

Theorem 4.2 *(Loss of decay) Let us assume $0 < \alpha_1, \alpha_2 < 1$, $\sigma_1, \sigma_2 \geq 1$ and $r_1, r_2 \geq 1$. We assume that $n \geq \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$. Moreover, for all $\epsilon > 0$ the exponents p_1 and p_2 satisfy the conditions*

$$p_1 < p^{r_1,r_2}_{\alpha_2,\sigma_2}(n,\epsilon) := \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\alpha_2}\Big\}$$

$$p_1 > \max\Big\{\frac{(\alpha_1+1)(-n+2\sigma_1 r_1)}{2\sigma_1 r_1(1-\alpha_2)}; \frac{\alpha_1}{1-\alpha_2}; \frac{2\sigma_2 r_2\alpha_1 + r_2 n(1+\alpha_2)}{n(1+\alpha_2) - 2\sigma_2 r_2};$$
$$r_2\frac{n[(1+\alpha_2)\sigma_1 r_1 - (1+\alpha_1)\sigma_2] + 2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2) - 2\alpha_2\sigma_2 r_2}; \frac{r_2}{r_1} - \epsilon; 1\Big\}$$

and

$$p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n,\epsilon) := \max\Big\{\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n,p_1); \frac{1}{1-\alpha_1}, \frac{r_1}{r_2}-\epsilon\Big\},$$

where

$$p^r_{\alpha,\sigma}(n) := 1 + \frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}.$$

$$\begin{aligned}\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n,p_1) = \max\Big\{&\frac{2\sigma_2 r_2}{n(\alpha_1+1)(p_1-r_2)-2\sigma_2 r_2(\alpha_1+p_1\alpha_2)};\\ &\frac{1}{p_1(1-\alpha_2)-\alpha_1}; \frac{2\sigma_1 r_1+n(\alpha_1+1)r_1}{(\alpha_1+1)(n-2\sigma_1 r_1)+2p_1\sigma_1 r_1(1-\alpha_2)};\\ &\sigma_2 r_2\frac{2\sigma_1 r_1+nr_1(\alpha_1+1)}{n[\sigma_2 r_2(\alpha_1+1)+\sigma_1 r_1(\alpha_2+1)(p_1-r_2)]-2\sigma_2 r_2\sigma_1 r_1[(\alpha_1+1)+p_1\alpha_2)]}\Big\}\end{aligned}$$

Then there exists a positive constant ε *such that for any data*

$$(u_0,v_0)\in\mathcal{A}^{r_2}_{r_1} := L^{r_1}(\mathbb{R}^n)\times L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}}\leq\varepsilon$*, we have a uniquely determined global (in time) Sobolev solution*

$$(u,v)\in C\big([0,\infty),L^{r_1}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)\times C\big([0,\infty),L^{r_2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). Moreover, the solution satisfies the following decay estimate for any $t\geq 0$ *and for all sufficiently small* $\delta>0$ *:*

$$\|u(t,\cdot)\|_{L^q}\leq C(1+t)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\alpha_1+\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}}\|u_0\|_{L^{r_1}} \text{ for all } q\in[r_1,\infty], \tag{4.4}$$

$$\|v(t,\cdot)\|_{L^q}\leq C(1+t)^{-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}+\alpha_2}\|v_0\|_{L^{r_2}} \text{ for all } q\in[r_2,\infty], \tag{4.5}$$

where

$$\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} = 1-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)$$

and

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big);1-\delta\Big\}.$$

The constant C *is independent of* u_0 *and* v_0*.*

4.1.2 The case $1\leq n<\min\big\{\frac{2\sigma_1 r_1}{1+\alpha_1},\frac{2\sigma_2 r_2}{1+\alpha_2}\big\}$

Theorem 4.3 *Let us assume* $0<\alpha_1,\alpha_2<1$, $1\leq\sigma_1<\frac{1+\alpha_1}{2\alpha_1}$, $1\leq\sigma_2<\frac{1+\alpha_2}{2\alpha_2}$, $1\leq r_1<\frac{1+\alpha_1}{2\alpha_1\sigma_1}$ *and* $1\leq r_2<\frac{1+\alpha_2}{2\alpha_2\sigma_2}$*. We assume that* $1\leq n<\min\big\{\frac{2\sigma_1 r_1}{1+\alpha_1},\frac{2\sigma_2 r_2}{1+\alpha_2}\big\}$*. Moreover, for all* $\epsilon>0$ *the exponents* p_1 *and* p_2 *satisfy the conditions*

$$p_1 > p^{r_1,r_2}_{\alpha_2,\sigma_2}(n) := \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n);\frac{r_2}{r_1}-\epsilon\Big\}$$

and

$$p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n) := \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{r_1}{r_2} - \epsilon\Big\},$$

where

$$p^r_{\alpha,\sigma}(n) := 1 + \frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}.$$

Then there exists a positive constant ε such that for any data

$$(u_0, v_0) \in \mathcal{A}^{r_2}_{r_1} := L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0, v_0)\|_{\mathcal{A}^{r_2}_{r_1}} \leq \varepsilon$, we have a uniquely determined global (in time) Sobolev solution

$$(u, v) \in C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). Moreover, the solution satisfies the following decay estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}+\alpha_1}\|u_0\|_{L^{r_1}} \text{ for all } q \in [r_1,\infty], \tag{4.6}$$

$$\|v(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}+\alpha_2}\|v_0\|_{L^{r_2}} \text{ for all } q \in [r_2,\infty], \tag{4.7}$$

and where

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big).$$

The constant C is independent of u_0 and v_0.

Theorem 4.4 *(Loss of decay) Let us assume $0 < \alpha_1, \alpha_2 < 1$, $1 \leq \sigma_1 \leq \frac{1+\alpha_1}{2\alpha_1}$, $1 \leq \sigma_2 < \frac{1+\alpha_2}{2\alpha_2}$, $1 \leq r_1 \leq \frac{1+\alpha_1}{2\alpha_1\sigma_1}$ and $1 \leq r_2 < \frac{1+\alpha_2}{2\alpha_2\sigma_2}$. We assume that $1 \leq n < \min\big\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\big\}$. Moreover, for all $\epsilon > 0$ the exponents p_1 and p_2 satisfy the conditions*

$$\max\Big\{1; r_2\frac{n[(1+\alpha_2)\sigma_1 r_1 - (1+\alpha_1)\sigma_2] + 2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2) - 2\alpha_2\sigma_2 r_2)}; \frac{r_2}{r_1} - \epsilon\Big\} < p_1 < p^{r_2}_{\alpha_2,\sigma_2}(n)$$

and

$$p_2 > p^{r_1,r_2}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n) := \max\Big\{\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n, p_1); \frac{r_1}{r_2} - \epsilon\Big\},$$

where

$$p^r_{\alpha,\sigma}(n) := 1 + \frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}.$$

and

$$\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n, p_1) =$$
$$\sigma_2 r_2\frac{2\sigma_1 r_1 + n r_1(\alpha_1+1)}{n[\sigma_2 r_2(\alpha_1+1) + \sigma_1 r_1(\alpha_2+1)(p_1 - r_2)] - 2\sigma_2 r_2\sigma_1 r_1[(\alpha_1+1) + p_1\alpha_2)]}$$

Then there exists a positive constant ε such that for any data

$$(u_0, v_0) \in \mathcal{A}_{r_1}^{r_2} := L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0, v_0)\|_{\mathcal{A}_{r_1}^{r_2}} \leq \varepsilon$, we have a uniquely determined global (in time) Sobolev solution

$$(u, v) \in C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). Moreover, the solution satisfies the following decay estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}+\alpha_1+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}} \|u_0\|_{L^{r_1}} \text{ for all } q \in [r_1, \infty], \tag{4.8}$$

$$\|v(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}+\alpha_2} \|v_0\|_{L^{r_2}} \text{ for all } q \in [r_2, \infty], \tag{4.9}$$

where

$$\gamma^{r_2}_{\alpha_2,p_1,\sigma_2} = 1 - p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2} - \alpha_2)$$

and

$$\beta^r_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r} - \frac{1}{q}\Big).$$

The constant C is independent of u_0 and v_0.

4.1.3 The case $\min\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\} \leq n < \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$

In the next section, we suppose that $\frac{2\sigma_1 r_1}{1+\alpha_1} < \frac{2\sigma_2 r_2}{1+\alpha_2}$

Theorem 4.5 *Let us assume $0 < \alpha_1, \alpha_2 < 1$, $\sigma_1 \geq 1$, $1 \leq \sigma_2 < \frac{1+\alpha_2}{2\alpha_2}$, $r_1 \geq 1$ and $1 \leq r_2 < \frac{1+\alpha_2}{2\alpha_2\sigma_2}$. We assume that $\frac{2\sigma_1 r_1}{1+\alpha_1} \leq n < \frac{2\sigma_2 r_2}{1+\alpha_2}$. Moreover, for all $\epsilon > 0$ the exponents p_1 and p_2 satisfy the conditions*

$$p_1 > p^{r_1,r_2}_{\alpha_2,\sigma_2}(n,\epsilon) := \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{r_2}{r_1} - \epsilon\Big\}$$

and

$$p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n,\epsilon) := \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\alpha_1}; \frac{r_1}{r_2} - \epsilon\Big\},$$

where

$$p^r_{\alpha,\sigma}(n) := 1 + \frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}.$$

Then there exists a positive constant ε such that for any data

$$(u_0, v_0) \in \mathcal{A}_{r_1}^{r_2} := L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0, v_0)\|_{\mathcal{A}_{r_1}^{r_2}} \leq \varepsilon$, we have a uniquely determined global (in time) Sobolev solution

$$(u, v) \in C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). Moreover, the solution satisfies the following decay estimate for any $t \geq 0$ and for all sufficiently small $\delta > 0$:

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\alpha_1}\|u_0\|_{L^{r_1}} \text{ for all } q \in [r_1,\infty], \tag{4.10}$$

$$\|v(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}+\alpha_2}\|v_0\|_{L^{r_2}} \text{ for all } q \in [r_2,\infty], \tag{4.11}$$

where

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big); 1-\delta\Big\}$$

and

$$\beta^{r}_{\alpha,q,\sigma} := \frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big).$$

The constant C is independent of u_0 and v_0.

