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Ministry of Higher Education and Scientific Research

Faculty of Exact Sciences and Computer Science

Department of Physics

Course Handout

Intended for students in the 2nd year of a Chemistry license

The Physics of Oscillations-Waves and Optics

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Preface

This handout presents oscillatory motions, propagation phenomena, and optical phenomena. It is particularly intended for second-year university students in sciences of matter (SM), particularly those in the Chemistry degree program, who are studying the "***Physics of Oscillations-Waves and optics***" module at the University of Chlef. This document brings together all the material from the lecture notes, as well as some corrected exercises. It has been prepared to minimize the difficulties inherent in scientific language while maintaining the required rigor. In addition, it provides students with benchmarks and the fundamentals of physics.

This handout is divided into eight chapters:

The first part (Chapters 1 to 4) deals with mechanical vibrations, concentrating on linear oscillators, their oscillation conditions and Lagrange's formalism. It deals with the notions of degrees of freedom and generalized coordinates, and is limited to low-amplitude linear oscillations. The study focuses on systems with one or two degrees of freedom, whose equations of motion are linear, making it possible to analyse phenomena such as resonance and to examine free and forced vibrations, with or without damping.

The second part (Chapters 5 and 6) deals with mechanical waves, introducing the fundamental notions of their propagation. The study begins with transverse waves on a string, covering the d'Alembert wave equation, the sinusoidal solution, the speed of the wave and the particle, and impedance. It concludes with the analysis of reflection and transmission coefficients in finite media. Finally, elastic waves in continuous media and acoustic waves in fluids are explored.

Finally, the third part (Chapters 7 and 8) deals with geometrical optics and optical instruments. It covers refraction, reflection, dispersion and image formation, as well as devices such as the prism. The study also includes the construction of images in a centered optical system to better comprehend the fundamental principles of optics, such as diopters, mirrors and thin lenses.

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Chapter I

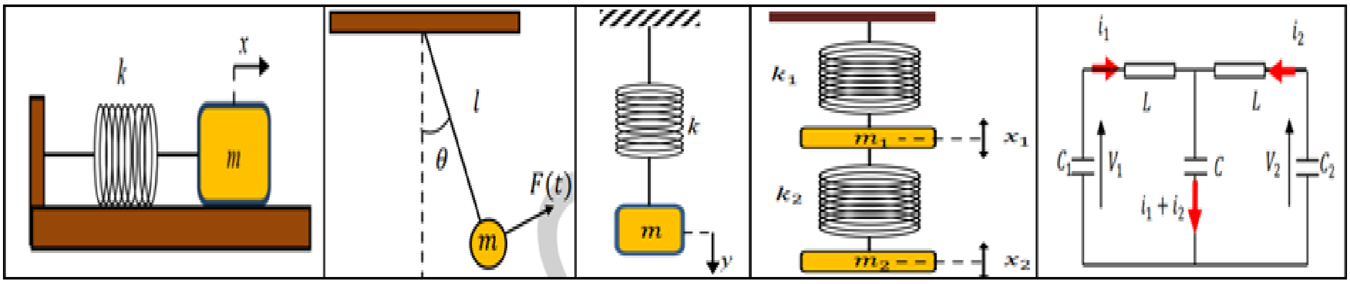
1. Second-order differential equations with constant coefficients

I. Generalities on oscillations

1.1 Definitions of vibration

- **Vibration:** is the oscillation of a moving body around its equilibrium position. Or it is a repetitive back-and-forth movement around a position of equilibrium.

Example of oscillations:



- **Verification of equilibrium and stability conditions**

There are two types of balance:

Stable and unstable equilibrium: The necessary condition is that

$$\frac{\partial^2 E_p}{\partial p^2}|_{p=0} > 0 \qquad \text{And} \qquad \frac{\partial^2 E_p}{\partial p^2}|_{p=0} < 0$$

1.2 Periodic movement (motion)

A periodic movement is one which repeats itself identically at successive time intervals of the same duration **T**.

1.2.1 The period T: Is defined as

$$T_0 = \frac{2\pi}{\omega_0} \qquad \text{Measured in seconds (s)}$$

Where $\boldsymbol{\omega_0}$ is called the pulsation of the movement.

1.2.2 The frequency f_0

Is the number of periods or oscillations per second, and is measured in Hertz (Hz) or s^{-1} is given by:

$$N = f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi}$$

1.2.3 The pulsation ω_0 :

Is the number of revolutions per second, measured in radians per second (rad/s) and is given by:

$$\omega_0 = 2\pi . f_0$$

1.3 Sinusoidal movement

Its time law (equation of motion) is a sinusoidal function of time ,it's written in the forms shown in Figure 1.

$$\boldsymbol{x(t) = X_m \cos(\omega_0 t + \varphi) \quad ou \quad y(t) = Y_m \cos(\omega_0 t + \varphi)}$$

With

$$\begin{cases} X_m \text{ ou } Y_m : amplitude\ , (m) always\ > 0\ (positive) \\ \omega_0 : natural\ pulsation, \left({rd}/{s} \right) > 0 \\ \varphi : initial\ phase\ at\ time\ t = 0s\ , (rd) \\ (\omega_0 t + \varphi) : The\ instantaneous\ phase\ at\ time\ t, (rd) \\ x(t)\ ou\ y(t) : Elongation\ or\ position\ at\ time\ t \\ x(t)\ between \quad +X_m and\ -X_m \end{cases}$$

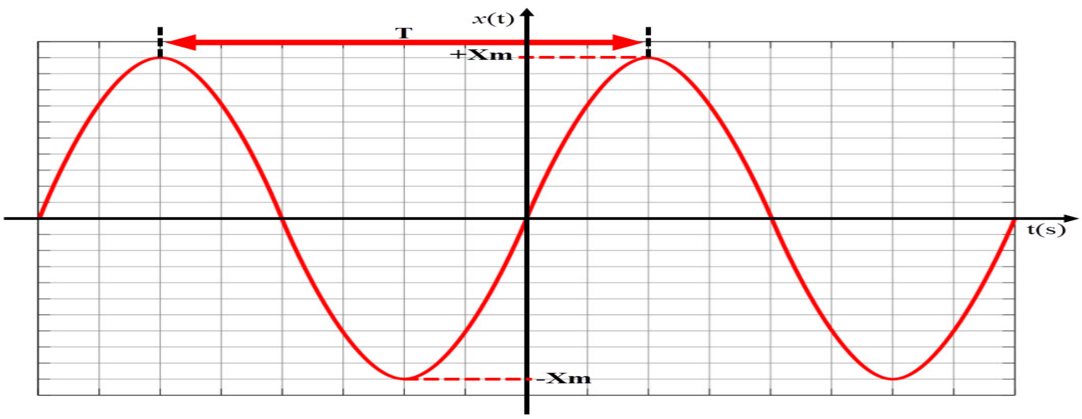


Figure 1: Sinusoidal movement (motion)

Example: The characteristic quantities of a harmonic vibration are:

Let the harmonic oscillator equation be:

$$x(t) = 2.10^{-2} \cos\left(100\pi t - \frac{\pi}{2}\right) \quad (m)$$

Is in the form $\quad x(t) = X_m \cos(\omega_0 t + \varphi)$ with :

$$X_m = 2.10^{-2} m, \quad \omega_0 = 100\pi \frac{rd}{s}, \quad \varphi = -\frac{\pi}{2} rd, \quad N = f_0 = 50Hz,$$
$$T_0 = 0.02s$$

1.4 General coordinates of a physical system

1.4.1 Definition

Generalized coordinates are the smallest number of independent coordinates (holonomic system) that can determine the motion of the system and the number of generalized coordinates is equal to the degree of freedom of the system and we denote them by: $\boldsymbol{q_1(t), q_2(t), \ldots q_n(t)}$.

Examples

Consider a free material point, the position of this point can be defined by Cartesian coordinates (x, y, z) or by the three spherical coordinates (ρ, θ, φ) or by the Three coordinates related to the Euler angles (φ, ψ, θ). And in general by three coordinates $(\boldsymbol{q_1}(t),\ \boldsymbol{q_2}(t),\ \boldsymbol{q_3}(t))$ called the generalized coordinates.

The generalized speeds are noted by: $\dot{q}_1(t),\ \dot{q}_2(t),\ \dot{q}_3(t)$.

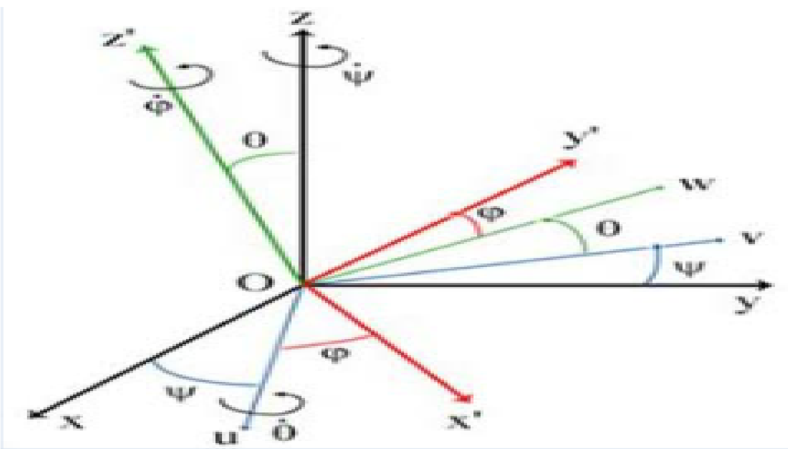


Figure 2: The different generalized coordinates of a moving system.

1.4.2 Degree of freedom

The number of degrees of freedom **N** is the number of independent motions of a physical system (determines the number of differential equations of motion.) and is given by the following relationship:

$$\mathbf{d = N - r}$$

$\boldsymbol{d}$: Degree of freedom.

$\boldsymbol{N}$: Number of generalized coordinates.

$\boldsymbol{r}$: Number of relations linking these coordinates together.

Exemples :

- **One degree of liberty system**: A disc of mass m and radius r rolls without slipping on a horizontal plane.

There are two generalized coordinates x and θ so $N = 2$

x and θ are linked with a relation: $x = R.\theta$ therefore: r =1.

The number of degrees of freedom $d = N - r = 2 - 1 = 1$ d l s

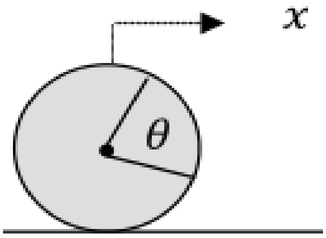


Figure 4: System with 1 d l s

- **Two degrees of liberty (freedom) system**: see the figure opposite. 2 generalized coordinates x_1 and x_2 so $N = 2$ and x_1 and x_2 are independent so $r = 0$. The number of degrees of freedom $\mathrm{d = N - r = 2 - 0 = 2}$ d l s

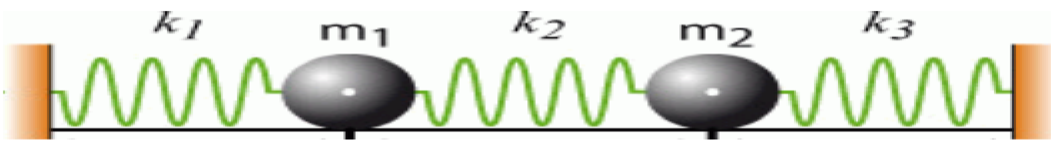


Figure 3: System with 2 d l s

1.5 Complex representation

1.5.1 Reminder:

Let be a complex number $\underline{Z} = x + jy$ algebraic form, and $\underline{Z} = r(\cos\theta + j\sin\theta)$, trigonometric form (in polar coordinates). Or

$$\underline{Z} = re^{j\theta} \qquad \text{Exponential form}$$

*- The complex conjugate of Z is any complex number of the form:

$$\underline{Z}^* = x - jy = re^{-j\theta}$$

*- module Z: $r = \|\underline{Z}\| = \sqrt{x^2 + y^2} = \sqrt{\underline{Z}\,\underline{Z}^*}$

*- Argument θ : $\tan\theta = \frac{Im\underline{Z}}{Re\underline{Z}} = \frac{y}{x} \Rightarrow \theta = arg\ \tan$

*- Trigonometric proportions:

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}\left(-\frac{\pi}{2}\right)$	π
$Cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	0	-1
$Sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	-1	0

Examples:

1/ Let the motion $x(t) = x_0\cos(3t + 5)$. Use the complex representation to find the speed $\dot{x}(t)$ and acceleration $\ddot{x}(t)$.