Theorem 4.6 *(Loss of decay) Let us assume $0 < \alpha_1, \alpha_2 < 1$,$\sigma_1 \geq 1$, $1 \leq \sigma_2 < \frac{1+\alpha_2}{2\alpha_2}$, and $1 \leq r_2 < \frac{1+\alpha_2}{2\alpha_2\sigma_2}$, We assume that $\frac{2\sigma_1 r_1}{1+\alpha_1} < n < \frac{2\sigma_2 r_2}{1+\alpha_2}$. Moreover, for all $\epsilon > 0$ the exponents p_1 and p_2 satisfy the conditions*

$$p_1 < p^{r_2}_{\alpha_2,\sigma_2}(n)$$

$$p_1 > \max\Big\{\frac{2\sigma_2 r_2(\delta+\alpha_1)+n(1+\alpha_2)r_2}{n(1+\alpha_2)-2\sigma_2 r_2\alpha_2}; \frac{r_2}{r_1}-\epsilon, 1$$

$$r_2\frac{n[(1+\alpha_2)\sigma_1 r_1-(1+\alpha_1)\sigma_2]+2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2)-2\alpha_2\sigma_2 r_2)}\Big\},$$

and

$$p_2 > p^{r_1,r_2}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n) := \max\Big\{\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n,p_1); \frac{r_1}{r_2}-\epsilon\Big\},$$

where

$$p^{r}_{\alpha,\sigma}(n) := 1+\frac{(n(r-1)+2\sigma r)(1+\alpha)}{(n-2\sigma r)(1+\alpha)+2\sigma r}.$$

and

$$\bar{p}^{(r_1,r_2)}_{(\alpha_1,\alpha_2),(\sigma_1\sigma_2)}(n,p_1) = \max\Big\{\frac{2\sigma_2 r_2}{n(\alpha_2+1)(p_1-r_2)-2\sigma_2 r_2(\alpha_1+p_1\alpha_2)}$$

$$;\sigma_2 r_2\frac{2\sigma_1 r_1+nr_1(\alpha_1+1)}{n[\sigma_2 r_2(\alpha_1+1)+\sigma_1 r_1(\alpha_2+1)(p_1-r_2)]-2\sigma_2 r_2\sigma_1 r_1[(\alpha_1+1)+p_1\alpha_2)]}\Big\},$$

Then there exists a positive constant ε such that for any data

$$(u_0,v_0) \in \mathcal{A}^{r_2}_{r_1} := L^{r_1}(\mathbb{R}^n) \times L^{r_2}(\mathbb{R}^n),$$

with $\|(u_0,v_0)\|_{\mathcal{A}_{r_1}^{r_2}} \leq \varepsilon$, *we have a uniquely determined global (in time) Sobolev solution*

$$(u,v) \in C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$$

to the Cauchy problem (4.1). *Moreover, the solution satisfies the following decay estimate for any* $t \geq 0$ *and for all sufficiently small* $\delta > 0$ *:*

$$\|u(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\alpha_1+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}} \|u_0\|_{L^{r_1}} \text{ for all } q \in [r_1,\infty], \quad (4.12)$$

$$\|v(t,\cdot)\|_{L^q} \leq C(1+t)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}+\alpha_2} \|v_0\|_{L^{r_2}} \text{ for all } q \in [r_2,\infty], \quad (4.13)$$

where

$$\gamma^{r_2}_{\alpha_2,p_1,\sigma_2} = 1 - p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2} - \alpha_2)$$

and

$$\beta^{r,\delta}_{\alpha,q,\sigma} := \min\Big\{\frac{n(1+\alpha)}{2\sigma}\Big(\frac{1}{r}-\frac{1}{q}\Big); 1-\delta\Big\}.$$

The constant C *is independent of* u_0 *and* v_0.

4.2 Treatment of the weakly coupled linear systems

we consider the following two Cauchy problems

$$\begin{cases} \partial_t^{1+\alpha_1} u + (-\Delta)^{\sigma_1} u = 0,\ u(0,x) = u_0(x)\ u_t(0,x) = 0, \\ \partial_t^{1+\alpha_2} v + (-\Delta)^{\sigma_2} v = 0,\ v(0,x) = v_0(x),\ v_t(0,x) = 0, \end{cases} \quad (4.14)$$

to prove the global (in time) existence of small data Sobolev solutions to the weakly coupled systems of semi-linear models (4.1). we write their solutions in the following form:

$$u^{ln}(t,x) := G^{\sigma_1}_{\alpha_1}(t,x) *_{(x)} u_0(x),\ \ v^{ln}(t,x) := G^{\sigma_2}_{\alpha_2}(t,x) *_{(x)} v_0(x). \quad (4.15)$$

Proposition 21 *Let* $u_0 \in L^{r_1}$, $v_0 \in L^{r_2}$, $r_1, r_2 \geq 1$. *Then the solution of the linear Cauchy problem* (4.14) *satisfies the following* $L^r - L^q$ *estimates with* $r = r_1$ *or* $r = r_2$*:*

$$\big\|u^{ln}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} (\|u_0\|_{L^{r_1}(\mathbb{R}^n)} + \|u_0\|_{L^q(\mathbb{R}^n)}) \text{ for all } q \in [r_1,\infty],$$
$$\big\|v^{ln}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}} (\|v_0\|_{L^{r_2}(\mathbb{R}^n)} + \|v_0\|_{L^q(\mathbb{R}^n)}) \text{ for all } q \in [r_2,\infty]. \quad (4.16)$$

Now, due to the Duhamel formula, we can recover equation (4.1) in the following equivalent form

$$\begin{aligned} u(t,x) &:= u^{ln}(t,x) + \textstyle\int_0^t G^{\sigma_1}_{\alpha_1}(t-\tau,\cdot) *_{(x)} |v(\tau,\cdot)|^{p_1} d\tau = (u^{ln} + u^{nl})(t,x), \\ v(t,x) &:= v^{ln}(t,x) + \textstyle\int_0^t G^{\sigma_2}_{\alpha_2}(t-\tau,\cdot) *_{(x)} |u(\tau,\cdot)|^{p_2} d\tau = (v^{ln} + v^{nl})(t,x). \end{aligned} \quad (4.17)$$

4.3 Proof of main results

4.3.1 Proof of main results fo the case $n \geq \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$

Proof of Theorem 4.1

Let $i = 1; 2$. For any $n \geq \frac{2\sigma_i r_i}{1+\alpha_i}$ and $\delta \in (0,1)$ is sufficiently small, there exists a parameter $\overline{q_i} = \overline{q_i}(\delta) \in (r_i, \infty)$ such that

$$\frac{n(1+\alpha_i)}{2\sigma_i}\Big(\frac{1}{r_i} - \frac{1}{\overline{q_i}}\Big) = 1 - \delta. \tag{4.18}$$

We define for all $T > 0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with its corresponding norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T}\Big\{M_1(t,u) + M_2(t,v)\Big\},$$

where

$$M_1(t,u) := (1+t)^{-\alpha_1}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{1-\delta-\alpha_1}\big(\|u(t,\cdot)\|_{L^{\overline{q_1}}} + \|u(t,\cdot)\|_{L^\infty}\big),$$
$$M_2(t,v) = (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{1-\delta-\alpha_2}\big(\|v(t,\cdot)\|_{L^{\overline{q_2}}} + \|v(t,\cdot)\|_{L^\infty}\big),$$

We define a mapping the operator N by

$$N \ : \ (u,v) \in X(t) \to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x) + (u,v)^{nl}(t,x).$$

In order to conclude the uniquenessand the global (intime)existence of small data solutions to (4.1)as a fixed point of the operator N,We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.19}$$
$$\|N(u,v) - N(\overline{u},\overline{v})\|_{X(T)} \lesssim \|(u,v) - (\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)}$$
$$+\|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big) \tag{4.20}$$

for any (u,v) and $(\overline{u},\overline{v}) \in X(T)$, uniformly with respect to $T \in [0,\infty)$. The estimates (4.19), (4.20) lead to a local (in time) well-posedness result for large data Sobolev solutions and a global (in time) well-posedness result for samll data Soboelv solutions as well.
Using the definition of the norm in $X(T)$ and Proposition 21, we may conclude

$$\|(u,v)^{ln}\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}}.$$

Hence, in order to complete the proof of (4.19) it is reasonable to show the following inequality:

$$\|(u,v)^{nl}\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}.$$

If $(u,v) \in X(T)$, then by interpolation we derive

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r_1,\infty],$$
$$\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_2-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r_2,\infty].$$

On the other hand, we obtain

$$\begin{aligned}\||u(t,\cdot)|^{p_2}\|_{L^q} &\le \|u(t,\cdot)\|^{p_2}_{L^{p_2q}} \lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1}-\alpha_1)}\|(u,v)\|^{p_2}_{X(T)}\\ &\lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1)}\|(u,v)\|^{p_2}_{X(T)}\end{aligned} \tag{4.21}$$

,

for any q such that $p_2q\in[r_1,\infty]$ and due to $\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1} \ge \beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}$.
Also

$$\begin{aligned}\||v(t,\cdot)|^{p_1}\|_{L^q} &\le \|v(t,\cdot)\|^{p_1}_{L^{p_1q}} \lesssim (1+t)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1q,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)}\\ &\lesssim (1+t)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)}\end{aligned} \tag{4.22}$$

for any q such that $p_1q\in[r_2,\infty]$ and due to $\beta^{r_2,\delta}_{\alpha_2,p_1q,\sigma_2} \ge \beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}$. Thanks to (4.22) we have for $q\in[r_1,\infty]$ the estimates

$$\begin{aligned}\|u^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}\||v(s,\cdot)|^{p_1}\|_{L^q}\,ds\,d\tau\\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}\|v(s,\cdot)\|^{p_1}_{L^{p_1q}}\,ds\,d\tau\\ &\lesssim \|(u,v)\|^{p_1}_{X(T)}I_1(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_1q\in[r_2,\infty],\end{aligned}$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds\,d\tau. \tag{4.23}$$

We are interested in estimating the right-hand side of (4.23). For this we apply Lemma 2. We put

$$\omega(\tau) = \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds.$$

Thanks to Lemma **??** we obtain $\omega(\tau) \lesssim (1+\tau)^{\alpha_1-1}$ if we assume that $p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2) > 1$. We notice that $p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2) > 1$ if and only if

$$p_1 > \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\delta-\alpha_2}\Big\}.$$

On other hand, the conditions $q \in [r_1,\infty]$ and $p_1 q \in [r_2,\infty]$ imply $p_1 \geq \frac{r_2}{r_1}$.
Hence,

$$I_1(t) \lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \omega(\tau)\, d\tau \lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} (1+\tau)^{\alpha_1-1}\, d\tau. \tag{4.24}$$

Once more we use Lemma 2 to estimes (4.24) then we have $I_1(t) \lesssim (1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}$.
Hence,

$$\|(u^{nl},0)\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)}.$$

Summarizing, from both estimates we may conclude for all $q \in [r_2,\infty]$

$$\|v^{nl}(t,\cdot)\|_{L^q} \lesssim \|(u,v)\|^{p_2}_{X(T)} I_2(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_2 q \in [r_1,\infty],$$

where

$$I_2(t) = \int_0^t (1+t-\tau)^{-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}} \int_0^\tau (\tau-s)^{\alpha_2-1}(1+s)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1)}\, ds\, d\tau. \tag{4.25}$$

we same way and under the condition $p_2 > \max\left\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\delta-\alpha_1}\right\}$, then $I_2(t) \lesssim (1+t)^{\alpha_2-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}}$.
Hence,

$$\|((0,v^{nl})\|_{X(T)} \lesssim \|(u,v)\|^{p_2}_{X(T)}.$$

Also, the conditions $q \in [r_2,\infty]$ and $p_2 q \in [r_1,\infty]$ imply $p_2 \geq \frac{r_1}{r_2}$.
To prove (4.20) we assume that (u,v) and $(\tilde{u},\tilde{v})$ are two vector-function belonging to $X(T)$. Then, we have

$$\begin{aligned} &N(u,v) - N(\tilde{u},\tilde{v}) \\ &= \Big(\int_0^t G^{\sigma_1}_{\alpha_1}(t-s) *_{(x)} I^{\alpha_1}_s(|v(s,\cdot)|^{p_1} - |\tilde{v}(s,\cdot)|^{p_1})(t,s,x)\, ds, \\ &\qquad \int_0^t G^{\sigma_2}_{\alpha_2}(t-s) *_{(x)} I^{\alpha_2}_s(|u(s,\cdot)|^{p_2} - |\tilde{u}(s,\cdot)|^{p_2})(t,s,x)\, ds \Big). \end{aligned}$$

We estimate for $q \in [r_1,\infty]$ as follows:

$$\begin{aligned} &\Big\| \int_0^t G^{\sigma_1}_{\alpha_1}(t-s) *_{(x)} I^{\alpha_1}_s(|v(s,\cdot)|^{p_1} - |\tilde{v}(s,\cdot)|^{p_1})(t,s,\cdot)\, ds \Big\|_{L^q} \\ &\quad \lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1} \big\| |v(s,\cdot)|^{p_1} - |\tilde{v}(s,\cdot)|^{p_1} \big\|_{L^q}\, ds d\tau. \end{aligned}$$

By apply Hölder's inequality ,we may show the estimates

$$\begin{aligned} &\big\| |v(s,\cdot)|^{p_1} - |\tilde{v}(s,\cdot)|^{p_1} \big\|_{L^q} \\ &\qquad \lesssim \big\| v(s,\cdot) - \tilde{v}(s,\cdot) \big\|_{L^{qp_1}} \big(\|v(s,\cdot)\|^{p_1-1}_{L^{qp_1}} + \|\tilde{v}(s,\cdot)\|^{p_1-1}_{L^{qp_1}} \big). \end{aligned}$$

By using the definition of the norm of the solution space $X(T)$ we obtain for $p_1 \geq \frac{r_2}{r_1}$ and $0 \leqslant s \leqslant t$ we derive the following estimates:

$$\begin{aligned}
&\big\|v(s,\cdot)-\tilde{v}(s,\cdot)\big\|_{L^{qp_1}} \lesssim (1+s)^{\alpha_2-\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}} M_2(s,v-\tilde{v}),\\
&\big\|v(s,\cdot)\big\|^{p_1-1}_{L^{qp_1}} \lesssim (1+s)^{(p_1-1)(\alpha_2-\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p-1}_{X(T)},\\
&\big\|\tilde{v}(s,\cdot)\big\|^{p_1-1}_{L^{qp_1}} \lesssim (1+s)^{(p_1-1)(\alpha_2-\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p-1}_{X(T)}.
\end{aligned}$$

Hence, we write

$$\begin{aligned}
&\big\||v(s,\cdot)|^{p_1}-|\tilde{v}(s,\cdot)|^{p_1}\big\|_{L^q}\\
&\quad\lesssim (1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)} M_2(s,v-\tilde{v})\Big(\|(u,v)\|^{p_1-1}_{X(T)}+\|(\tilde{u},\tilde{v})\|^{p_1-1}_{X(T)}\Big)\\
&\quad\lesssim (1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)} \|(u,v)-(\tilde{u},\tilde{v})\|_{X(T)}\Big(\|(u,v)\|^{p_1-1}_{X(T)}+\|(\tilde{u},\tilde{v})\|^{p_1-1}_{X(T)}\Big).
\end{aligned}$$

Using the same approach, we obtain for $p_2 \geq \frac{r_1}{r_2}$ and $0 \leqslant s \leqslant t$ we may conclude:

$$\begin{aligned}
&\big\||u(s,\cdot)|^{p_2}-|\tilde{u}(s,\cdot)|^{p_2}\big\|_{L^q}\\
&\quad\lesssim (1+s)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1)} \|(u,v)-(\tilde{u},\tilde{v})\|_{X(T)}\Big(\|(u,v)\|^{p_2-1}_{X(T)}+\|(\tilde{u},\tilde{v})\|^{p_2-1}_{X(T)}\Big).
\end{aligned}$$

So, for

$$p_1 > \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\delta-\alpha_2}; \frac{r_2}{r_1}-\epsilon\Big\}$$

and

$$p_2 > \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\delta-\alpha_1}; \frac{r_1}{r_2}-\epsilon\Big\}$$

we obtain the desired estimate (4.20).
Notice that

$$p_1 > \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\delta-\alpha_2}; \frac{r_2}{r_1}-\epsilon\Big\}$$

and

$$p_2 > \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\delta-\alpha_1}; \frac{r_1}{r_2}-\epsilon\Big\}$$

for all $\delta > 0$ if and only if

$$p_1 > p^{r_1,r_2}_{\alpha_2,\sigma_2}(n,\epsilon) := \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\alpha_2}; \frac{r_2}{r_1}-\epsilon\Big\}$$

and

$$p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n,\epsilon) := \max\Big\{p^{r_1}_{\alpha_1,\sigma_1}(n); \frac{1}{1-\alpha_1}; \frac{r_1}{r_2}-\epsilon\Big\}.$$

Remark 3 *All estimates (4.19) and (4.20) are uniformly with respect to* $T \in (0,\infty)$ *if* $p_1 > p^{r_1,r_2}_{\alpha_2,\sigma_2}(n,\epsilon)$ *and* $p_2 > p^{r_1,r_2}_{\alpha_1,\sigma_1}(n,\epsilon)$ *for all* $\epsilon > 0$.

From (4.19) it follows that N maps $X(T)$ into itself for all T and for small data. By standard contraction arguments, the estimates (4.19) and (4.20) lead to the existence of a unique fixed point for $(u,v) = N(u,v)$. Consequently, we arrive at well-posedness of Sobolev solutions to (4.1) satisfying the desired decay estimates. Since all constants are independent of T, after letting T tend to ∞ we obtain a global (in time) well-posedness result for small data Sobolev solutions to (4.1). This completes the proof.

Proof of Theorem 4.2

We introduce for all $T > 0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T} \Big\{ M_1(t,u) + M_2(t,v) \Big\},$$

where

$$\begin{aligned} M_1(t,u) &= (1+t)^{-\alpha_1-\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{1-\delta-\alpha_1-\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}} \\ &\qquad\qquad\qquad \times \big(\|u(t,\cdot)\|_{L^{\overline{q_1}}} + \|u(t,\cdot)\|_{L^\infty}\big), \\ M_2(t,v) &= (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{1-\delta-\alpha_2}\big(\|v(t,\cdot)\|_{L^{\overline{q_2}}} + \|v(t,\cdot)\|_{L^\infty}\big), \end{aligned}$$

where $\overline{q_1}$ and $\overline{q_2}$ are defined as in the proof of Theorem 4.1. Finally, the operator N by

$$N \;:\; (u,v) \in X(t) \to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x) + (u,v)^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (4.1) as a fixed point of the operator N. We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.26}$$

$$\begin{aligned} \|N(u,v) - N(\overline{u},\overline{v})\|_{X(T)} &\lesssim \|(u,v) - (\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)} \\ &\quad + \|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big) \end{aligned} \tag{4.27}$$

for any (u,v) and $(\overline{u},\overline{v}) \in X(T)$, uniformly with respect to $T \in [0,\infty)$. The estimates (4.26), (4.27) lead to a local (in time) well-posedness result for large data Sobolev solutions and a global (in time) well-posedness result for samll data Soboelv solutions as well.
Using the definition of the norm in $X(T)$ and Proposition 21, we may conclude

$$\|(u,v)^{ln}\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}}.$$

Hence, in order to complete the proof of (4.26) it is reasonable to show the following inequality:

$$\|(u,v)^{nl}\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}.$$

If $(u,v) \in X(T)$, then by interpolation we derive for all $t \geq 0$

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q \in [r_1,\infty],$$
$$\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_2-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q \in [r_2,\infty].$$