The solution:

$x(t) = x_0\cos(3t + 5) \longrightarrow \underline{x}(t) = x_0 e^{j(3t+5)}$

$\dot{x}(t) = 3x_0\cos\left(3t + 5 + \frac{\pi}{2}\right) \longleftarrow \underline{\dot{x}}(t) = 3jx_0 e^{j(3t+5)} = 3x_0 e^{j\left(3t+5+\frac{\pi}{2}\right)}$ Because $j = e^{j\frac{\pi}{2}}$

$\ddot{x}(t) = -9x_0\cos(3t + 5) \longleftarrow \underline{\ddot{x}}(t) = -9x_0 e^{j(3t+5)}$

2/ Write complex numbers in exponential form, $Z = re^{j\varphi}$, $Z = 1 + 2j$, $Z = 1$, $Z = -1$, $Z = -j$, $Z = \frac{\sqrt{3}}{2} + j\frac{1}{2}$, (1+j2)(3+j4)

Solution: $Z = 1 + 2j$, Module of $Z = \sqrt{1^2 + 2^2} = \sqrt{5}$, Argument θ is $\tan\theta = \frac{y}{x}$

$$\tan\theta = \frac{2}{1} \Rightarrow \theta = arc\tan 2 \quad so\ Z = \sqrt{5}e^{j\,arc\tan 2}$$

In the same way: $1 = e^{j2\pi}$, $-1 = e^{-j2\pi} = e^{j\pi}$, $-j = e^{j\frac{3\pi}{2}} = e^{-j\frac{\pi}{2}}$

1.6 Superposition of periodic quantities

1.6.1 Sinusoidal quantities of the same pulsation

Let the two sinusoidal quantities be:

$x_1(t) = a\cos(\omega t + \varphi_1)$ And $x_2(t) = b\cos(\omega t + \varphi_2)$

On utilise la représentation complexe pour trouver :

$$X(t) = x_1(t) + x_2(t)$$

$$x_1(t) + x_2(t) = a\cos(\omega t + \varphi_1) + b\cos(\omega t + \varphi_2) = ae^{j(\omega t+\varphi_1)} + be^{j(\omega t+\varphi_2)}$$

$$X(t) = ae^{j\varphi_1}e^{j(\omega t)} + be^{j\varphi_2}e^{j(\omega t)} = \left(ae^{j\varphi_1} + be^{j\varphi_2}\right)e^{j(\omega t)} = Ce^{j\varphi}e^{j(\omega t)}$$

$$X(t) = Ce^{j(\omega t+\varphi)} = C\cos(\omega t + \varphi)$$

C'est une grandeur sinusoïdale de pulsation ω.

- **L'amplitude C**

$$C = \left\|ae^{j\varphi_1} + be^{j\varphi_2}\right\| = \sqrt{(ae^{j\varphi_1} + be^{j\varphi_2})(ae^{-j\varphi_1} + be^{-j\varphi_2})}$$

$$C = \sqrt{a^2 + b^2 + 2ab\cos(\varphi_1 - \varphi_1)}$$

- **La Phase φ**

$$\tan\varphi = \frac{Im\left(ae^{j\varphi_1} + be^{j\varphi_2}\right)}{Re(ae^{j\varphi_1} + be^{j\varphi_2})} = \frac{a\sin\varphi_1 + b\sin\varphi_2}{a\cos\varphi_1 + b\cos\varphi_2}$$

So the superposition of two sinusoidal quantities with the same pulse ω is a sinusoidal quantity with a pulse ω.

1.6.2 Sinusoidal quantities of the same amplitude

Consider the two sinusoidal quantities

$x_1(t) = a\cos(\omega_1 t)$ And $x_2(t) = a\cos(\omega_2 t)$

$$X(t) = a\cos(\omega_1 t) + a\,cos(\omega_2 t) = 2a\cos\left(\frac{\omega_1 - \omega_2}{2}\,t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}\,t\right)$$

The superposition of two sinusoidal quantities of the same amplitude is an amplitude-modulated sinusoidal quantity if the two pulsations are different.

1.6.3 Any sinusoidal quantities

The superposition of two sinusoidal quantities of different pulsations $\omega_1$and $\omega_2$will only be a periodic quantity if the ratio between their periods is a rational number

$$\frac{T_1}{T_2} = \frac{k}{m} \Longrightarrow T = mT_1 = kT_2 \ll$$

Consider the two sinusoidal quantities

$$x_1(t) = 6\cos(5t + 3) \quad \text{And} \quad x_2(t) = 3\cos(7t + 4)$$

The superposition of $x_1(t)$ and $x_2(t)$ is

$$X(t) = x_1(t) + x_1(t)$$

As $\frac{T_1}{T_2} = \frac{k}{m} = \frac{2\pi/5}{2\pi/7} = \frac{7}{5}$, it is a periodic quantity with period

Hence

$$T = mT_1 = kT_2 = m(2\pi/5) = k(2\pi/7) = 2\pi\ (s)$$

1.7. Fresnel representation

The Fresnel representation is the geometric version of the complex representation. This quantity is associated with a rotating vector known as the Fresnel vector, the characteristics of which are as follows:

- Its rotation speed is equal to ω.
- Its norm is equal to the amplitude of the sinusoidal quantity.
- The angle with respect to the phase origin is equal to the instantaneous value $(\omega_0 t + \varphi)$ of the sinusoidal quantity.

For example:

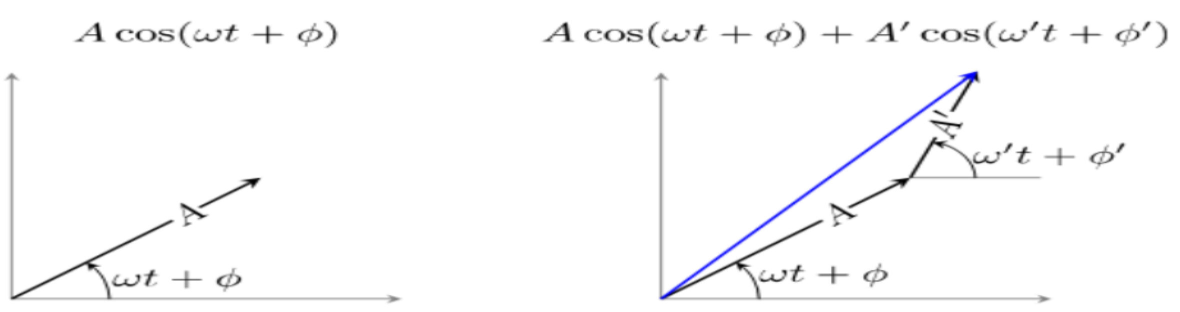


Figure 5: Fresnel representation

We find ourselves:

$$x(t) = A\cos(\omega t + \Phi) + A'\cos(\omega' t + \Phi') = A_1\cos(\omega_1 t + \Phi_1)$$

1.8 Fourier series

It is shown mathematically that any periodic oscillation is decomposed into a sum of harmonic oscillations (Fourier decomposition). To simplify the study, the oscillator is said to be **anharmonic**. For small variations, it is defined by:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

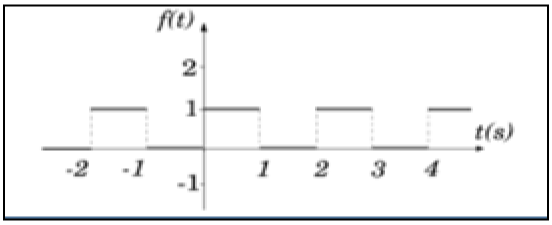
Where

- a_0 , a_n and b_n are called Fourier coefficients.
- The pulsation: $\omega = \frac{2\pi}{T}$ is called the fundamental pulsation.
- The higher pulsations $n\omega$ (multiples of ω) are called the harmonics.
- The Fourier coefficients are defined by:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \ , \ a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt \ , \ b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

Example: Let the function be defined as follows

$$f(t) = \begin{cases} 1 & si \quad t \in [0,1] \\ 0 & si \quad t \in [1,2] \end{cases}$$



We can deduce the period of the graph $T = 2s, \omega = \frac{2\pi}{T} = \pi$ rd/s , the Fourier coefficients of this function are given by :

$$a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{2}\int_0^1 dt = \frac{1}{2}[t]_0^1 = \frac{1}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt = \frac{2}{2}\int_0^1 \cos(n\pi t)\,dt = \frac{1}{n\pi}[\sin(\pi n t)]_0^1 = \frac{\sin(n\pi)}{n\pi} = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt = \frac{2}{2}\int_0^1 \sin(n\omega t)\,dt = -\frac{1}{n\pi}[\cos(\pi n t)]_0^1 = \frac{1-\cos(\pi n)}{n\pi}$$

$$b_n = \begin{cases} 0 \ \ si\ n\ pair \\ \frac{2}{n\pi} \ \ si\ n\ impair \end{cases} = \frac{2}{(2k+1)\pi} \quad , \qquad k = 0, 1, 2 \ldots\ldots$$

The function $f(t)$ is written as

$$f(t) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi}\sin(2k+1)\,\pi\, t$$

II. Second-order differential equations with constant coefficients

1.9 Solutions of second-order differential equations with constant coefficients

1.9.1 Homogeneous equation (without second member)

- **General solution**

If the differential equation of movement of the form

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0 \ \ldots\ldots\ldots\ldots\ldots \ (1)$$

This is a homogeneous second-order differential equation, and we choose a trial solution of the type.

We rename the fractions to get a tidier expression:

$$\begin{cases} \frac{\alpha}{m} = 2\lambda \\ \frac{k}{m} = \omega_0^2 \end{cases}$$

The equation now becomes is of the form:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \ \ldots\ldots\ldots\ldots \ (2)$$

Where λ is the damping coefficient and $\omega_0$is the proper pulsation proper pulsation

Then the general solution $x(t)$ in complex form is of the form: $x(t) = Ae^{rt}$or of the form $x(t) = A\sin(\omega_0 t + \varphi)$ or $x(t) = A\cos(\omega_0 t + \varphi)$, where A, φ are coefficients to be determined by the initial conditions.

We look for the characteristic equation (polynomial):

$$r^2 + 2\lambda r + \omega_0^2 = 0$$

There arise three different types of solutions, depending on the discriminant:

$$\Delta' = \lambda^2 - \omega_0^2$$

<u>1st case:</u> Supercritical damping, overdamping

If $\Delta' > 0 \Rightarrow \lambda > \omega_0$

$$\Rightarrow \begin{cases} r_1 = -\lambda - \sqrt{\lambda^2 - \omega_0^2} \\ r_2 = -\lambda + \sqrt{\lambda^2 - \omega_0^2} \end{cases}$$, then the general solution of this equation

$$x(t) = e^{-\lambda t}\left(A_1 e^{\left(\sqrt{\lambda^2-\omega_0^2}\right)t} + A_2 e^{-\left(\sqrt{\lambda^2-\omega_0^2}\right)t}\right)$$

The system is called aperiodic (highly damped) regime. See 1st Figure 6,

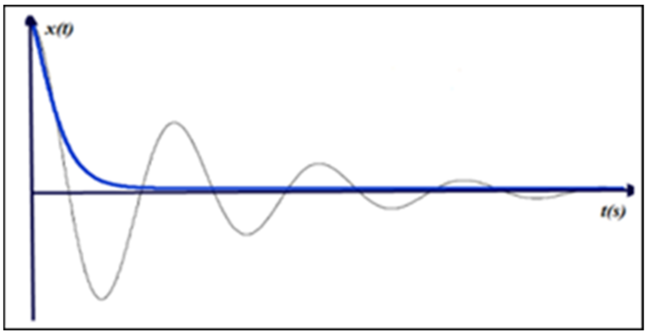


Figure 6: aperiodic regime

<u>2nd case:</u> Critical regime

If $\Delta' = 0 \Rightarrow \lambda = \omega_0 \Rightarrow r_1 = r_2 = \boldsymbol{r} = \frac{\boldsymbol{-\lambda}}{\boldsymbol{1}} = \boldsymbol{-\lambda}$, then the general solution of this equation

$$x(t) = e^{-\lambda t}(Bt + A)$$

Where B, A to be determined by the initial conditions. The system is called critical regime where damping is critical. See 2nd Figure 7.

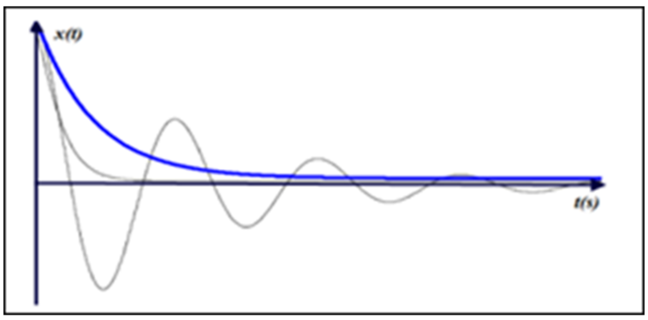


Figure 7: Critical regime

3rd case: **Sub-critical damping**

If $\Delta' < 0 \Rightarrow \lambda < \omega_0 \Rightarrow$ there are 2 complex solutions

$$\begin{cases} r_1 = -\lambda - j\sqrt{\omega_0^2 - \lambda^2} = -\lambda - j\omega_a \\ r_2 = -\lambda + j\sqrt{\omega_0^2 - \lambda^2} = -\lambda + j\omega_a \end{cases}$$

The solution for sub-critical damping can be put in a simpler form:

$$x(t) = e^{-\lambda t}\boldsymbol{A}\cos(\omega_a t + \varphi)$$

where $\omega_a = \sqrt{\omega_0^2 - \lambda^2}$ is called pseudo pulsation (is a real number).

Such that λ, A and φ are constants to be determined by the initial conditions,

$$T_a = \frac{2\pi}{\omega_a}$$

The regime is called pseudo period or the damping is low amplitude (weakly damped). See 3rd Figure 8.