On the other hand, we have

$$\begin{aligned} \||u(t,\cdot)|^{p_2}\|_{L^q} &\leq \|u(t,\cdot)\|^{p_2}_{L^{p_2 q}} \lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1}-\alpha_1-\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)} \\ &\lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1-\gamma^{r_2,\delta}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)} \end{aligned} \tag{4.28}$$

,

for any q such that $p_2q \in [r_1,\infty]$ and due to $\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1} \geq \beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}$.
Also

$$\begin{aligned} \||v(t,\cdot)|^{p_1}\|_{L^q} &\leq \|v(t,\cdot)\|^{p_1}_{L^{p_1 q}} \lesssim (1+t)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1q,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \\ &\lesssim (1+t)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \end{aligned} \tag{4.29}$$

for any q such that $p_1q \in [r_2,\infty]$ and due to $\beta^{r_2,\delta}_{\alpha_2,p_1q,\sigma_2} \geq \beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}$. Thanks to (4.22) we have for $q \in [r_1,\infty]$ the estimates

$$\begin{aligned} \|u^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}\||v(s,\cdot)|^{p_1}\|_{L^q}\,ds\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}\|v(s,\cdot)\|^{p_1}_{L^{p_1q}}\,ds\,d\tau \\ &\lesssim \|(u,v)\|^{p_1}_{X(T)} I_1(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_1q \in [r_2,\infty], \end{aligned}$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds\,d\tau. \tag{4.30}$$

We are interested in estimating the right-hand side of (4.30). For this we apply Lemma 2. We put

$$\omega(\tau) = \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds.$$

Thanks to Lemma 2 we obtain $\omega(\tau) \lesssim (1+\tau)^{\alpha_1-p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}$ if we assume that $p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2) < 1$. We notice that $p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2) < 1$ if and only if

$$p_1 < \max\Big\{p^{r_2}_{\alpha_2,\sigma_2}(n); \frac{1}{1-\delta-\alpha_2}\Big\}.$$

On other hand, the conditions $q \in [r_1, \infty]$ and $p_1 q \in [r_2, \infty]$ imply $p_1 \geq \frac{r_2}{r_1}$.
Hence,

$$\begin{aligned} I_1(t) &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} \omega(\tau)\, d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}} (1+\tau)^{\alpha_1 - p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2)}\, d\tau. \end{aligned} \tag{4.31}$$

Once more we apply Lemma 2 to (4.31) to obtain

$$I_1(t) \lesssim (1+t)^{\alpha_1 - \beta^{r_1,\delta}_{\alpha_1,q,\sigma_1} + 1 - p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2)}.$$

Hence,

$$\|(u^{nl}, 0)\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)}.$$

By the same way, we have for $q \in [r_2, \infty]$

$$\|v^{nl}(t,\cdot)\|_{L^q} \lesssim \|(u,v)\|^{p_2}_{X(T)} I_2(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_2 q \in [r_1, \infty],$$

where

$$I_2(t) = \int_0^t (1+t-\tau)^{-\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}} \int_0^\tau (\tau - s)^{\alpha_2 - 1} (1+s)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1} - \alpha_1 - 1 + p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2))}\, ds\, d\tau. \tag{4.32}$$

If

$$p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1} - \alpha_1 - 1 + p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2)) > 1,$$

then $I_2(t) \lesssim (1+t)^{\alpha_2 - \beta^{r_2,\delta}_{\alpha_2,q,\sigma_2}}$.
Hence,

$$\|((0, v^{nl})\|_{X(T)} \lesssim \|(u,v)\|^{p_2}_{X(T)}.$$

The condition

$$p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1} - \alpha_1 - 1 + p_1(\beta^{r_2,\delta}_{\alpha_2,p_1,\sigma_2} - \alpha_2)) > 1,$$

is equivalent to

$$\begin{aligned} p_2 > \max\Big\{ &\frac{2\sigma_2 r_2}{n(\alpha_1+1)(p_1 - r_2) - 2\sigma_2 r_2(\delta + \alpha_1 + p_1\alpha_2)}; \frac{1}{-\delta - \alpha_1 + p_1(1-\delta-\alpha_2)} \\ &; \sigma_2 r_2 \frac{2\sigma_1 r_1 + n r_1(\alpha_1+1)}{n[\sigma_2 r_2(\alpha_1+1) + \sigma_1 r_1(\alpha_2+1)(p_1 - r_2)] - 2\sigma_2 r_2 \sigma_1 r_1[(\alpha_1+1) + p_1\alpha_2)]}; \\ &\frac{2\sigma_1 r_1 + n(\alpha_1+1) r_1}{(\alpha_1+1)(n - 2\sigma_1 r_1) + 2p_1\sigma_1 r_1(1-\delta-\alpha_2)} \Big\} \end{aligned}$$

and

$$\begin{aligned} p_1 > \max\Big\{ &\frac{(\alpha_1+1)(-n+2\sigma_1 r_1)}{2\sigma_1 r_1(1-\delta-\alpha_2)}; \frac{\delta+\alpha_1}{1-\delta-\alpha_2}; \frac{2\sigma_2 r_2(\delta+\alpha_1) + r_2 n(1+\alpha_2)}{n(1+\alpha_2) - 2\sigma_2 r_2}; \\ &r_2 \frac{n[(1+\alpha_2)\sigma_1 r_1 - (1+\alpha_1)\sigma_2] + 2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2) - 2\alpha_2\sigma_2 r_2} \Big\}. \end{aligned}$$

Also, the conditions $q \in [r_2, \infty]$ and $p_2 q \in [r_1, \infty]$ imply $p_2 \geq \frac{r_1}{r_2}$.
The proof of (4.27) is similar to the proof of (4.20) of the proof to Theorem 4.1. This completes the proof.

4.3.2 Proof main results for the case $n < \min\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$

Proof of Theorem 4.3

If $1 \leq n < \min\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$, then for all $i = 1;2$ and for all $q \in [r_i, \infty]$ we obtain

$$\frac{n(1+\alpha_i)}{2\sigma_i}\Big(\frac{1}{r_i} - \frac{1}{q}\Big) < 1 - \frac{n(1+\alpha_i)}{2q\sigma_i} \leq 1. \tag{4.33}$$

Hence, we can choose a positive δ such that there does not exist any $\overline{q} \in [r_i, \infty]$ which satisfies (4.18). For this reason,

$$\beta^{r_i,\delta}_{\alpha_i,q,\sigma_i} = \beta^{r_i}_{\alpha_i,q,\sigma_i} := \frac{n(1+\alpha_i)}{2\sigma_i}\Big(\frac{1}{r_i} - \frac{1}{q}\Big).$$

We introduce for all $T > 0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T}\Big\{M_1(t,u) + M_2(t,v)\Big\},$$

where

$$M_1(t,u) := (1+t)^{-\alpha_1}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{\beta^{r_1}_{\alpha_1,\infty,\sigma_1}-\alpha_1}\|u(t,\cdot)\|_{L^\infty},$$
$$M_2(t,v) = (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{\beta^{r_2}_{\alpha_2,\infty,\sigma_2}-\alpha_2}\|v(t,\cdot)\|_{L^\infty},$$

where $\beta^r_{\alpha,\infty,\sigma} = \frac{n(1+\alpha)}{2\sigma r}$. For any $(u,v) \in X(T)$ the operator N

$$N \ : \ (u,v) \in X(t) \to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x) + (u,v)^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (4.1) as a fixed point of the operator N. We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.34}$$

$$\|N(u,v) - N(\overline{u},\overline{v})\|_{X(T)} \lesssim \|(u,v) - (\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)} + \|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big) \tag{4.35}$$

for any (u,v) and $(\overline{u},\overline{v}) \in X(T)$, uniformly with respect to $T \in [0,\infty)$. The estimates (4.34), (4.35) lead to a local (in time) well-posedness result for large data Sobolev solutions and a global (in time) well-posedness result for samll data Soboelv solutions as well.
Using the definition of the norm in $X(T)$ and Proposition 21, we may conclude

$$\|(u,v)^{ln}\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}}.$$

Hence, in order to complete the proof of (4.26) it is reasonable to show the following inequality:

$$\|(u,v)^{nl}\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}.$$

If $(u,v) \in X(T)$, then by interpolation we derive

$$\begin{aligned} &\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r_1,\infty],\\ &\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_2-\beta^{r_2}_{\alpha_2,q,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad q\in[r_2,\infty]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \||u(t,\cdot)|^{p_2}\|_{L^q} &\le \|u(t,\cdot)\|^{p_2}_{L^{p_2q}} \lesssim (1+t)^{-p_2(\beta^{r_1}_{\alpha_1,p_2q,\sigma_1}-\alpha_1)}\|(u,v)\|^{p_2}_{X(T)}\\ &\lesssim (1+t)^{-p_2(\beta^{r_1}_{\alpha_1,p_2,\sigma_1}-\alpha_1)}\|(u,v)\|^{p_2}_{X(T)} \end{aligned} \tag{4.36}$$

,

for any q such that $p_2q\in[r_1,\infty]$ and due to $\beta^{r_1}_{\alpha_1,p_2q,\sigma_1} \ge \beta^{r_1}_{\alpha_1,p_2,\sigma_1}$.
Also

$$\begin{aligned} \||v(t,\cdot)|^{p_1}\|_{L^q} &\le \|v(t,\cdot)\|^{p_1}_{L^{p_1q}} \lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1q,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)}\\ &\lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \end{aligned} \tag{4.37}$$

for any q such that $p_1q\in[r_2,\infty]$ and due to $\beta^{r_2}_{\alpha_2,p_1q,\sigma_2} \ge \beta^{r_2}_{\alpha_2,p_1,\sigma_2}$. Thanks to (4.37) we have for $q\in[r_1,\infty]$ the estimates

$$\begin{aligned} \|u^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}\||v(s,\cdot)|^{p_1}\|_{L^q}\,ds\,d\tau\\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}\|v(s,\cdot)\|^{p_1}_{L^{p_1q}}\,ds\,d\tau\\ &\lesssim \|(u,v)\|^{p_1}_{X(T)}I_1(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_1q\in[r_2,\infty], \end{aligned}$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds\,d\tau. \tag{4.38}$$