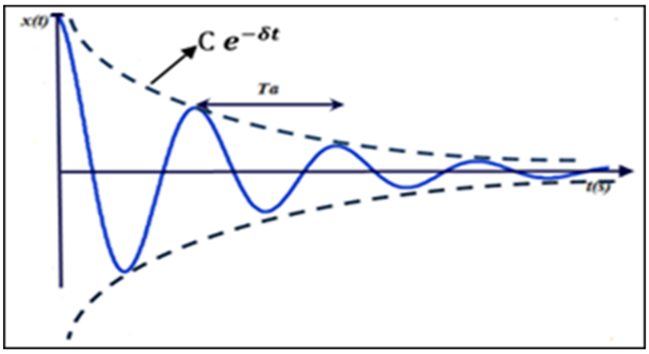
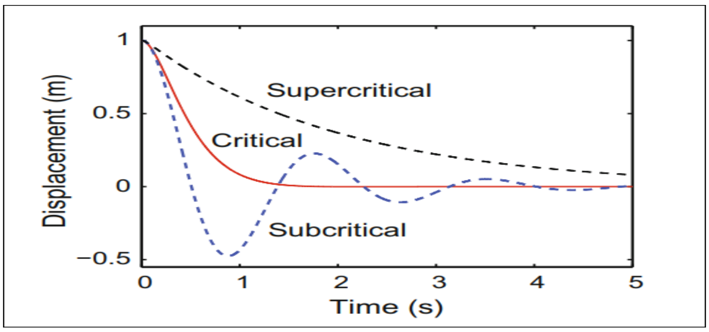


Figure 8: Pseudo-period regime

Examples of overcritical, critical and sub-critical damping of an oscillation that would be simple harmonic in the absence of friction. The friction is increased by a factor of four from one curve to another: sub-critical, critical and overcritical damping



Of the equilibrium point while the amplitude decreases to zero. The oscillation frequency is lower than when there is no damping (something that is to be expected since the friction acts to slow down all movement).

1.9.2 Differential equations with second member

We study the equations as follows

$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = A_0 \cos \Omega t$ Or $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = A_0 \sin \Omega t$

The differential equation of motion can therefore be written as

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos \Omega t \iff \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) \quad \ldots\ldots\ldots. (1.6)$$

This is a non-homogeneous linear second-order differential equation with constant coefficients and a second member

With

$$\begin{cases} \lambda = \dfrac{\alpha}{2m} \ldots\ldots\ldots (3-5) \\ \omega_0^2 = \dfrac{k}{m} \ldots\ldots\ldots (3-6) \\ A(t) = \dfrac{f_0}{m}\cos \Omega t \equiv A_0 \cos \Omega t \end{cases} \quad \ldots\ldots\ldots. (1.7)$$

1.9.3 Solution of the differential equation of motion

The general solution of this equation It has two solutions

- A solution of the equation without a second member: homogeneous solution $X_H(t)$.
- A solution to the equation with second member: particular solution $X_P(t)$

The total or general solution is :

$$X_g(t) = X(t) = X_H(t) + X_P(t) \quad \ldots\ldots\ldots\ldots\ldots (1.8)$$

1.9.4 Homogeneous solution (The transitional solution)

We have already studied the equation without a second member (damped harmonic oscillator system)

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0 \iff \ddot{x} + 2\lambda\,\dot{x} + \omega_0^2 x = 0 \quad \ldots\ldots\ldots\ldots (1.9)$$

$$\Downarrow$$

$$r^2 + 2\lambda r + \omega_0^2 = 0 \text{ .... (1.10)}$$

$$\Delta' = \lambda^2 - \omega_0^2$$

There are three cases depending on the sign of the reduced discriminant

$\Delta' > 0 \implies \lambda > \omega_0$ The system is highly damped

$\Delta' < 0 \implies \lambda < \omega_0$ Low damping

$\Delta' = 0 \implies \lambda = \omega_0$ Damping is critical

The solution to the differential equation is of the form:

$$X_H(t) = A' e^{-\lambda t} \cos(\omega_a t + \varphi) \text{ .... (1.11)}$$

With: $\omega_a = \sqrt{\omega_0^2 - \lambda^2}$ pseudo-pulsation, we say that the system with a damped oscillatory movement corresponds to a **transient regime**, (free regime in the case of weakly damped oscillations).

Remark:

1. The general solution of the equation without a second member corresponds to a transitory regime (which only lasts for a certain time).
2. If $\lambda < \omega_0$: the limit of $X_H(t)$ is always zero when t tends to infinity:

$$t \longrightarrow +\infty \implies e^{(-\lambda).(+\infty)} = 0$$

The homogeneous solution becomes negligible compared to the **particular solution** (the permanent regime or the permanent solution, or stationary regime).

The general solution becomes:

$$X_g(t) = X(t) = X_P(t)$$

1.9.5 Particular solution of a second member

$X_P(t)$: is the permanent (stationary) solution of the non-homogeneous equation with second term $F(t)$. Then the particular solution $X_P(t)$ will be a sinusoidal function of same pulsation , if $F(t)$ is a sinusoidal function of pulsation Ω, the solution it is of the form :

$$X_P(t) = A\cos(\Omega\,\mathrm{t} + \Phi) \quad ou \quad X_P(t) = A\sin(\Omega\,\mathrm{t} + \Phi) \text{ .... (1.12)}$$

We find A and Φ using the complex representation as follows

$$F(t) = f_0 \cos \Omega t \quad \longrightarrow \quad f_0 e^{j\Omega t}$$

$$q_P(t) = X_P(t) = A\cos(\Omega\, \mathrm{t} + \Phi) \longrightarrow \underline{q}_P(t) = \underline{X}_P(t) = Ae^{j(\Omega t + \Phi)}$$

$$\underline{X}_P(t) = Ae^{j\Phi}e^{j\Omega t}$$

$$\underline{X}_P(t) = \underline{A}e^{j\Omega t}$$

With : $\underline{A} = Ae^{j\Phi}$

We calculate $\underline{\dot{X}}_p(t)\ et\ \underline{\ddot{X}}_p(t)$

$$\underline{\dot{X}}_p(t) = \underline{\dot{q}}_p(t) = \frac{d\underline{X}_P(t)}{dt} = j\Omega\underline{A}e^{j\Omega t} \text{ ………….. (1.13)}$$

$$\underline{\ddot{X}}_p(t) = \underline{\ddot{q}}_p(t) = \frac{d^2\underline{X}_P(t)}{dt^2} = -\Omega^2\underline{A}e^{j\Omega t} \text{ ………….. (1.14)}$$

So the equation

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos\Omega t = A_0\cos\Omega t \iff \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) = A_0\cos\Omega t$$

Such as:

$$A_0 = \frac{f_0}{m} \text{ ………… (1.15)}$$

Determining the quantities A and Φ is the same as finding the modulus of the complex amplitude.

(1.6) becomes the equation:

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) = A_0 e^{j\Omega t}$$

After replacing: $\underline{X}_P(t),\ \underline{\dot{X}}_p(t)\ \ et\ \underline{\ddot{X}}_p(t)$, we find

$$\left(-\Omega^2\underline{A} + 2\lambda j\Omega\,\underline{A} + \omega_0^2\,\underline{A}\right)e^{j\Omega t} = A_0 e^{j\Omega t}$$

Divide by "$e^{j\Omega t}$" and find

$$(-\Omega^2 + 2\lambda j\Omega + \omega_0^2)\underline{A} = A_0$$

$$\underline{A} = \frac{A_0}{(\omega_0^2 - \Omega^2) + j2\lambda\Omega} = Ae^{j\Phi}$$

- **Calculating the amplitude A**

The amplitude of the movement is therefore:

$$A = |\underline{A}| = \frac{\|A_0\|}{\|(\omega_0^2 - \Omega^2) + j2\lambda\Omega\,\|}$$

$$A = \frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+4\lambda^2\Omega^2}} = Cte \text{ ………….. (1.16)}$$

The phase (phase shift) Φ of the movement between q(t) and F(t) is given by:

We have

$$\underline{A} = \frac{A_0}{(\omega_0^2 - \Omega^2) + j2\lambda\Omega} = Ae^{j\Phi}$$

The conjugate of this equation is

$$\underline{A}^* = \frac{A_0}{(\omega_0^2 - \Omega^2) - j2\lambda\Omega} = Ae^{-j\Phi}$$

We multiply the formula for $\underline{A}$ inverted and find:

$$\underline{A} = \frac{A_0}{[(\omega_0^2 - \Omega^2) + j2\lambda\Omega]} * \frac{[(\omega_0^2 - \Omega^2) - j2\lambda\Omega]}{[(\omega_0^2 - \Omega^2) - j2\lambda\Omega]} = Ae^{j\Phi}$$

$$\underline{A} = \frac{A_0(\omega_0^2 - \Omega^2) - A_0(j2\lambda\Omega)}{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2} = Ae^{j\Phi}$$

$$\underline{A} = \frac{A_0(\omega_0^2 - \Omega^2)}{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2} - j\frac{2A_0\lambda\Omega}{(\omega_0^2 - \Omega^2)^2 + (2\lambda\Omega)^2}$$

Hence

$$\tan\Phi = \frac{Im\,\underline{A}}{Re\,\underline{A}} = -\frac{2\lambda\Omega}{(\omega_0^2 - \Omega^2)}$$

- **Calculating phase Φ**

$$\Phi = \text{arc }\tan\frac{2\lambda\Omega}{(\Omega^2-\omega_0^2)} \text{ …………… (1.17)}$$

So the steady-state solution is

$$X_P(t) = q_P(t) = A\cos(\Omega t + \Phi) \text{ .... (1.18)}$$

$$X_P(t) = \frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+4\lambda^2\Omega^2}}\cos\left(\boldsymbol{\Omega} t + \boldsymbol{arc\ tan}\frac{2\lambda\Omega}{(\Omega^2-\omega_0^2)}\right) \text{ .... (1.19)}$$

If the homogeneous solution is not negligible, the general solution is given by

$$X_g = A'e^{-\lambda t}\cos(\omega_a t + \varphi) + \frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+4\lambda^2\Omega^2}}\cos\left(\boldsymbol{\Omega} t + \boldsymbol{arc\ tan}\frac{2\lambda\Omega}{(\Omega^2-\omega_0^2)}\right) \dots \text{ (1.20)}$$

With

$$\begin{cases} \boldsymbol{\omega_a} = \sqrt{\omega_0^2 - \lambda^2} \\ \boldsymbol{T_a} = \frac{2\pi}{\omega_a} = \frac{2\pi}{\sqrt{\omega_0^2-\lambda^2}} \\ \boldsymbol{A_0} = \frac{f_0}{m} \end{cases} \text{ .... (1.21)}$$

Exercises and solutions

Find the solution to each of the following differential equations.

$5\ddot{x} + 3x = 0$, $3\ddot{x} + 5\dot{x} + 4x = 0$, $2\ddot{x} + 6\dot{x} + 2x = 0$, $2\ddot{x} + \sqrt{48}\dot{x} + 6x = 0$

Solutions: we have

a.

$5\ddot{x} + 3x = 0 \Leftrightarrow \ddot{x} + \frac{3}{5}x = 0$. The equation is $\ddot{x} + \omega_0^2 x = 0$ with $\omega_0^2 = \frac{3}{5}$

The general solution to the equation:

$x(t) = A\sin\left(\sqrt{\frac{3}{5}}\ t + \emptyset\right)$ Or $x(t) = A\cos\left(\sqrt{\frac{3}{5}}\ t + \emptyset\right)$.

b.

$3\ddot{x} + 5\dot{x} + 4x = 0 \Leftrightarrow \ddot{x} + \frac{5}{3}\dot{x} + \frac{4}{3}x = 0$. The equation is $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$ with $\lambda = \frac{5}{6}$ et $\omega_0^2 = \frac{4}{3}$.

Since $\lambda^2 - \omega_0^2 = -\frac{23}{36} < 0$, the general solution of the equation is of the form :

$$x(t) = Ae^{-\frac{5}{2}t}\cos\left(\sqrt{\frac{23}{36}}t + \emptyset\right)$$

c.

$2\ddot{x} + 6\dot{x} + 2x = 0 \Leftrightarrow \ddot{x} + 3\dot{x} + x = 0$. The equation is $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$ with $\lambda = \frac{\sqrt{3}}{2}$ and $\omega_0^2 = 1$

Since $\lambda^2 - \omega_0^2 = \frac{5}{4} > 0$, the general solution of the equation is of the form:

$$x(t) = e^{-\frac{\sqrt{3}}{2}t}(A_1 e^{-\sqrt{\frac{5}{4}}t} + A_2 e^{+\sqrt{\frac{5}{4}}t}).$$

d.

$2\ddot{x} + \sqrt{48}\,\dot{x} + 6x = 0 \;\Leftrightarrow\; \ddot{x} + \sqrt{12}\dot{x} + 3x = 0$. The equation is $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0$ with $\lambda = \sqrt{3}$ and $\omega_0^2 = 3$.

Since $\lambda^2 - \omega_0^2 = 0$, the general solution of the equation is of the form:

$$x(t) = e^{-\sqrt{3}t}(A_1 + A_2 t).$$

Where the quantities ϕ, A, A_1, A_2 are determined by the initial conditions.

Chapter II

Free oscillations of systems with one degree of freedom

I. Undamped oscillations

1. Introduction

- We are interested in the oscillations caused by a restoring force about an equilibrium position.

- An oscillator which is not subject to any exciting force is said to be a free oscillator.

- The degrees of freedom are the independent variables of the system, variables of the system.