We are interested in estimating the right-hand side of (4.38). For this we apply Lemma 2. We put

$$\omega(\tau) = \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds.$$

Thanks to Lemma 2 we obtain $\omega(\tau) \lesssim (1+\tau)^{\alpha_1-1}$ if we assume that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)>1$. We notice that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)>1$ if and only if

$$p_1 > p^{r_2}_{\alpha_2,\sigma_2}(n) := 1 + \frac{(n(r_2-1)+2\sigma_2 r_2)(1+\alpha_2)}{(n-2\sigma_2 r_2)(1+\alpha_2)+2\sigma r_2}$$

under the assumptions $1 \leq \sigma_2 < \frac{1+\alpha_2}{2\alpha_2}$ and $1 \leq r_2 < \frac{1+\alpha_2}{2\alpha_2\sigma_2}$.
On other hand, the conditions $q \in [r_1,\infty]$ and $p_1 q \in [r_2,\infty]$ imply $p_1 \geq \frac{r_2}{r_1}$.
Hence,

$$I_1(t) \lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}} \omega(\tau)\,d\tau \lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}} (1+\tau)^{\alpha_1-1}\,d\tau. \tag{4.39}$$

Once more we apply Lemma 2 to (4.39) to obtain $I_1(t) \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}}$.
Hence,

$$\|(u^{nl},0)\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)}.$$

By the same way, we have for $q \in [r_2,\infty]$

$$\|v^{nl}(t,\cdot)\|_{L^q} \lesssim \|(u,v)\|^{p_2}_{X(T)} I_2(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_2 q \in [r_1,\infty],$$

where

$$I_2(t) = \int_0^t (1+t-\tau)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}} \int_0^\tau (\tau-s)^{\alpha_2-1}(1+s)^{-p_2(\beta^{r_1}_{\alpha_1,p_2,\sigma_1}-\alpha_1)}\,ds\,d\tau. \tag{4.40}$$

If

$$p_2 > p^{r_1}_{\alpha_1,\sigma_1}(n) := 1 + \frac{(n(r_1-1)+2\sigma_1 r_1)(1+\alpha_1)}{(n-2\sigma_1 r_1)(1+\alpha_1)+2\sigma r_1},$$

then

$$I_2(t) \lesssim (1+t)^{\alpha_2-\beta^{r_2}_{\alpha_2,q,\sigma_2}}.$$

Hence,

$$\|((0,v^{nl})\|_{X(T)} \lesssim \|(u,v)\|^{p_2}_{X(T)}.$$

Also, the conditions $q \in [r_2,\infty]$ and $p_2 q \in [r_1,\infty]$ imply $p_2 \geq \frac{r_1}{r_2}$.
The proof of (4.35) is similar as the proof of (4.20) of Theorem 4.1.
The proof is complete.

Proof of Theorem 4.4

We introduce for all $T>0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T}\Big\{M_1(t,u)+M_2(t,v)\Big\},$$

where

$$M_1(t,u) := (1+t)^{-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{\beta^{r_1}_{\alpha_1,\infty,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|u(t,\cdot)\|_{L^\infty},$$
$$M_2(t,v) = (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{\beta^{r_2}_{\alpha_2,\infty,\sigma_2}-\alpha_2}\|v(t,\cdot)\|_{L^\infty},$$

where $\beta^r_{\alpha,\infty,\sigma} = \frac{n(1+\alpha)}{2\sigma r}, \gamma^{r_2}_{\alpha_2,p_1,\sigma_2} = 1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)$.

For any $(u,v)\in X(T)$ the operator N

$$N\ :\ (u,v)\in X(t)\to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x)+(u,v)^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (4.1) as a fixed point of the operator N. We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.41}$$
$$\begin{aligned}\|N(u,v)-N(\overline{u},\overline{v})\|_{X(T)} &\lesssim \|(u,v)-(\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)}\\ &\quad+\|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big)\end{aligned} \tag{4.42}$$

for any (u,v) and $(\overline{u},\overline{v})\in X(T)$, uniformly with respect to $T\in[0,\infty)$. The estimates (4.41), (4.42) lead to a local (in time) well-posedness result for large data Sobolev solutions and a global (in time) well-posedness result for samll data Soboelv solutions as well.
Using the definition of the norm in $X(T)$ and Proposition 21, we may conclude

$$\|(u,v)^{ln}\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}}.$$

Hence, in order to complete the proof of (4.41) it is reasonable to show the following inequality:

$$\|(u,v)^{nl}\|_{X(T)} \lesssim \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}.$$

If $(u,v)\in X(T)$, then by interpolation we derive for all $t\geq 0$

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q\in[r_1,\infty],$$
$$\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{\alpha_2-\beta^{r_2}_{\alpha_2,q,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q\in[r_2,\infty].$$

On the other hand, we have

$$\begin{aligned}\||u(t,\cdot)|^{p_2}\|_{L^q} &\leq \|u(t,\cdot)\|^{p_2}_{L^{p_2q}} \lesssim (1+t)^{-p_2(\beta^{r_1}_{\alpha_1,p_2q,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)}\\ &\lesssim (1+t)^{-p_2(\beta^{r_1}_{\alpha_1,p_2,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)},\end{aligned} \tag{4.43}$$

for any q such that $p_2q \in [r_1,\infty]$ and due to $\beta^{r_1}_{\alpha_1,p_2q,\sigma_1} \geq \beta^{r_1}_{\alpha_1,p_2,\sigma_1}$.
Also

$$\begin{aligned} \||v(t,\cdot)|^{p_1}\|_{L^q} &\leq \|v(t,\cdot)\|^{p_1}_{L^{p_1q}} \lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1q,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \\ &\lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \end{aligned} \tag{4.44}$$

for any q such that $p_1q \in [r_2,\infty]$ and due to $\beta^{r_2}_{\alpha_2,p_1q,\sigma_2} \geq \beta^{r_2}_{\alpha_2,p_1,\sigma_2}$. Thanks to (4.44) we have for $q \in [r_1,\infty]$ the estimates

$$\begin{aligned} \|u^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}\||v(s,\cdot)|^{p_1}\|_{L^q}\,ds\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}\|v(s,\cdot)\|^{p_1}_{L^{p_1q}}\,ds\,d\tau \\ &\lesssim \|(u,v)\|^{p_1}_{X(T)} I_1(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_1q\in[r_2,\infty], \end{aligned}$$

where

$$I_1(t) = \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}} \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds\,d\tau. \tag{4.45}$$

We are interested in estimating the right-hand side of (4.45). For this we apply Lemma 2. We put

$$\omega(\tau) = \int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds.$$

Thanks to Lemma 2 we obtain $\omega(\tau) \lesssim (1+\tau)^{\alpha_1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}$ if we assume that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2} - \alpha_2) < 1$. We notice that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2} - \alpha_2) < 1$ if and only if

$$p_1 < p^{r_2}_{\alpha_2,\sigma_2}(n) := 1 + \frac{(n(r_2-1)+2\sigma_2r_2)(1+\alpha_2)}{(n-2\sigma_2r_2)(1+\alpha_2)+2\sigma r_2}$$

under the assumptions $1 \leq \sigma_2 < \frac{1+\alpha_2}{2\alpha_2}$ and $1 \leq r_2 < \frac{1+\alpha_2}{2\alpha_2\sigma_2}$.
On other hand, the conditions $q \in [r_1,\infty]$ and $p_1q \in [r_2,\infty]$ imply $p_1 \geq \frac{r_2}{r_1}$.
Hence,

$$\begin{aligned} I_1(t) &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}}\omega(\tau)\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1}_{\alpha_1,q,\sigma_1}}(1+\tau)^{\alpha_1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,d\tau. \end{aligned} \tag{4.46}$$

Once more we apply Lemma 2 to (4.46) to obtain

$$I_1(t) \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}+1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)} \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}.$$

Hence,

$$\|(u^{nl},0)\|_{X(T)} \lesssim (1+t)^{\alpha_1-\beta^{r_1}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|(u,v)\|^{p_1}_{X(T)}.$$

By the same way, we have for $q \in [r_2, \infty]$

$$\|v^{nl}(t,\cdot)\|_{L^q} \lesssim \|(u,v)\|^{p_2}_{X(T)} I_2(t) \quad \text{for all} \quad t \in [0,T] \quad \text{and} \quad p_2 q \in [r_1, \infty],$$

where

$$I_2(t) = \int_0^t (1+t-\tau)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}} \int_0^\tau (\tau-s)^{\alpha_2-1}(1+s)^{-p_2(\beta^{r_1}_{\alpha_1,p_2,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\, ds\, d\tau. \tag{4.47}$$

If

$$p_2(\beta^{r_1}_{\alpha_1,p_2,\sigma_1} - \alpha_1 - 1 + p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2} - \alpha_2)) > 1, \tag{4.48}$$

Then

$$I_2(t) \lesssim (1+t)^{\alpha_2 - \beta^{r_2}_{\alpha_2,q,\sigma_2}}.$$

The condition (4.48) is equivalent to

$$p_1 > r_2 \frac{n[(1+\alpha_2)\sigma_1 r_1 - (1+\alpha_1)\sigma_2] + 2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2) - 2\alpha_2\sigma_2 r_2)}$$

and

$$p_2 > \frac{1}{2\sigma_1} \frac{2\sigma_1 + n(1+\alpha_1)}{n[(1+\alpha_1)\sigma_2 r_2 + \sigma_1 r_1(1+\alpha_2)(p_1 - r_2)] - 2\sigma_1 r_1 \sigma_2 r_2(p_1+\alpha_1+1)}.$$

Hence,

$$\|((0, v^{nl})\|_{X(T)} \lesssim \|(u,v)\|^{p_2}_{X(T)}.$$

Also, the conditions $q \in [r_2, \infty]$ and $p_2 q \in [r_1, \infty]$ imply $p_2 \geq \frac{r_1}{r_2}$.
The proof of (4.42) is similar as the proof of (4.20) of Theorem 4.1.
The proof is complete.

4.3.3 Proof main results for the case $\min\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\} \leq n < \max\left\{\frac{2\sigma_1 r_1}{1+\alpha_1}, \frac{2\sigma_2 r_2}{1+\alpha_2}\right\}$

Proof of Theorem 4.5

For any $n \geq \frac{2\sigma_1 r_1}{1+\alpha_1}$ and $\delta \in (0,1)$ is sufficiently small, there exists a parameter $\overline{q_1} = \overline{q_1}(\delta) \in (r_1, \infty)$ such that

$$\frac{n(1+\alpha_1)}{2\sigma_1}\Big(\frac{1}{r_1} - \frac{1}{\overline{q_1}}\Big) = 1 - \delta. \tag{4.49}$$

If $n < \frac{2\sigma_2 r_2}{1+\alpha_2}$, then for all $q \in [r_2, \infty]$ we obtain

$$\frac{n(1+\alpha_2)}{2\sigma_2}\Big(\frac{1}{r_2} - \frac{1}{q}\Big) < 1 - \frac{n(1+\alpha_2)}{2q\sigma_2} \leq 1. \tag{4.50}$$

Hence, we can choose a positive δ such that there does not exist any $\overline{q} \in [r_2, \infty]$ which satisfies (4.18). For this reason,

$$\beta^{r_2,\delta}_{\alpha_2,q,\sigma_2} = \beta^{r_2}_{\alpha_2,q,\sigma_2} := \frac{n(1+\alpha_2)}{2\sigma_2}\Big(\frac{1}{r_2} - \frac{1}{q}\Big).$$

We introduce for all $T > 0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T} \Big\{ M_1(t,u) + M_2(t,v)\Big\},$$

where

$$M_1(t,u) := (1+t)^{-\alpha_1}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{1-\delta-\alpha_1}\big(\|u(t,\cdot)\|_{L^{\overline{q_1}}} + \|u(t,\cdot)\|_{L^\infty}\big),$$
$$M_2(t,v) = (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{\beta^{r_2}_{\alpha_2,\infty,\sigma_2}-\alpha_2}\|v(t,\cdot)\|_{L^\infty},$$

where $\beta^r_{\alpha_2,\infty,\sigma_2} = \frac{n(1+\alpha_2)}{2\sigma r_2}$, and the operator N by

$$N \ : \ (u,v) \in X(t) \to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x) + (u,v)^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (4.1) as a fixed point of the operator N. We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.51}$$

$$\|N(u,v) - N(\overline{u},\overline{v})\|_{X(T)} \lesssim \|(u,v) - (\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)} + \|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big) \tag{4.52}$$

The proof of (4.51) and (4.52) is similar to the proof in Theorem 4.1 and Theorem 4.3. This completes the proof.

Proof of Theorem 4.6

We introduce for all $T > 0$ the space $X(T)$ as follows:

$$X(T) := C\big([0,\infty), L^{r_1}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \times C\big([0,\infty), L^{r_2}(\mathbb{R}^n)\big) \cap L^\infty(\mathbb{R}^n)\big)$$

with the norm

$$\|(u,v)\|_{X(T)} := \sup_{0\leqslant t\leqslant T} \Big\{ M_1(t,u) + M_2(t,v)\Big\},$$

where

$$\begin{aligned} M_1(t,u) &:= (1+t)^{-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|u(t,\cdot)\|_{L^{r_1}} + (1+t)^{1-\delta-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}} \\ &\qquad \times \big(\|u(t,\cdot)\|_{L^{\overline{q_1}}} + \|u(t,\cdot)\|_{L^\infty}\big), \\ M_2(t,v) &:= (1+t)^{-\alpha_2}\|v(t,\cdot)\|_{L^{r_2}} + (1+t)^{\beta^{r_2}_{\alpha_2,\infty,\sigma_2}-\alpha_2}\|v(t,\cdot)\|_{L^\infty}, \end{aligned}$$

where $\beta^{r}_{\alpha_2,\infty,\sigma_2} = \frac{n(1+\alpha_2)}{2\sigma r_2}$. The operator N is defined by

$$N \;:\; (u,v)\in X(t) \to N(u,v) = N(u,v)(t,x) := (u,v)^{ln}(t,x) + (u,v)^{nl}(t,x).$$

To get the global (in time) existence and uniqueness of Sobolev solutions in $X(T)$, we can consider a global (in time) Sobolev solution to (4.1) as a fixed point of the operator N. We will prove that the operator N satisfies the following two inequalities:

$$\|N(u,v)\|_{X(T)} \lesssim \|(u_0,v_0)\|_{\mathcal{A}^{r_2}_{r_1}} + \|(u,v)\|^{p_1}_{X(T)} + \|(u,v)\|^{p_2}_{X(T)}, \tag{4.53}$$

$$\begin{aligned} \|N(u,v)-N(\overline{u},\overline{v})\|_{X(T)} &\lesssim \|(u,v)-(\overline{u},\overline{v})\|_{X(T)}\big(\|(u,v)\|^{p_1-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_1-1}_{X(T)} \\ &\quad + \|(u,v)\|^{p_2-1}_{X(T)} + \|(\overline{u},\overline{v})\|^{p_2-1}_{X(T)}\big) \end{aligned} \tag{4.54}$$

If $(u,v)\in X(T)$, then by interpolation we derive for all $t\geq 0$

$$\begin{aligned} \|u(t,\cdot)\|_{L^q} &\lesssim (1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q\in[r_1,\infty], \\ \|v(t,\cdot)\|_{L^q} &\lesssim (1+t)^{\alpha_2-\beta^{r_2}_{\alpha_2,q,\sigma_2}}\|(u,v)\|_{X(T)} \quad \text{for all} \quad q\in[r_2,\infty]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \||u(t,\cdot)|^{p_2}\|_{L^q} &\leq \|u(t,\cdot)\|^{p_2}_{L^{p_2q}} \lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)} \\ &\lesssim (1+t)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\|(u,v)\|^{p_2}_{X(T)} \end{aligned} \tag{4.55}$$

,

for any q such that $p_2q\in[r_1,\infty]$ and due to $\beta^{r_1,\delta}_{\alpha_1,p_2q,\sigma_1}\geq\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}$.
Also

$$\begin{aligned} \||v(t,\cdot)|^{p_1}\|_{L^q} &\leq \|v(t,\cdot)\|^{p_1}_{L^{p_1q}} \lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1q,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \\ &\lesssim (1+t)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\|(u,v)\|^{p_1}_{X(T)} \end{aligned} \tag{4.56}$$

for any q such that $p_1q\in[r_2,\infty]$ and due to $\beta^{r_2}_{\alpha_2,p_1q,\sigma_2}\geq\beta^{r_2}_{\alpha_2,p_1,\sigma_2}$. Thanks to (4.56) we have for $q\in[r_1,\infty]$ the estimates

$$\begin{aligned} \|u^{nl}(t,\cdot)\|_{L^q} &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau(\tau-s)^{\alpha_1-1}\||v(s,\cdot)|^{p_1}\|_{L^q}\,ds\,d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau(\tau-s)^{\alpha_1-1}\|v(s,\cdot)\|^{p_1}_{L^{p_1q}}\,ds\,d\tau \\ &\lesssim \|(u,v)\|^{p_1}_{X(T)}I_1(t) \quad \text{for all} \quad t\in[0,T] \quad \text{and} \quad p_1q\in[r_2,\infty], \end{aligned}$$

where

$$I_1(t)=\int_0^t (1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds\,d\tau. \tag{4.57}$$

We are interested in estimating the right-hand side of (4.57). For this we apply Lemma 2. We put

$$\omega(\tau)=\int_0^\tau (\tau-s)^{\alpha_1-1}(1+s)^{-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,ds.$$

Thanks to Lemma 2 we obtain $\omega(\tau)\lesssim(1+\tau)^{\alpha_1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}$ if we assume that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)<1$. We notice that $p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)<1$ if and only if

$$p_1<p^{r_2}_{\alpha_2,\sigma_2}(n):=1+\frac{(n(r_2-1)+2\sigma_2r_2)(1+\alpha_2)}{(n-2\sigma_2r_2)(1+\alpha_2)+2\sigma_2r_2}$$

under the assumptions $1\le\sigma_2<\frac{1+\alpha_2}{2\alpha_2}$ and $1\le r_2<\frac{1+\alpha_2}{2\alpha_2\sigma_2}$.

On other hand, the conditions $q\in[r_1,\infty]$ and $p_1q\in[r_2,\infty]$ imply $p_1\ge\frac{r_2}{r_1}$.
Hence,

$$\begin{aligned}I_1(t)&\lesssim\int_0^t(1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}\omega(\tau)\,d\tau\\&\lesssim\int_0^t(1+t-\tau)^{-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}}(1+\tau)^{\alpha_1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\,d\tau.\end{aligned} \tag{4.58}$$

Once more we apply Lemma 2 to (4.58) to obtain

$$I_1(t)\lesssim(1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+1-p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2)}\lesssim(1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}.$$

Hence,

$$\|(u^{nl},0)\|_{X(T)}\lesssim(1+t)^{\alpha_1-\beta^{r_1,\delta}_{\alpha_1,q,\sigma_1}+\gamma^{r_2}_{\alpha_2,p_1,\sigma_2}}\|(u,v)\|^{p_1}_{X(T)}.$$

By the same way, we have for $q\in[r_2,\infty]$

$$\|v^{nl}(t,\cdot)\|_{L^q}\lesssim\|(u,v)\|^{p_2}_{X(T)}I_2(t)\quad\text{for all}\quad t\in[0,T]\quad\text{and}\quad p_2q\in[r_1,\infty],$$

where

$$I_2(t)=\int_0^t(1+t-\tau)^{-\beta^{r_2}_{\alpha_2,q,\sigma_2}}\int_0^\tau(\tau-s)^{\alpha_2-1}(1+s)^{-p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1-\gamma^{r_2}_{\alpha_2,p_1,\sigma_2})}\,ds\,d\tau. \tag{4.59}$$