2. Linear oscillator

2.1 Differential equation of movement

The equation of motion for a **conservative** system can be determined by three methods:

- Newton's law of Dynamics (PFD):

$$\sum \vec{f} = m\vec{a} = m\ddot{x} \;\ldots\ldots\ldots\ldots\ldots\; (2.1)$$

f Represents the set of forces applied to the object, $\vec{a}$ its acceleration and m is the mass of the solid.

- Principle of conservation of (total) mechanical energy:

$$E_T = E_c + E_p \;\Rightarrow \frac{dE_T}{dt} = 0 \;\ldots\ldots\ldots\ldots\; (2.2)$$

Where E_c represents the kinetic energy of the system and E_p the potential energy.

- Lagrange formalism (Lagrange-Euler method)

We define the Lagrange function

$$L = T - U \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2.3)$$

The Euler-Lagrange equation for a conservative system is given as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0 \quad i = 1, \ldots n \;\ldots\ldots\ldots\ldots\ldots\; (2.4)$$

- L : Lagrange fonction or Lagrangien.
- T : The kinetic energy of the system.

- U : The potential energy of the system.
- q_i : The generalized coordinate.
- $\dot{q}_i$: The generalized speed of the system

The **Euler-Lagrange** equation for a **non-conservative system** (dissipative system) is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)=\sum \vec{f}_{ext} \quad i=1,\dots n,\ en\ translation\ \dots\dots\dots (2.5)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\left(\frac{\partial L}{\partial q_i}\right)=\sum \mathcal{M}_{/\Delta}\left(\vec{f}_{ext}\right) \quad i=1,\dots n,\ en\ rotation\ \dots\dots (2.6)$$

Where

$\vec{f}_{ext}$: are the external forces applied to the system.

$\mathcal{M}_{/\Delta}\left(\vec{f}_{ext}\right)$: Are the external moments applied to the system.

In this case the forces do not derive from a potential

- For one-dimensional motion x, Lagrange's equation is written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0\ \dots\dots\dots\dots\dots\dots\dots (2.7)$$

- For rotational motion, Lagrange's equation can be written as follows

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\left(\frac{\partial L}{\partial \theta}\right)=0\ \dots\dots\dots\dots\dots\dots (2.8)$$

2.1.1 Harmonic oscillators

Example 1: Vertical mass-spring system (vertical elastic pendulum)

See Figure 1

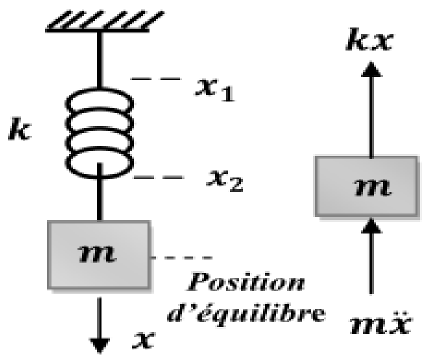


Figure 1: Vertical elastic pendulum

The kinetic energy is written: $T = \frac{1}{2}m\dot{x}^2$ ………………. (2.9)

The potential energy for small oscillations is written as: $U = \frac{1}{2}kx^2$ ……… (2.10)

The Lagrangian of the system is written as:

$$L = T - U \text{ ……… (2.11)}$$

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \text{ ………………….. (2.12)}$$

The equation of motion is of the form

$$\begin{cases} \frac{\partial L}{\partial \dot{x}} = m\dot{x} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \text{ … … (2.13)} \\ \frac{\partial L}{\partial x} = -kx \text{ … … … … … … … … .. (2.14)} \end{cases} \Rightarrow \quad (2.13) - (2.14) = 0$$

$\Rightarrow m\ddot{x} + kx = 0,$ Dividing by m, we find

$$\ddot{x} + \frac{k}{m}x = 0 \quad \Leftrightarrow \quad \ddot{x} + \omega_0^2 x = 0 \text{ ……………. (2.15)}$$

This is a second-order differential equation with its general solution in the form

$$x(t) = A\sin(\omega_0 t + \varphi) \quad Ou \quad x(t) = A\cos(\omega_0 t + \varphi) \text{ ………. (2.16)}$$

Where A, φ are coefficients to be determined by the initial conditions

- The proper pulsation is: $\omega_0 = \sqrt{\frac{k}{m}}$ $(\frac{rd}{s})$ ……………. (2.17)
- The period proper is: $T_0 = 2\pi\sqrt{\frac{m}{k}}$ (s) ……………….(2.18)

2.1.2 Simple pendulum

A simple pendulum is constructed as shown in Figure 2.

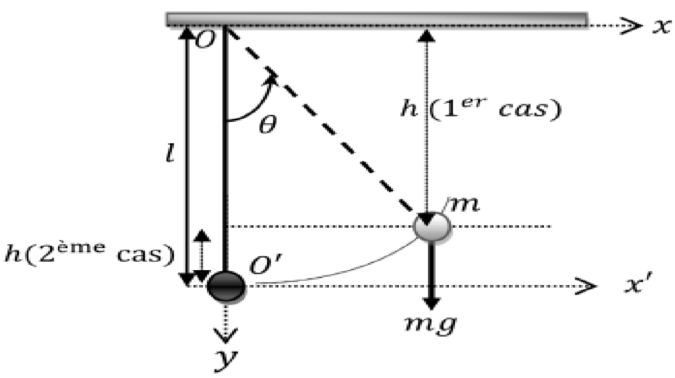


Figure 2: Simple pendulum

- The kinetic energy of the system: $T = \frac{1}{2} m\, V_m^2$ ……… (2.19)

The system coordinates are:

$$m \begin{cases} x = l \sin\theta \Longrightarrow \dot{x} = l\dot{\theta}\cos\theta \\ y = l\cos\theta \Longrightarrow \dot{y} = l\dot{\theta}\sin\theta \end{cases} \text{………….. (2.20)}$$

$$V_m^2 = \dot{x}^2 + \dot{y}^2 = \left(l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2 = l^2\dot{\theta}^2 \text{ …………. (2.21)}$$

From (2.19) $\Leftrightarrow$ $T = \frac{1}{2} m V_m^2 = \frac{1}{2} m l^2 \dot{\theta}^2 = \frac{1}{2} j_{/\Delta}\dot{\theta}^2$ ……… (2.22)

Where $j_{/\Delta} = ml^2$: Represents the moment of inertia

Hence

$$T = \frac{1}{2} j_{/\Delta}\dot{\theta}^2 \text{ ……………. (2.23)}$$

- **<u>1 st case</u>:** The potential energy of the system is: $U = mgh$ .We need to choose a reference point to calculate the potential energy, for example we choose the axis $(O'x')$ as the origin of the potential energies $(U(h=0))$, h becomes

$$h = l - l\cos\theta \text{ ……………… (2.24)}$$

The potential energy is:

$$U = mgl(1 - \cos\theta) \text{ ……………. (2.25)}$$

☞ The Lagrangian of the system is written in the form:

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta) \text{ ……………. (2.26)}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} \text{ … … … … … … … (2.27)} \\ \left(\frac{\partial L}{\partial \theta}\right) = -mgl\sin\theta \text{ … … … … … (2.28)} \end{cases}$$

$$(2.27) - (2.28) \Longrightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$$

The following approximation is used: $\sin\theta \approx \theta$

From this

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \text{ Dividing by } ml^2 \text{ we find}$$

$$\ddot{\theta} + m\frac{g}{l}\theta = 0 \text{ ……………….. (2.29)}$$

Are a second-order linear differential equation with constant coefficients and no second member of the form

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

Is its solution of form

$$\theta(t) = \theta_0 \sin(\omega_0 t + \emptyset) \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.30)}$$

With $\omega_0 = \sqrt{\frac{g}{l}}$: Is the proper pulsation of a simple pendulum.

- **<u>2 nd case</u>:** If we take the reference (the reference frame) as the origin of the potential energies ($U(h = 0) = 0$) on the axis ($O\ x$), h becomes

  $$h = -l \cos\theta \ldots\ldots\ldots\ldots\ldots \text{(2.31)}$$

  The potential energy becomes

  $$U = -mgl \cos\theta \ldots\ldots\ldots\ldots\ldots \text{(2.32)}$$

  Lagrange becomes

$$L = T - U = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta \ldots\ldots\ldots\ldots \text{(2.33)}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.34)} \\ \left(\frac{\partial L}{\partial \theta}\right) = -mgl \sin\theta \ldots\ldots\ldots\ldots\ldots. \text{(2.35)} \end{cases}$$

$$(2.34) - (2.35) \Longrightarrow\ m l^2 \ddot{\theta} + mgl \sin\theta = 0$$

In the case of weak oscillations, the angles are very small, so: $\begin{cases} \sin\theta \approx \theta \\ \cos\theta \approx 1 - \frac{\theta^2}{2} \end{cases}$

So

$$m l^2 \ddot{\theta} + mgl \sin\theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(2.36)}$$

Dividing by ml^2 we find

$$\ddot{\theta} + m \frac{g}{l} \theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.37)}$$

Like the previous equation, it is a differential equation of the form;

$$\ddot{\theta} + \omega_0^2 \theta = 0$$

Is its solution of form

$$\theta(t) = \theta_0 \sin(\omega_0 t + \emptyset) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.38)$$

2.1.3 Solution of the differential equation of movement

The differential equation of motion is of the form:

$$\ddot{x} - \omega_0^2 x = 0$$

This is a second-order differential equation without a second member whose solution in complex form is of the form: $x(t) = Ae^{\lambda t}$or of the form

$$x(t) = A\sin(\omega_0 t + \varphi) \quad \text{Or} \quad x(t) = A\cos(\omega_0 t + \varphi) \quad \ldots\ldots\ldots\ldots (2.39)$$

Where A, φ are coefficients to be determined by the initial conditions.

2.1.4 Electrical oscillations:

Consider an electrical circuit (L, C) through which a current $i(t)$ flows, as shown in the diagram opposite:

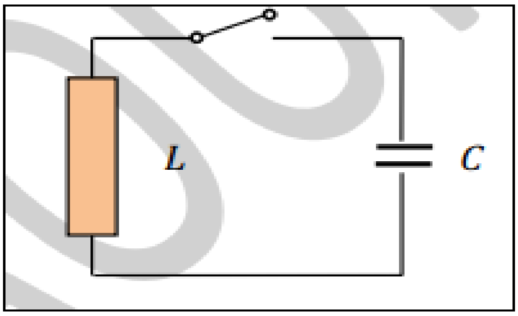


Figure 3: Circuit (L, C).

Using Kirchhoff's law of meshes, the voltage balance is written as follows

$$U_L + U_C = 0$$

$$L\frac{di(t)}{dt} + \frac{1}{c}\int i(t)dt = 0$$

Dividing by L we find

$$\frac{di(t)}{dt} + \frac{1}{Lc}\int i(t)dt = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.40)$$

Such as: the current $i(t)$for a time dt gives a charge dq:

$$i(t) = \frac{dq(t)}{dt} \Rightarrow q(t) = \int i(t)dt \quad and \quad \frac{di(t)}{dt} = \frac{d^2q(t)}{dt^2}$$

Becomes:

$$\frac{d^2q(t)}{dt^2} + \frac{1}{Lc}\, q(t) = 0 \qquad \Leftrightarrow \qquad \ddot{q}(t) + \omega_0^2 q(t) = 0 \text{ .... (2.41)}$$

Note that this equation is equivalent to the equation of harmonic oscillatory motion.

$$\ddot{q}(t) + \frac{1}{Lc} q(t) = 0 \qquad \Leftrightarrow \qquad \ddot{x} + \frac{k}{m} x = 0 \text{ .... (2.42)}$$

- **The proper pulsation and the proper period** as follows $\begin{cases} \omega_0 = \sqrt{\frac{1}{Lc}} \\ T_0 = 2\pi\sqrt{Lc} \end{cases}$ … (2.43)

The general solution of the equation for weak oscillations is written

$$q(t) = A\, cos(\omega_0 t + \varphi) \text{ .... (2.44)}$$

2.1.5 Analogy between the mechanical system and the electrical system

Mechanical system		Electrical system	
Displacement:	$x(t)$	$q(t)$:	Electrical charge
Mass :	m	L :	Winding inductance
Rigidity constant:	k	$1/c$:	Inverse of capacity

II. Free and Damped Oscillations (Damped Oscillations) systems with one degree of liberty (freedom) (1.D.O.L)

In this part we must take into account the influence of the viscous frictional force, which is proportional to the speed of oscillation of the system.

2.2 Damping force

A system subjected to friction is said to be damped. The simplest form of friction is viscous friction. Viscous friction is of the form:

$$\vec{f} = -\alpha\vec{V} \qquad or \qquad f_q = -\alpha\dot{q} \text{ .... (2.45)}$$

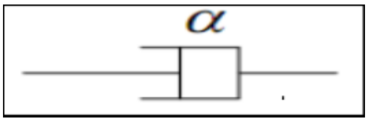
Where α: is the coefficient of friction and v: is the speed of the moving body or particle.

f_q : The viscous friction force,

q : The generalized coordinate of the system

$\dot{q}$: The generalized velocity of the system.

In mechanics, the shock absorber is represented by



2.2.1 Differential equation of movement

Lagrange equation

If friction exists $f_q = -\alpha\dot{q}$, lagrange's quation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = -\alpha\dot{q} \quad i = 1, \dots n \quad \dots\dots\dots\dots\dots\dots (2.46)$$

We define the dissipation function D

$$D = \frac{1}{2}\alpha\dot{q}^2 \dots\dots\dots\dots\dots\dots\dots (2.47)$$

The friction force becomes

$$f_q = -\frac{\partial D}{\partial \dot{q}} \dots\dots\dots\dots\dots\dots\dots (2.48)$$

The friction force becomes The dissipation function D in electricity:

$$D = \frac{1}{2}Ri^2 \dots\dots\dots\dots\dots\dots\dots (2.49)$$

The Lagrange quation in a generalised coordinate $q_i$can be written as :

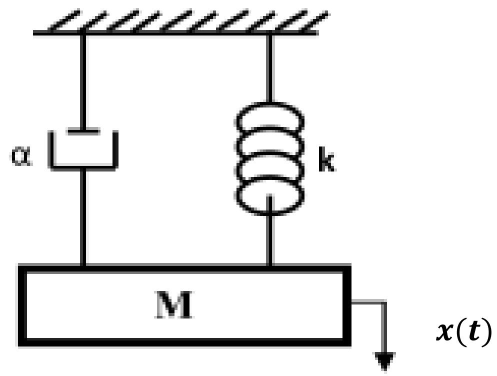
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \left(\frac{\partial D}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \dots\dots\dots\dots\dots\dots\dots (2.50)$$

With $$L(q, \dot{q}) = T - U$$

2.2.2 Mass-spring-damper system

The damped oscillator is studied in the same way as above, but with the addition of the viscous friction force.

- The kinetic energy of the mass m: $T = \frac{1}{2}m\dot{x}^2$ ……………….. (2.51)
- The energy stored in the spring: $U = \frac{1}{2}kx^2$…………………. (2.52)
- The dissipation function: $D = \frac{1}{2}\alpha\dot{x}^2$………………….. (2.53)
- Lagrangian is written



$$L = T - U \quad \Longrightarrow \quad L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \;\; \text{………………} \; (2.54)$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \dots\dots (2.55) \\ \frac{\partial L}{\partial x} = -kx \dots\dots\dots . (2.56) \\ \frac{\partial D}{\partial \dot{x}} = \alpha\dot{x} \dots\dots\dots\dots (2.57) \end{cases}$$

By replacing in the Lagrange equation we get: $(2.55) + (2.57) - (2.56) = 0$

$$m\ddot{x} + \alpha\dot{x} + kx = 0 \qquad \Longrightarrow \qquad \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0 \;\; \text{…………………} \; (2.58)$$

The differential equation of movement of a damped free system is of the form

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = 0$$

With

$$\begin{cases} \lambda = \frac{\alpha}{2m} : Damping\ factor \\ \omega_0^2 = \frac{k}{m} : is\ the\ natural\ pulsation\ or\ the\ proper\ pulsation \end{cases} \;\; \text{…………} \; (2.59)$$

2.2.3 Solution the equation of movement (motion)

The equation $\ddot{q} + 2\lambda\dot{q} + {\omega_0}^2 q = 0$ is a second-order linear differential equation with constant coefficients and no second member.

$x(t) = Ae^{rt}$: is a solution of this differential equation. Injecting this into The equation $\ddot{q} + 2\lambda\dot{q} + {\omega_0}^2 q = 0$ we obtain the characteristic equation:

$$r^2Ae^{rt} + 2\lambda rAe^{rt} + \omega_0^2Ae^{rt} = 0 \quad \Longrightarrow \quad \begin{cases} r^2 + 2\lambda r + \omega_0^2 = 0 \\ Ae^{rt} \neq 0 \end{cases} \quad \text{………… (2.60)}$$

To solve the characteristic equation we need to calculate its reduced discriminant

$$\Delta' = \lambda^2 - \omega_0^2$$

There are three cases depending on the sign of the reduced discriminant.

1st case: Supercritical damping, overdamping

If $\lambda^2 - \omega_0^2 > 0 \Longrightarrow \lambda^2 > \omega_0^2$ Or $Q < 0.5$, (high amortization).

Two real solutions for the characteristic equation:

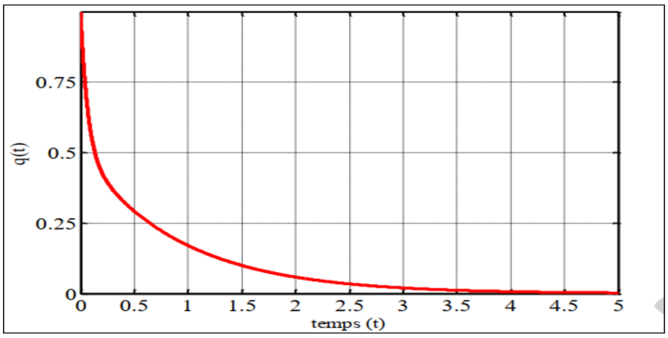
$$r_1 = -\lambda - \sqrt{\lambda^2 - \omega_0^2} \quad \text{et} \quad r_2 = -\lambda + \sqrt{\lambda^2 - \omega_0^2}$$

The resulting movement is: $q(t) = A_1e^{r_1t} + A_2e^{r_2t}$

Hence:

$$q(t) = e^{-\lambda t}\left(A_1e^{-\left(\sqrt{\lambda^2-\omega_0^2}\right)t} + A_2e^{+\left(\sqrt{\lambda^2-\omega_0^2}\right)t}\right) \quad \text{…………….. (2.61)}$$

The movement is said to be **aperiodic.**



2nd case: Critical damping

If $\lambda^2 - \omega_0^2 = 0 \Longrightarrow \lambda^2 = \omega_0^2$ Or $Q = 0$ (critical damping).

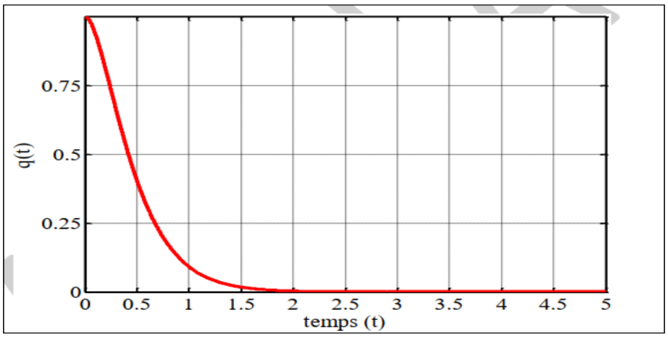
A dual solution for the characteristic equation :

$$r_1 = r_0 = r = -\lambda$$

The resulting movement is :

$$q(t) = (A + Bt)e^{rt} = (A + Bt)e^{-\lambda t} \quad \text{.................} \quad (2.62)$$

The movement is said to be in **critical mode**



3rd case: Subcritical damping, underdamping

If $\lambda^2 - \omega_0^2 < 0 \implies \lambda^2 < \omega_0^2$ Or $Q = 0$ (low damping)

Two complex solutions for the characteristic :

$$r_1 = -\lambda - j\sqrt{\omega_0^2 - \lambda^2} \qquad et \qquad r_2 = -\lambda + j\sqrt{\omega_0^2 - \lambda^2}$$

The resulting movement is :

$$q(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t}$$

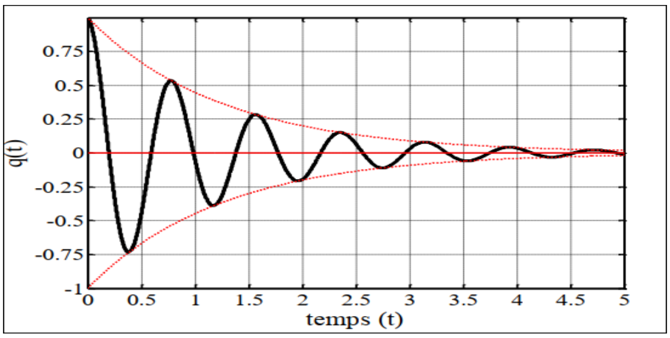
Hence :

$$q(t) = e^{-\lambda t}\,\mathbf{A}\cos(\omega_a t + \emptyset) \quad \text{.............} \quad (2.63)$$

The movement is said to be **pseudo-periodic**.

With

$$\begin{cases} \omega_a = \sqrt{\omega_0^2 - \lambda^2} & Is\ called\ pseudo - pulsation. \\ T_a = \dfrac{2\pi}{\omega_a} = \dfrac{2\pi}{\sqrt{\omega_0^2 - \lambda^2}} & Is\ called\ a\ pseudo - period \end{cases}$$



2.2.4 Logarithmic decrement:

This is the logarithm of the ratio of two successive amplitudes of damped oscillation, to evaluate the exponential decrease in the amplitude of the pseudo-periodic movement.

$$\delta = \ln \frac{q(t)}{q(t+T)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.64)$$

By using

$$q(t) = Ae^{-\lambda t}\cos(\omega_a t + \emptyset)$$

For $\Delta' < 0$ and $\omega_a = \sqrt{\omega_0^2 - \lambda^2}$. we find

$$\delta = \ln \frac{Ae^{-\lambda t}}{Ae^{-\lambda(t+T)}} = \lambda T_a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.65)$$

- **The quality factor of the mass-spring-damper system** (m, k, α) is given by :

$$Q = \frac{\omega_0}{2\lambda}$$

- Electrical system, **the factor quality system** $(R.L.C)$ is given by:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

- **Dissipation degree** :

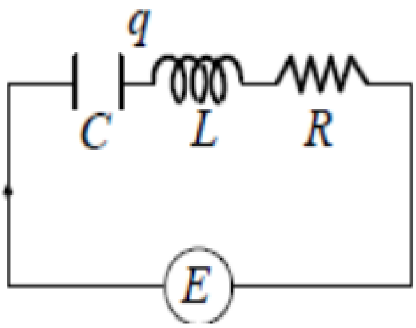
$$D = \frac{\omega_0}{2\lambda}$$

- **Time constant** :

$$\tau = \frac{1}{\lambda}$$

2.2.5 Electrical oscillator

An oscillating circuit $(R.L.C)$ shown opposite



Using Kirchhoff's law of meshes, the voltage balance is written as follows:

$$U_L + U_C + U_R = 0 \quad \ldots\ldots\ldots\ldots (2.66)$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{c}\int i(t)dt = 0$$

Divided by L, we find :

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) + \frac{1}{Lc}\int i(t)dt = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.67)$$

With: the current $i(t)$for a time dt gives a charge dq:

$$i(t) = \frac{dq(t)}{dt} \Rightarrow q(t) = \int i(t)dt \quad and \quad \frac{di(t)}{dt} = \frac{d^2q(t)}{dt^2}$$

(2.67) Become:

$$\frac{d^2q(t)}{dt^2} + \frac{R}{L}\dot{q}(t) + \frac{1}{Lc}\,q(t) = 0$$

$$\ddot{q}(t) + \frac{R}{L}\dot{q}(t) + \frac{1}{Lc}q(t) = 0 \quad \ldots\ldots\ldots\ldots\ldots (2.68)$$

$$\Leftrightarrow$$

$$\ddot{q}(t) + 2\lambda\dot{q}(t) + \omega_0^2 q(t) = 0$$

This equation is equivalent to the equation for damped oscillatory motion represented as follows:

$$\ddot{q}(t) + \frac{R}{L}\dot{q}(t) + \frac{1}{Lc}q(t) = 0 \qquad \Leftrightarrow \qquad \ddot{x}(t) + \frac{\alpha}{m}\dot{x}(t) + \frac{k}{m}x\,(t) = 0 \quad \ldots\ldots\ldots (2.68)$$

- **The proper pulsation and the proper period** are given as follows

$$\begin{cases} \omega_0 = \sqrt{\frac{1}{Lc}} \\ T_0 = 2\pi\sqrt{Lc} \\ \lambda = \frac{R}{2L} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.69)$$

For weak oscillations, the general solution of the equation is written:

$$q(t) = Ae^{-\lambda t}\cos(\omega_a t + \emptyset) \quad \ldots\ldots\ldots\ldots\ldots\ (2.70)$$

Remark:

For critical damping, $\lambda = \omega_0 \quad \Longrightarrow \quad \frac{R}{2L} = \sqrt{\frac{1}{Lc}}$

The critical resistance is given by

$$R = R_c = 2\sqrt{\frac{L}{C}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (2.71)$$

2.2.6 Analogy between mechanical and electrical systems

We obtain

Mechanical system	Electrical system
Displacement : $x(t)$	$q(t)$: Electrical charge
Masse : m	L : Coil inductance
Rigidity constant: k	$1/c$: Inverse of capacity
Coefficient of friction: α	R : Resistance

2.2.6.1 Capacitor energy

The energy stored in the capacitor at any particular time t is given by:

$$E_C(t) = \frac{1}{2}Vq(t) = \frac{1}{2C}q^2(t) = \frac{1}{2C}q_0^2\cos^2(\omega_a t + \emptyset) \quad \ldots\ldots\ldots\ldots\ (2.72)$$

2.2.6.2 Coil energy:

From electromagnetism we know that the energy stored in an inductor is given by the expression:

$$E_L(t) = \frac{1}{2}L\,I^2 = \frac{1}{2}L\left(\frac{dq(t)}{dt}\right)^2 = \frac{1}{2C}q_0^2\sin^2(\omega_a t + \emptyset) \quad \ldots\ldots\ldots\ldots\ (2.73)$$

2.2.6.3 The total energy, found by summing the two contributions, is thus: (Where the total energy is given by):

$$E_{Tot}(t) = E_c(t) + E_L(t) = \frac{1}{2C}q_0^2\left(\cos^2(\omega_a t + \emptyset) + \sin^2(\omega_a t + \emptyset)\right) = \frac{1}{2C}q_0^2 \ldots (2.74)$$

Chapter III

Forced oscillations of one degree of freedom systems

3. Introduction: In this chapter, we study a mechanical system forced to oscillate by the application of an external force varying harmonically with time. The amplitude of the oscillations, which is shown to depend on the frequency of the external force, reaches its peak value when the frequency of the applied force is close to the natural frequency of the system, a phenomena called resonance. However, details depend on the energy loss in the system, a property described by a quality factor Q, and the phase difference is described by so-called phasors. Emphasis is placed on how the system behaves when the external force starts vanishes.