If

$$p_2(\beta^{r_1,\delta}_{\alpha_1,p_2,\sigma_1}-\alpha_1-1+p_1(\beta^{r_2}_{\alpha_2,p_1,\sigma_2}-\alpha_2))>1, \tag{4.60}$$

then

$$I_2(t) \lesssim (1+t)^{\alpha_2-\beta^{r_2}_{\alpha_2,q,\sigma_2}}.$$

The condition (4.60) is equivalent to

$$p_1 > \max\Big\{\frac{2\sigma_2 r_2(\delta+\alpha_1)+n(1+\alpha_2)r_2}{n(1+\alpha_2)-2\sigma_2 r_2\alpha_2}; \frac{r_2}{r_1}-\epsilon; 1$$
$$; r_2\frac{n[(1+\alpha_2)\sigma_1 r_1-(1+\alpha_1)\sigma_2]+2\sigma_1 r_1(\alpha_1+1)\sigma_2}{\sigma_1 r_1(n(1+\alpha_2)-2\alpha_2\sigma_2 r_2)}\Big\},$$

and

$$p_2 > \max\Big\{\frac{2\sigma_2 r_2}{n(\alpha_2+1)(p_1-r_2)-2\sigma_2 r_2(\delta+\alpha_1+p_1\alpha_2)}$$
$$; \sigma_2 r_2\frac{2\sigma_1 r_1+n r_1(\alpha_1+1)}{n[\sigma_2 r_2(\alpha_1+1)+\sigma_1 r_1(\alpha_2+1)(p_1-r_2)]-2\sigma_2 r_2\sigma_1 r_1[(\alpha_1+1)+p_1\alpha_2)]}\Big\},$$

The proof of (4.54) is similar as the proof of (4.20) of Theorem 4.6

CHAPTER 5

APPENDIX

In this chapter we present results that we have already used in the proof of results of the previous chapters.

5.0.1 Useful Lemmas

Lemma 3 *(see [2])Let g be a function defined on $\mathbb{R}^+$ with real values. Suppose there exist two positive constants α_1 and α_2 such that :*

$$\begin{cases} f(t) \leq C_0 & \text{if } 0 \leq t \leq t_0 \\ f(t) \leq C_1 t^{-\alpha} & \text{if } t \geq t_0. \end{cases}$$

Where α and t_0 are positive reals. So there is a positive constant $C_2 = (\alpha, t_0, C_0, C_1)$ such that:

$$f(t) \leq C_2(1+t)^{-\alpha}$$

Proof: in first the case : If $0 \leq t \leq t_0$. Then we get :

$$\begin{aligned} t \leq t_0 \quad &\Rightarrow \quad (1+t_0)^{-\alpha} \leq (1+t)^{-\alpha} \\ &\Rightarrow \quad 1 \leq (1+t_0)^{\alpha}(1+t)^{-\alpha} \\ &\Rightarrow \quad C_0 \leq C_0(1+t_0)^{\alpha}(1+t)^{-\alpha} \\ &\Rightarrow \quad f(t) \leq C_0(1+t_0)^{\alpha}(1+t)^{-\alpha} \\ &\Rightarrow \quad f(t) \leq C_3(1+t)^{-\alpha}, \end{aligned}$$

with $C_3 = C_0(1+t_0)^{\alpha}$.

in the second case : If $t \geq t_0$.Consequently, we obtain

$$\begin{aligned} t \geq t_0 \quad &\Rightarrow \quad \frac{1}{t} \leq \frac{1}{t_0} \Rightarrow \left(1+\frac{1}{t_0}\right)^{-\alpha} \leq \left(1+\frac{1}{t}\right)^{-\alpha} \\ &\Rightarrow \quad t^{-\alpha} \leq \left(1+\frac{1}{t_0}\right)^{\alpha} (1+t)^{-\alpha} \\ &\Rightarrow \quad f(t) \leq C_1 \left(1+\frac{1}{t_0}\right)^{\alpha} (1+t)^{-\alpha} \\ &\Rightarrow \quad f(t) \leq C_4(1+t)^{-\alpha}, \end{aligned}$$

where $C_4 = C_1\left(1+\frac{1}{t_0}\right)^{\alpha}$. So, it is sufficient to take $C_2 = \max\{C_3, C_4\}$.

Lemma 4 *[2] Let $0 \leq \theta < 1$, $a \geq 0$, and $b \geq 0$. Then there exists a constant $C > 0$ depending only on a, b, and θ such that for all $t > 0$, the following inequality holds:*

$$\begin{aligned} &\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau \\ &\qquad \leq \begin{cases} C(1+t)^{-\min\{a+\theta,b\}}, & if \max\{a+\theta,b\} > 1, \\ C(1+t)^{-\min\{a+\theta,b\}}\ln(2+t), & if\ \max\{a+\theta,b\} = 1, \\ C(1+t)^{1-a-\theta-b}, & if \max\{a+\theta,b\} < 1. \end{cases} \end{aligned} \tag{5.1}$$

Proof: First, we assume that $t \leq 2$, Then :

$$\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \leq \int_0^t (t-\tau)^{-\theta}d\tau = \frac{t^{1-\theta}}{1-\theta} \leq \frac{2^{1-\theta}}{1-\theta}. \tag{5.2}$$

Next, we assume $t \geq 2$. Then :

$$\int_0^t (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau = \int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t .$$

For the first integral on the right side, we have :

$$\int_0^{\frac{t}{2}} (t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \leq \begin{cases} \left(\frac{t}{2}\right)^{-(a+\theta)} \int_0^{\infty}(1+\tau)^{-b}d\tau, & \text{if } b > 1, \\ \left(\frac{t}{2}\right)^{-(a+\theta)} \ln\left(1+\frac{t}{2}\right), & \text{if } b = 1, \\ \frac{1}{1-b}\left(\frac{t}{2}\right)^{-(a+\theta)}\left(1+\frac{t}{2}\right)^{1-b}, & \text{if } b < 1. \end{cases} \tag{5.3}$$

For the second integral, we have :

$$\int_{\frac{t}{2}}^{t}(t-\tau)^{-\theta}(1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \leq \left(1+\frac{t}{2}\right)^{-b}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\theta}(1+t-\tau)^{-a}d\tau$$

$$= \frac{1}{1-\theta}\left(\frac{t}{2}\right)^{1-\theta}\left(1+\frac{t}{2}\right)^{-a-b} + \frac{a}{1-\theta}\left(1+\frac{t}{2}\right)^{-b}\int_{0}^{\frac{t}{2}}\tau^{1-\theta}(1+\tau)^{-a-1}d\tau$$

$$\leq \frac{1}{1-\theta}\left(\frac{t}{2}\right)^{1-\theta}\left(1+\frac{t}{2}\right)^{-a-b} + \frac{a}{1-\theta}\left(1+\frac{t}{2}\right)^{-b}\int_{0}^{\frac{t}{2}}(1+\tau)^{-a-\theta}d\tau$$

$$\leq \begin{cases} \frac{1}{1-\theta}\left(\frac{t}{2}\right)^{1-\theta-a-b} + \frac{a}{1-\theta}\left(\frac{t}{2}\right)^{-b}\int_{0}^{\infty}(1+\tau)^{-a-\theta}d\tau, & \text{if } a+\theta>1, \\ \frac{1}{1-\theta}\left(\frac{t}{2}\right)^{1-\theta-a-b} + \frac{a}{1-\theta}\left(\frac{t}{2}\right)^{-b}\ln\left(1+\frac{t}{2}\right), & \text{if } a+\theta=1, \\ \frac{1}{1-\theta}\left(\frac{t}{2}\right)^{1-\theta-a-b} + \frac{a}{(1-\theta)(1-\theta-a)}\left(1+\frac{t}{2}\right)^{1-\theta-a-b}, & \text{if } a+\theta<1. \end{cases} \tag{5.4}$$

Now,(5.1)follows easily from (5.2)-(5.4).Q.E.D.

5.0.2 Some fixed point arguments

Proposition 22 *The operator N admits a unique fixed point $u \in X(t)$ when the following conditions hold:*

$$||Nu||_{X(T)} \leq C_0(t)||(u_0,u_1)||_{\mathcal{A}} + C_1(t)||u||^p_{X(T)}, \tag{5.5}$$

$$||Nu-Nv||_{X(T)} \leq C_2||u-v||_{X(T)}\big(||u||^{p-1}_{X(T)} + ||v||^{p-1}_{X(T)}\big), \tag{5.6}$$

Where $C_0(t), C_1(t), C_2(t) \longrightarrow 0$ for $t \longrightarrow +0$ and $C_0(t), C_1(t), C_2(t) \leq C$ for all $t \in [0,\infty)$

Proof: Let $L(t) := ||u||_{X(T)}$. We prove that for any $\xi \in [0,\xi_0[$ we have

$$L(t) \leq C_0\xi + C_1(L(t))^p \text{ implies } L(t) \leq 2C_0\xi.$$

Let $\phi(x) = x - C_1x^p$.Then $\phi(x) \leq x$ for any $x \geq 0$ and $\phi'(x) \geq \frac{1}{2}$ for $x \in [0,\overline{x}]$, where $\overline{x} = (\frac{1}{2C_1p})^{\frac{1}{p-1}}$. Consequently,

$$\phi(x) \leq x \leq 2\phi(x) \text{ for } x \in [0,\overline{x}].$$

Let $\xi_0 = min(\overline{x}, \frac{\overline{x}}{2C_0})$. Then $L(0) \leq \overline{x}$ for $||(u_0, u_1)||_{\mathcal{A}} = \xi$. Thanks to (5.5) we get

$$\phi(L(t)) \leq C_0\xi \leq C_0\xi_0 \leq \frac{\overline{x}}{2} \leq \phi(\overline{x}).$$

So, $L(t) \in [0, \overline{x}]$ and $L(t) \leq 2\phi(L(t)) \leq 2C_0\xi$. This proves that N maps $X(t)$ into itself
Let us define the sequence $(u_j)_{j\geq 0}$ inductively by

$$u_0 \equiv 0 \text{ and } u_j \equiv N(u_{j-1}).$$

Then $||u_j||_{X(T)} \leq 2C_0\xi = C\xi$, where $\xi \in [0, \xi_0]$ and C is independent of t. Applying (5.6) we have for $\xi \leq \frac{1}{2C}$ sufficiently small the following estimate:

$$\begin{aligned} ||u_{j+1} - u_j||_{X(T)} &= ||N(u_j) - N(u_{j-1})||_{X(T)} \\ &\leq \frac{1}{2}||u_j - u_{j-1}||_{X(T)} \leq \cdots \leq \frac{1}{2^j}||u_1||_{X(T)}. \end{aligned}$$

This implies that $(u_j)_j$ is a Cauchy sequence in the Banach space $X(T)$. This sequence converges to the unique fixed point $N(u) = u$.