3.1 Definitions

We define a forced oscillation as any system in motion under the action of an external force to compensate for energy losses due to damping α, see Figure 1.

Example: Vibratory system with a single forced degree of freedom and damped.

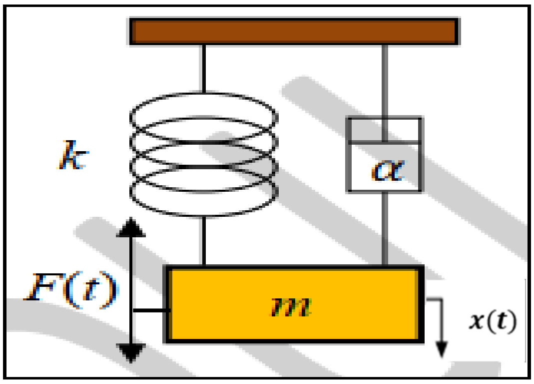


Figure 1: Forced and damped system

To study this type of oscillation, we consider a damped harmonic oscillator acted on by an external force, called a sinusoidal **excitation**:

$$F(t) = f_0 \cos \Omega t \quad Or \quad F(t) = f_0 \sin \Omega t \qquad (3\text{-}1)$$

3.2 Differential equation of forced systems

Recall the general form of the Lagrange equation for systems with one degree of freedom in a generalized string q is written as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)+\left(\frac{\partial D}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=\sum \vec{f}_{qext}=F(t) \qquad \text{In translation .... (3.2)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)+\left(\frac{\partial D}{\partial \dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=\sum \mathcal{M}_{/\Delta}\left(\vec{f}_{qext}\right) \qquad \text{In rotation .... (3.3)}$$

$\mathcal{M}_{/\Delta}\left(\vec{f}_{qext}\right)$: Is the moment of the applied force $[N.m]$ is equal to $\vec{f}_{qext}.L$

L: The lever arm: is the straight distance at which the force acts.

3.2.1 Mass-spring-damper system

In the figure 1 au-dessus, the mass m is attached to a spring of stiffness constant k and a damper of coefficient α

System kinetic energy: $T=\frac{1}{2}m\dot{x}^2=\frac{1}{2}m\dot{y}^2$

System potential energy: $U=\frac{1}{2}kx^2=\frac{1}{2}ky^2$

The dissipation function: $D=\frac{1}{2}\alpha\dot{x}^2=\frac{1}{2}\alpha\dot{y}^2$

The Lagrange function:

$$L=T-U \quad \Longrightarrow \quad L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2$$

☞ The Lagrangian formalism is written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)+\left(\frac{\partial D}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=F(t)$$

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=m\ddot{x} \ldots\ldots (3.4)\\ \frac{\partial L}{\partial x}=-kx \ldots\ldots\ldots. (3.5)\\ \frac{\partial D}{\partial \dot{x}}=\alpha\dot{x} \ldots\ldots\ldots\ldots (3.6)\\ F(t)=f_0\cos\Omega t \ldots (3.7)\end{cases}$$

$$(3.4)+(3.6)-(3.5)=(3.7) \qquad \Leftrightarrow \qquad m\ddot{x}+\alpha\dot{x}+kx=f_0\cos\Omega t$$

$$\ddot{x}+\frac{\alpha}{m}\dot{x}+\frac{k}{m}x=\frac{f_0}{m}\cos\Omega t$$

The differential equation of motion can therefore be written as:

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos\Omega t \quad \Leftrightarrow \quad \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) \text{ .... (3.8)}$$

This is an inhomogeneous (non-homogeneous) linear second order differential equation with constant coefficients and a second member.

With:
$$\begin{cases} \lambda = \dfrac{\alpha}{2m} \\ \omega_0^2 = \dfrac{k}{m} \\ A(t) = \dfrac{f_0}{m}\cos\Omega t \equiv A_0\cos\Omega t \end{cases} \text{ .... (3.9)}$$

3.3 Solution of the differential equation of motion

This is an inhomogeneous second order differential equation, and its general solution may be written as:

The general solution to this equation there are two solutions:

- A solution to the equation without a second member: is the general solution of the corresponding homogeneous equation $X_H(t)$.
- A solution to the equation with second member: is a particular solution to the inhomogeneous equation itself $X_P(t)$

The general solution may be written as:

$$X_g(t) = X(t) = X_H(t) + X_P(t) \text{ .... (3.10)}$$

3.3.1 Homogeneous solution (the transitional solution)

We have already found in Chap. 2 the general solution of the corresponding homogeneous equation, so the challenge is to find a particular solution.

We know that the solution of the homogeneous equation decreases with time to zero. Therefore, after a long time from start, the movement will be dominated by the external periodic force.

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0 \quad \Leftrightarrow \quad \ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = 0 \text{ .... (3.11)}$$

$$\Downarrow$$

$$r^2 + 2\lambda r + \omega_0^2 = 0$$

$$\Downarrow$$

$$\Delta' = \lambda^2 - \omega_0^2$$

There are three cases depending on the sign of the reduced discriminant.

$\Delta' > 0 \implies \lambda > \omega_0$: The system is highly damped

$\Delta' < 0 \implies \lambda < \omega_0$: Low depreciation

$\Delta' = 0 \implies \lambda = \omega_0$: Damping is critical

The solution to the differential equation is of the form:

$$X_H(t) = Be^{-\lambda t}\cos(\omega_a t + \varphi)$$

With: $\omega_a = \sqrt{\omega_0^2 - \lambda^2}$ In the case of pseudo-pulsation, the system is said to have a damped oscillatory movement corresponding to a **transient regime** (free-running in the case of weakly damped oscillations).

Remark:

If $\lambda < \omega_0$: the limit of $X_H(t)$ is always zero when $\boldsymbol{t}$ tends to infinity:

$$t \longrightarrow +\infty \implies e^{(-\lambda).(+\infty)} = 0$$

The homogeneous solution becomes negligible compared to the particular solution (the permanent regime or the permanent solution or stationary regime).

The general solution becomes:

$$X_g(t) = X(t) = X_P(t)$$

3.3.2 Particular solution (Special solution)

$X_P(t)$: is the permanent (stationary) solution of the non-homogeneous equation with second term $F(t)$. Then the particular solution $X_P(t)$ will be a sinusoidal function of the same pulsation Ω , if $F(t)$ is a sinusoidal function of pulsation Ω , the solution is of the form :

$$X_P(t) = A\cos(\Omega\,\mathrm{t} + \Phi) \;\; or \;\; X_P(t) = A\sin(\Omega\,\mathrm{t} + \Phi) \;\ldots\ldots\ldots (3.12)$$

A and Φ can be found using the complex representation as follows:

$$f_0\cos\Omega t \quad \longrightarrow \quad f_0 e^{j\Omega t}$$

$$q_P(t) = X_P(t) = A\cos(\Omega\,\mathrm{t} + \Phi) \quad \longrightarrow \quad \underline{q}_P(t) = \underline{X}_P(t) = Ae^{j(\Omega t + \Phi)}$$

$$\underline{X}_P(t) = Ae^{j\Phi}e^{j\Omega t}$$

$$\underline{X}_P(t) = \underline{A}e^{j\Omega t}$$

With: $\underline{A} = Ae^{j\Phi}$

We calculate $\dot{X}_p(t)\ and\ \ddot{X}_p(t)$

$$\begin{cases} \underline{\dot{X}}_p(t) = \underline{\dot{q}}_p(t) = \frac{dX_P(t)}{dt} = j\Omega\underline{A}e^{j\Omega t} \\ \underline{\ddot{X}}_p(t) = \underline{\ddot{q}}_p(t) = \frac{d^2\underline{X}_P(t)}{dt^2} = -\Omega^2\underline{A}e^{j\Omega t} \end{cases} \quad \text{.........} (3.13)$$

So the equation

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos\Omega t = A_0\cos\Omega t \iff \ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) = A_0\cos\Omega t$$

Such that

$$A_0 = \frac{f_0}{m}$$

Becomes equation (3.8)

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = A(t) = A_0 e^{j\Omega t}$$

After replacing the $X_P(t), \dot{X}_p(t)\ and\ \ddot{X}_p(t)$ we find:

$$\left(-\Omega^2\underline{A} + 2\lambda j\Omega\,\underline{A} + \omega_0^2\,\underline{A}\right)e^{j\Omega t} = A_0 e^{j\Omega t}$$

Divide by $e^{j\Omega t}$ and find:

$$(-\Omega^2 + 2\lambda j\Omega + \omega_0^2)\underline{A} = A_0$$

$$\underline{A} = \frac{A_0}{(\omega_0^2 - \Omega^2) + j2\lambda\Omega} = Ae^{j\Phi}$$

- **Calculating the amplitude A**

The amplitude of the movement is therefore:

$$A = |\underline{A}| = \frac{\|A_0\|}{\|(\omega_0^2 - \Omega^2) + j2\lambda\Omega\|}$$

$$A = \frac{A_0}{\sqrt{(\omega_0^2-\Omega^2)^2+4\lambda^2\Omega^2}} = Cte \ \ldots\ldots\ldots\ldots\ldots \ (3.14)$$

The phase (phase shift) Φ of the motion between $x(t)$ or $q(t)$and $F(t)$is given by:

We have: $$\underline{A} = \frac{A_0}{(\omega_0^2-\Omega^2)+j2\lambda\Omega} = Ae^{j\Phi}$$

The conjugate of this equation is:

$$\underline{A}^* = \frac{A_0}{(\omega_0^2-\Omega^2)-j2\lambda\Omega} = Ae^{-j\Phi}$$

We multiply the formulae of $\underline{A}$ in Inverted and we find:

$$\underline{A} = \frac{A_0}{[(\omega_0^2-\Omega^2)+j2\lambda\Omega]} * \frac{[(\omega_0^2-\Omega^2)-j2\lambda\Omega]}{[(\omega_0^2-\Omega^2)-j2\lambda\Omega]} = Ae^{j\Phi}$$

$$\underline{A} = \frac{A_0(\omega_0^2-\Omega^2)-A_0(j2\lambda\Omega)}{(\omega_0^2-\Omega^2)^2+(2\lambda\Omega)^2} = Ae^{j\Phi}$$

$$\underline{A} = \frac{A_0(\omega_0^2-\Omega^2)}{(\omega_0^2-\Omega^2)^2+(2\lambda\Omega)^2} - j\frac{2A_0\lambda\Omega}{(\omega_0^2-\Omega^2)^2+(2\lambda\Omega)^2}$$

- **Calculating the phase Φ**

$$\tan\Phi = \frac{Im\ \underline{A}}{Re\ \underline{A}} = -\frac{2\lambda\Omega}{(\omega_0^2-\Omega^2)}$$

$$\Phi = \text{Arc}\ \tan\frac{2\lambda\Omega}{(\Omega^2-\omega_0^2)} \ \ldots\ldots\ldots\ldots \ (3.15)$$

So the **permanent regime** solution is :

$$X_P(t) = q_P(t) = A\cos(\Omega\,t+\Phi) \ \ldots\ldots\ldots\ldots\ldots \ (3.16)$$

Hence

$$\boldsymbol{X_P(t)} = \frac{\boldsymbol{A_0}}{\sqrt{(\boldsymbol{\omega_0^2-\Omega^2})^2+4\boldsymbol{\lambda^2\Omega^2}}}\cos\left(\boldsymbol{\Omega}\,t+\boldsymbol{Arc\ tan}\frac{\boldsymbol{2\lambda\Omega}}{(\boldsymbol{\Omega^2-\omega_0^2})}\right) \ \ldots\ldots\ldots\ldots \ (3.17)$$

If the homogeneous solution is not negligible, the general solution is given by:

$$X_g = \boldsymbol{B}e^{-\lambda t}\cos(\boldsymbol{\omega_a}t+\boldsymbol{\varphi}) + \frac{\boldsymbol{A_0}}{\sqrt{(\boldsymbol{\omega_0^2-\Omega^2})^2+4\boldsymbol{\lambda^2\Omega^2}}}\cos\left(\boldsymbol{\Omega}\,t+\boldsymbol{Arc\ tan}\frac{\boldsymbol{2\lambda\Omega}}{(\boldsymbol{\Omega^2-\omega_0^2})}\right) \ldots \ (3.18)$$

With:

$$\begin{cases} \boldsymbol{\omega_a} = \sqrt{\omega_0^2 - \lambda^2} \\ \boldsymbol{T_a} = \frac{2\pi}{\omega_a} = \frac{2\pi}{\sqrt{\omega_0^2-\lambda^2}} \\ \boldsymbol{A_0} = \frac{f_0}{m} \end{cases} \quad \ldots\ldots\ldots\ldots (3.19)$$

Remark:

- Steady state (stationary) because it remains as long as the external force (F_{ext}) is applied. We note the dependence of the amplitude A on the pulsation $\boldsymbol{\omega}$.
- It should be noted that at the start of the movement $X(t)$ represents the transient state. Over time the solution $X_H(t)$ becomes negligible compared with the solution $X_P(t)$, which defines the steady state (permanent regime), as shown in Figure 2:

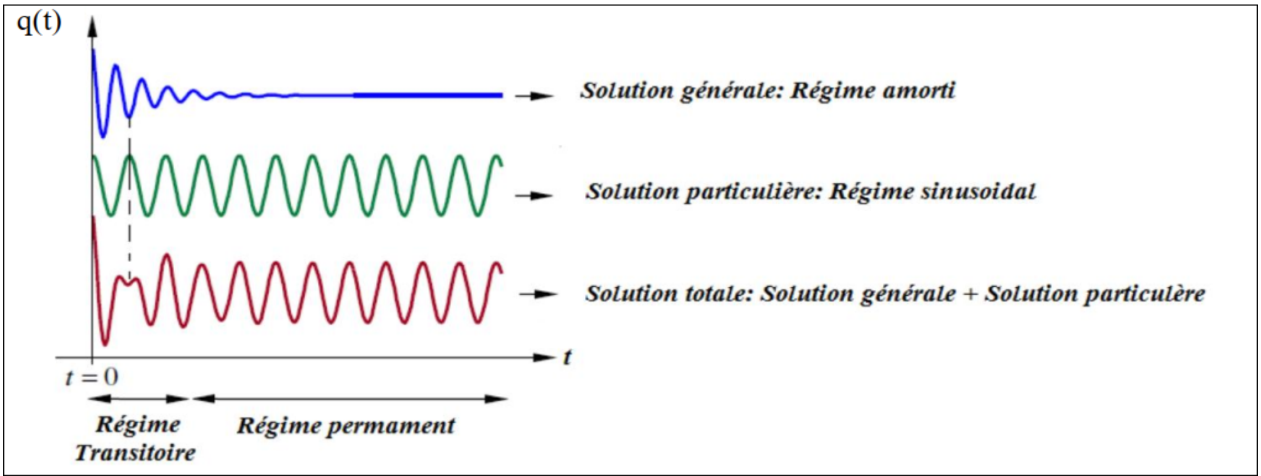


Figure 2: Superposition of transient and steady state conditions.

- We see that the amplitude is greatest when the frequency of the applied force is nearly the same as the natural frequency of oscillation in the same system when the applied force and damping are both absent. We call this phenomenon resonance, and it will be discussed in more detail in the next section.

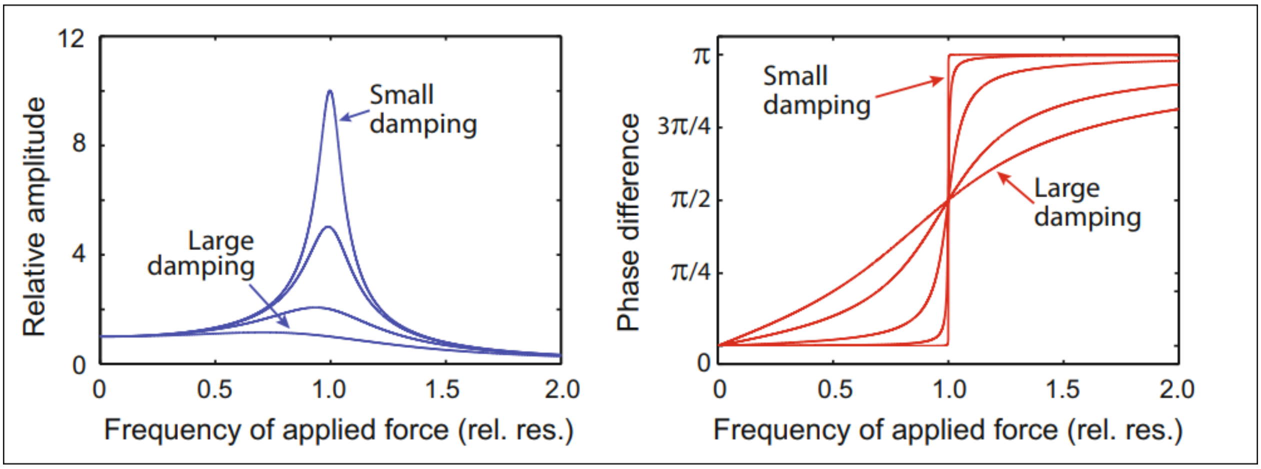


Figure 3: The amplitude of a forced oscillation (***left***) and the phase difference between the output and the applied force (***right***) as a function of the frequency of the applied force

3.4 Resonance phenomenon

3.4.1 Study of the variations in amplitude $\underline{A}$ and phase Φ as a function of the excitation pulsation

3.4.1.1 Amplitude resonance

One sees from Eq. (3.14) that the amplitude of the forced oscillations varies with the frequency of the applied force. When the frequency is such that the amplitude is greatest, the system is said to be at resonance.

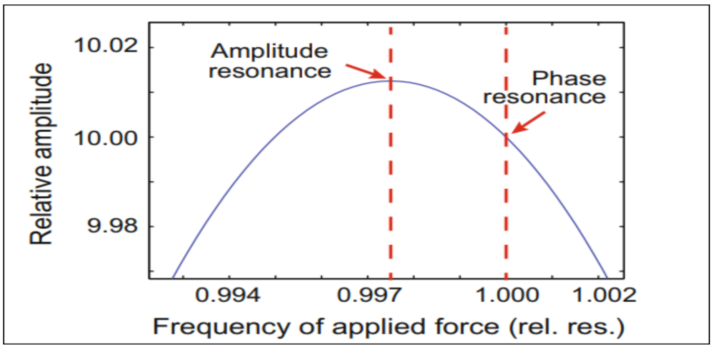


Figure 4: A close-up view of the relative amplitude in a forced oscillation as a function of the frequency of the applied force. Note the numbers along the axes.

1- Mechanical resonance occurs when the period of the exciter is close to the natural period of the resonator.

2- The phenomenon of resonance is observed in the context of forced oscillations. The excitation pulsation Ω is called the **resonance** pulsation Ω_R when the amplitude of the oscillations A is maximum.

Let us find mathematical expressions for the two resonance frequencies.

The amplitude resonance frequency can be found by differentiating the expression for the amplitude given by Eq. (3.4) (a common procedure for finding extreme values). We calculate the Ω_R angular frequency at which. A Is maximum when:

$$\frac{dA(\Omega)}{d\Omega} = 0 \text{ ……………. (3.20)}$$

We have

$$A(\Omega) = \left|\underline{A}\right| = \frac{A_0}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2}}$$

$$\frac{dA(\Omega)}{d\Omega} = 0 \quad \Leftrightarrow \quad \frac{A_0[-4\Omega(\omega_0^2 - \Omega^2) + 8\lambda^2\Omega\,]}{2[(\omega_0^2 - \Omega^2)^2 + 4\lambda^2\Omega^2]^{3/2}} = 0$$

$$\Leftrightarrow \begin{cases} A_0 \neq 0 \\ -4[(\omega_0^2 - \Omega^2) + 2\lambda^2]\Omega = 0 \end{cases}$$

Either

$$\begin{cases} \Omega_1 = 0 \\ \Omega_2 = \Omega_R = \sqrt{\omega_0^2 - 2\lambda^2} \quad \Leftrightarrow \quad \Omega_R = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \end{cases} \text{ ………………. (3.21)}$$

The quality factor is given by :

$$Q = \frac{\omega_0}{2\lambda}$$

The amplitude resonance frequency is:

$$f_{amp.res} = \frac{1}{2\pi}\sqrt{\omega_0^2 - 2\lambda^2} = \frac{1}{2\pi}\sqrt{\omega_0^2 - \frac{\alpha^2}{2m^2}} \text{ ……………. (3.22)}$$

In this case, we have the phenomenon of Resonance

In this case; the maximum amplitude is given by

$$A_{max} = \frac{A_0}{\sqrt{4\lambda^2\omega_0^2 - 4\lambda^4}}$$

$$A_{max} = \frac{f_0}{m\,\omega_0^2}\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.23)$$

To determine the maximum and minimum of A

- ☞ For $\Omega_1 = 0 \Rightarrow A(\Omega_1)$: the amplitude is minimum (study the sign of the second derivative)
- ☞ For $\Omega_2 = \sqrt{\omega_0^2 - 2\lambda^2} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \Rightarrow A(\Omega_2)$: the amplitude is maximum $\longrightarrow$ we obtain the maximum response of the system.

So in this case we have the phenomenon of **resonance**.

$$\Omega_2 = \Omega_R = \sqrt{\omega_0^2 - 2\lambda^2} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.24)$$

For there to be resonance:

$$\omega_0^2 - 2\lambda^2 > 0 \quad \Rightarrow \quad Q > \frac{1}{\sqrt{2}}$$

At resonant frequency Ω_R, the amplitude is given by :

$$A_{max}(\Omega_R) = \frac{A_0}{\sqrt{(\omega_0^2-\Omega_R^2)^2+4\lambda^2\Omega_R^2}} \quad \ldots\ldots\ldots\ldots\ldots (3.25)$$

After replacing $\Omega_R = \sqrt{\omega_0^2 - 2\lambda^2}$ we obtain:

$$A_{max}(\Omega_R) = \frac{A_0}{2\lambda\sqrt{\omega_0^2-\lambda^2}} \quad \ldots\ldots\ldots\ldots (3.26)$$

For very low damping $\lambda \ll \Omega_R \approx \omega_0$

λ: Negligible in front of ω_0 Becomes the maximum amplitude is equal to:

$$A_{max}(\Omega_R) = \frac{A_0}{2\lambda\omega_0} \quad for\ \lambda \ll \omega_0 \quad \ldots\ldots\ldots\ldots\ldots (3.27)$$

The variation of the amplitude as a function of the pulsation of the excitation force for different values of Q is shown in Figure 5.

The greater the quality factor (low damping), the sharper the curve and the greater the maximum.

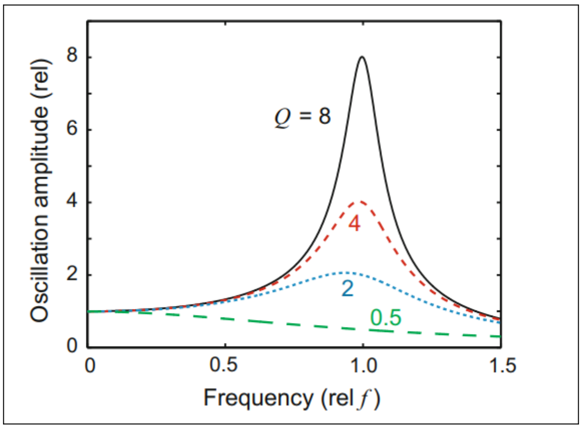


Figure 5: The variation of amplitude as a function of pulsation or frequency

When the frequency of the applied force changes relative to the natural frequency of the system, the amplitude is greatest when the two frequencies are almost equal. The higher the quality factor Q (i.e. the lower the loss), the higher the amplitude of the resonance.

3.4.1.2 Phase resonance

The phase (phase shift) Φ of the movement becomes: On at

When $\Omega_R \approx \omega_0$: This pulsation is called the phase resonance pulsation.

$$\tan\Phi = \frac{Im\,\underline{A}}{Re\,\underline{A}} = -\frac{2\lambda\Omega_R}{(\omega_0^2 - \Omega_R^2)} = -\infty \Longrightarrow \Phi = -\frac{\pi}{2}$$

The oscillator is in phase resonance

The phase resonance frequency is:

$$f_{ph.res} = \frac{1}{2\pi}\,\omega_0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (3.28)$$

- The oscillator is always out of phase with the force and this delay increases as the pulsation increases. $(-\pi < \Phi < 0)$

If $\lambda = 0 \implies \tan\Phi = 0 \quad \Leftrightarrow \quad \Phi = 0$ ou $\Phi = -\pi$

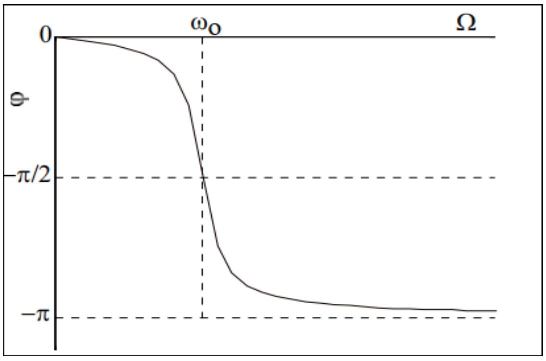


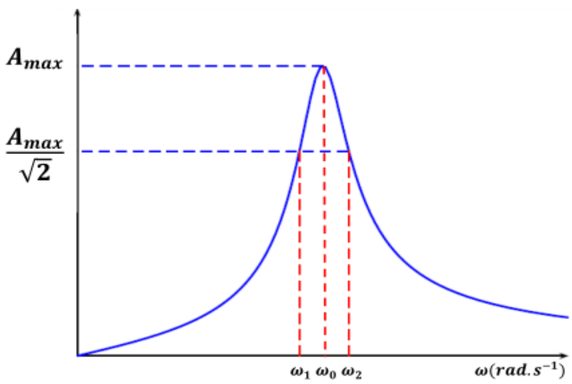
Figure 6: Phase shift Φ as a function of Ω

Remark

We observe that the two resonance frequencies coincide only when $\alpha = 0$ (no damping).