5.0.3 Useful inequalities

Young's inequality

Lemma 5 *Let be $f \in L^p(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$ with $1 \leq p, r \leq \infty$ then $f * g \in L^q(\mathbb{R}^n)$. Where: $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ and*

$$||f * g||_{L^q} \leq ||f||_{L^p}||g||_{L^r}$$

.

5.0.4 Duhamel's principle

(see [23]) The classical Duhamel principle reduces the Cauchy problem for an inhomogeneous partial differential equation to Cauchy problem for corresponding homogeneous equation. In [23, 24], authors have established fractional analog of Duhamel principle. Here we introduce a simple version of fractional Duhamel principle that helps us to reduce the problem (5.7) to an equivalent integral equation.

$$\begin{cases} \partial_t^{1+\alpha} u + (-\Delta)^\sigma u + m^2 u = |u|^p, \\ u(0,x) = u_0(x), \quad u_t(0,x) = 0. \end{cases} \tag{5.7}$$

where $\alpha \in (0,1)$, $m > 0$, $p > 1$, $(t,x) \in [0,\infty) \times \mathbb{R}^n$, $\partial_t^{1+\alpha} u = D_t^\alpha(u_t)$ with

$$D_t^\alpha(f) = \partial_t(I_t^{1-\alpha} f) \;\; \textit{and} \;\; I_t^\beta f = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s), \quad \text{for} \quad \beta > 0.$$

for any fractional numbers $\sigma \geq 1$ and for the nonlinearities with powers $p, q > 1$. The operator $(-\Delta)^\sigma$ is defined by

$$((-\Delta)^\sigma u))(x) = F^{-1}_{\xi\to x}(|\xi|^{2\sigma} F_{x\to\xi}(u)(\xi))(x),$$

where F represents the usual Fourier transform in L^1.

Consider the initial value problem for linear inhomogeneous diffusion-wave equation (5.7) By linear properties of the operator $Lu := \partial_t^{1+\alpha} u(x,t) + (-\Delta)^\sigma u(x,t) + m^2 u(x,t)$ the solution $u(x,t)$ of (5.7) is the sum of solutions $v(x,t)$ and $w(x,t)$ of problems

$$\begin{cases} \partial_t^{1+\alpha} v(x,t) + (-\Delta)^\sigma v(x,t) + m^2 v(x,t) = 0, \\ v(0,x) = u_0(x), \quad v_t(0,x) = 0. \end{cases} \tag{5.8}$$

and

$$\begin{cases} \partial_t^{1+\alpha} w(x,t) + (-\Delta)^\sigma w(x,t) + m^2 w(x,t) = |u|^p, \\ w(0,x) = 0, \quad w_t(0,x) = 0. \end{cases} \tag{5.9}$$

The representation of the solution of the linear integro-differential equation associated to (5.8) with $\sigma \geq 1$ and $m \geq 0$ (and without the term $|u|^p$) is given by

$$v(t,x) = (G^m_{\alpha,\sigma}(t) * u_0)(t,x). \tag{5.10}$$

Consider the cauchy problem (5.9) with homogeneous initial conditions

Further, let a sufficiently smooth function $q(x,t,\tau)$, $t \geq \tau$, $\tau \geq 0$, $x \in \mathbb{R}^n$, be for $t > \tau$ a solution of the homogeneous equation

$$\partial_t^{1+\alpha} q(x,t,\tau) + (-\Delta)^\sigma q(x,t,\tau) + m^2 q(x,t,\tau) = 0, \tag{5.11}$$

satisfying the following conditions ; Where $F(x,t) = |v|^p$:

$$q(x,t,\tau)|_{t=\tau} = 0, \quad \frac{\partial q}{\partial t}(x,t,\tau)|_{t=\tau} = I^\alpha_\tau(F(x,\tau)). \tag{5.12}$$

Then a solution of the Cauchy problem (5.9) is given by means of the Duhamel's integral

$$w(x,t) = \int_0^t q(x,t,\tau)\, d\tau.$$

The formulated statement has been entitled as the "Duhamel's principle".

A function $u \in X$, where X is a suitable space, is a solution to (5.7) if, and only if, it satisfies the equality $u(t,x) = u^{ln}(t,x) + u^{nl}(t,x)$, *in* X, where we set $u^{ln} = u^{hom} = v$, $u^{nl} = w$, so that

$$u^{nl}(t,x) = \int_0^t \left(G(t-s) * I^\alpha_s\left(|u|^p\right)\right)(t,s,x) ds.$$

with

$$G(t,x)=\int_{\mathbb{R}^n} e^{ix\cdot\xi} E_{\alpha+1}\left(-t^{\alpha+1}\langle\xi\rangle_{m,\sigma}^2\right)d\xi,$$

where $\{G^m_{\alpha,\sigma}(t)\}_{t\geq 0}$ denotes the semigroup of operators which is defined via Fourier transform by $(\widehat{G^m_{\alpha,\sigma}(t)*f})(t,\xi)=E_{\alpha+1}\left(-t^{\alpha+1}\langle\xi\rangle_{m,\sigma}^2\right)\widehat{f}(\xi)$ with $\langle\xi\rangle^2_{m,\sigma}=|\xi|^{2\sigma}+m^2$.

Here

$$E_\beta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)},\ \beta\in\mathbb{C}\ \ with\ \ \mathrm{Re}(\beta)>0.$$

5.0.5 Some properties of Mittag-Leffler function

The Mittag-Leffler function E_β allows the following implicit definition:

$$\frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}E_\beta(\lambda s^\beta)ds=E_\beta(\lambda t^\beta)-1. \tag{5.13}$$

The Mittag-Leffler function $E_\beta(-t^\beta\langle\xi\rangle_m^2)$ with

$$E_\beta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)},\beta\in\mathbb{C}\ avec\ Re\beta>0,$$

can be decomposed in the following form :

$$E_\beta(-t^\beta\langle\xi\rangle_m^2)=\tfrac{1}{\beta}\Big(\exp(a_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_m))+\exp(b_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_m))\Big)+l_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_m).$$

Or

$$a_\beta(y)=y^{\frac{2}{\beta}}\exp(\frac{\pi i}{\beta})\ for\ y\geqslant 0,$$

$$b_\beta(y)=y^{\frac{2}{\beta}}\exp(-\frac{\pi i}{\beta})\ for\ y\geqslant 0,$$

and

$$l_\beta(y)=\begin{cases}\frac{\sin(\beta\pi)}{\pi}\int_0^\infty\dfrac{y^2s^{\beta-1}\exp(-s)}{s^{2\beta}+2y^2s^\beta\cos(\beta\pi)+y^4}ds=\frac{\sin(\beta\pi}{\beta\pi}\int_0^\infty\dfrac{\exp(-y^{\frac{2}{\beta}}s^{\frac{1}{\beta}})}{s^2+2s\cos(\beta\pi)+1}ds\quad for\,y>0,\\ \qquad 1-\frac{2}{\beta}\ for\ y=0.\end{cases} \tag{5.14}$$

$\beta=1+\alpha$ The proof is found in the document [8].

Remark 2 *We have also the relation*

$$\exp\left(a_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_m)\right)+\exp\left(b_\beta(t^{\frac{\beta}{2}}\langle\xi\rangle_m)\right)=2e^{t\langle\xi\rangle_m^{\frac{2}{1+\alpha}}\cos(\frac{\pi}{1+\alpha})}\cos\left(t\langle\xi\rangle_m^{\frac{2}{1+\alpha}}\sin\left(\frac{\pi}{1+\alpha}\right)\right)$$
$$=2e^{-ct\langle\xi\rangle_m^{\frac{2}{1+\alpha}}}\cos\left(t\langle\xi\rangle_m^{\frac{2}{1+\alpha}}\sqrt{1-c^2}\right),\quad \textit{where}\ c=-\cos\left(\frac{\pi}{1+\alpha}\right).$$

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Abstract

In this theisis, we are interested to study Global existence of small data solutions to some semi-linear σ-evolution models. The main goal of this study is to clarify the effect of the influence of parameters and the data onthe rang and qualitative properties of solutions. Using modified Bessel functions and the Mittag-Leffler function, we show some polynomial decay $L^m - L^q$ estimates of Sobolev solutions to related linear models with vanishing right-hand side. We explain connections between the fractional orders and the exponents, which allow to prove the global (in time) existence of small-data Sobolev solutions by applying the fixed-point argument.

الملخص

في هذه الأطروحة، نهتم بدراسة الوجود الكلي لحلول البيانات الصغيرة لبعض النماذج شبه الخطية للتطور من نوع سيغما. الهدف الرئيسي من هذه الدراسة هو توضيح تأثير المعاملات والبيانات على نطاق وخصائص الحلول النوعية. باستخدام دوال بيسل المعدلة ودالة ميتاغليفلر، نعرض بعض التقديرات متعددة الحدود من النوع $L^m - L^q$ لحلول سوبوليف للنماذج الخطية ذات الصلة مع طرف أيمن معدوم. نوضح الروابط بين الرتب الكسرية والأسس، مما يسمح بإثبات الوجود الكلي (بالنسبة الزمن) لحلول سوبوليف ذات البيانات الصغيرة من خلال تطبيق منهج النقطة الثابتة.