3.4.1.3 Bandwidth and quality factor

We give the graph of $A(\Omega)$ as a function of the excitation pulsation Ω or ω, Here (ω ≡ Ω) is :



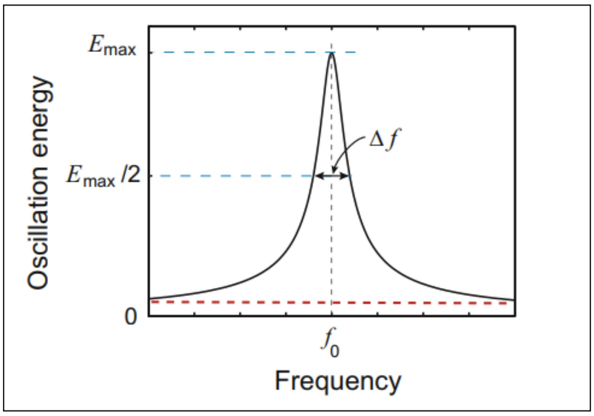


Figure 7: Resonance curves in ω and f.

To characterize the sharpness (intensity) of an oscillator's response as a function of pulsation, a pass band (width of the pass band) ΔΩ or Δω is defined as follows:

Where

$$\Delta\Omega = \Omega_2 - \Omega_1 \quad or \quad \Delta\omega = \omega_2 - \omega_1 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (3.29)$$

Ω_1, Ω_2 are pulsations deduced by the intersection of the amplitude curve of the system response $A(\Omega)$ and the straight line:

$\Omega_R = \omega_0$: **Resonance** pulsation.

❖ **The quality factor *Q*** for low damping is

The factor tells us something about how easy it is to make the system oscillate, or how long the system will continue to oscillate after the driving force is removed. This is more or less equivalent to how small loss/friction is in the system.

The quality factor for a **spring oscillator** is given by:

$$Q = \frac{m\omega_0}{\alpha} = \frac{\omega_0}{2\lambda} \text{................ (3.30)}$$

3.5 Electrical oscillations

Consider the oscillating circuit (R, L, C) supplied by an external sinusoidal voltage source $E(t)$ such that:

$$E(t) = U_0\, cos(\Omega t) = U_0 e^{j\Omega t} \text{..................... (3.31)}$$

An oscillating circuit $(R.L.C.)$ shown below in Figure 8, powered by an external voltage source.

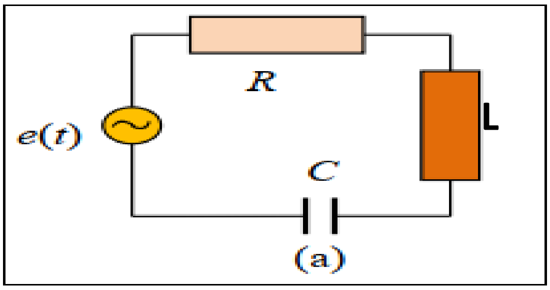


Figure 8: A series $R.C.L$ circuit driven by a harmonically varying applied.

Using Kirchhoff's law of meshes, the voltage balance is written as follows:

$$U_L + U_C + U_R = E(t) \text{ .... (3.32)}$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{c}\int i(t)dt = E(t)$$

Dividing by L we find

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) + \frac{1}{Lc}\int i(t)dt = \frac{E(t)}{L}$$

Such as: the current $i(t)$for a time dt gives a charge dq:

$$i(t) = \frac{dq(t)}{dt} \Rightarrow q(t) = \int i(t)dt \quad and \quad \frac{di(t)}{dt} = \frac{d^2q(t)}{dt^2}$$

$$\frac{d^2q(t)}{dt^2} + \frac{R}{L}\frac{dq(t)}{dt} + \frac{1}{Lc}q(t) = \frac{E(t)}{L} \quad \text{................ (3.33)}$$

Note that equation (3-33) is equivalent to the equation for forced oscillatory motion as follows :

$$\ddot{q}(t) + \frac{R}{L}\dot{q}(t) + \frac{1}{Lc}q(t) = \frac{E(t)}{L} \quad \Leftrightarrow \quad \ddot{x}(t) + \frac{\alpha}{m}\dot{x}(t) + \frac{k}{m}x\,(t) = \frac{F(t)}{m} \quad \text{........ (3.34)}$$

$$\ddot{q} + 2\lambda\dot{q} + \omega_0^2 q = B(t) = B_0 e^{j\Omega t} \quad and \quad B_0 = \frac{U_0}{L}$$

With

$$\begin{cases} \lambda = \frac{R}{2L} \\ \omega_0^2 = \frac{1}{\sqrt{LC}} \end{cases} \quad \text{.................. (3.35)}$$

The connection between $R.C.L$, current and phase can be elegantly illustrated by means of phasors.

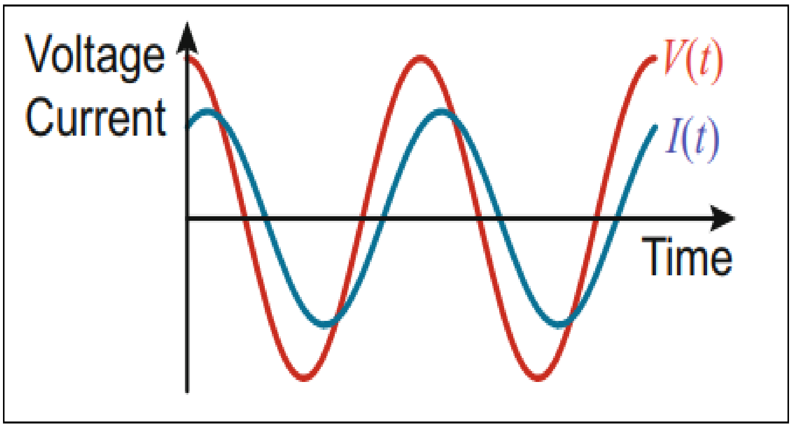


Figure 9: A time plot in which the current slightly leads the applied voltage

Figure 9 shows the time development of voltage and current in a time plot. The current in the circuit is slightly leading the applied voltage. For a series RCL circuit with an applied voltage, this means that the applied frequency is lower than the resonant frequency of the circuit. Note that phasors can be used only after the initial rather complicated oscillatory pattern is over, and we have a steady sinusoidal output corresponding to the particular solution of differential equation.

3.6 Analogy between mechanical and electrical systems

Is of the following form:

Mechanical system	**Electrical system**
Déplacement : $x(t)$	$q(t)$: Charge électrique
Masse : m	L : Inductance de bobine
Constante de raideur : k	$1/c$: Inverse de la capacité
Coefficient de frottement : α	R : Résistance
La tension extérieure : $E(t)$	$F(t)$ la force excitation extérieure

❖ **Electrical system (RLC)**

✍ **impedance** : is equal to

$$Z_{éq} = R + j\left(L\omega - \frac{1}{C\omega}\right)$$

✍ The **modulus** of the current is written

$$I_0 = \frac{|E(t)|}{|Z_{éq}|} = \frac{U_0}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}}$$

✍ The current is maximum for

$$L\omega - \frac{1}{C\omega} = 0 \quad \text{Or} \quad L\Omega - \frac{1}{C\,\Omega} = 0$$

$$I_0 = \frac{U_0}{R}$$

☞ This gives the value of the corresponding pulsation:

$$\omega_R = \frac{1}{\sqrt{LC}} = \omega_0$$

- The **resonance pulsation**, which depends only on the inductance and capacitance, corresponding to the value:

$$\omega_R = \omega_0 = \frac{1}{\sqrt{LC}}$$

- **Bandwidth** is defined:

$$\Delta\omega = \Omega_2 - \Omega_1 = \omega_2 - \omega_1 = \frac{R}{L}$$

- The **quality factor** is written:

$$Q = \frac{\Omega_R}{\Omega_2 - \Omega_1} = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{\omega_0}{2\lambda} = \frac{L\omega_0}{R}$$

Chapter IV

4. Free oscillations of systems with two degrees of freedom

4. Introduction

A system with several degrees of freedom is a system that requires several independent coordinates to describe its motion. The number of degrees of freedom determines the number of Lagrange equations and the number of modes propes.

4.1 Systems with 2 degrees of freedom

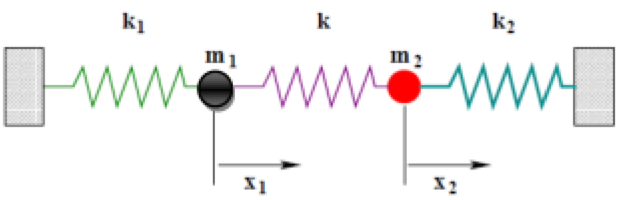
To study systems with two degrees of freedom, it is necessary to write two differential equations of motion that can be obtained from Lagrange's equations (in the absence of any friction):

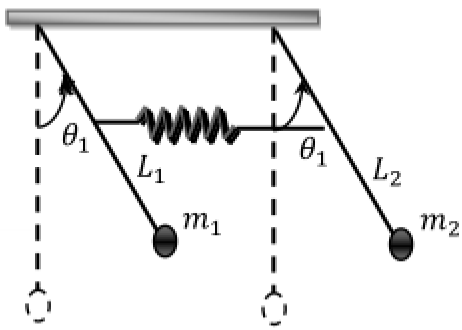
$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \left(\frac{\partial L}{\partial q_2}\right) = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$

A system with 2 degrees of freedom has 02 generalized coordinates, 02 differential equations and 02 Eigen pulsations (ω_1, ω_2)

4.2 Type of coupling

4.2.1 Elastic coupling





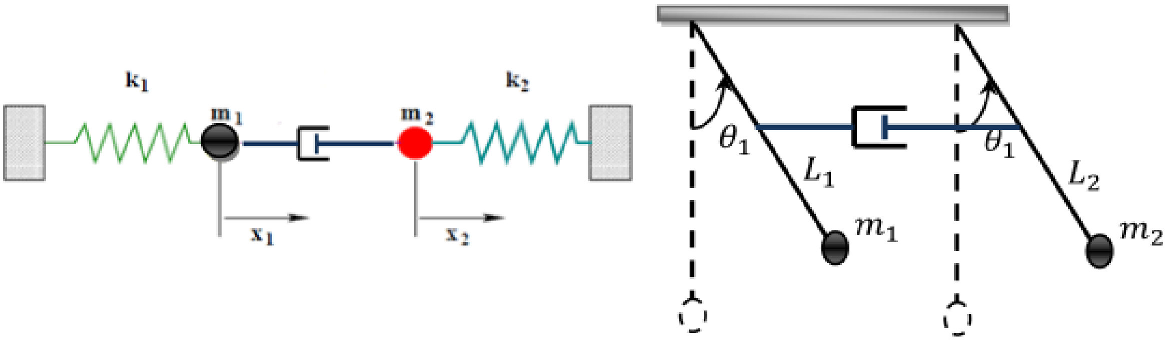
The equations of motion are written (elastic coupling state):

$$\begin{cases} \ddot{q}_1 + a_1 q_1 - a_2 q_2 = 0 \\ \ddot{q}_2 + a_1 q_2 - a_2 q_1 = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.2)$$

Such that: $q_1 \equiv x_1 , q_2 \equiv x_2$ $\quad Or \quad$ $q_1 \equiv \theta_1 , q_2 \equiv \theta_2$.

$\boldsymbol{a_2 q_2}$ and $\boldsymbol{a_4 q_1}$ are the coupling terms. $\boldsymbol{a_1, a_2, a_3, a_4}$ Are constant.

4.2.2 Viscous coupling

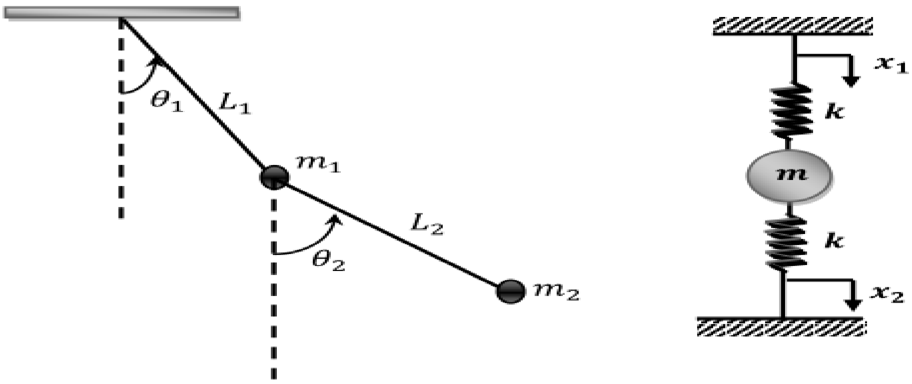


The corresponding differential equations are:

$$\begin{cases} \ddot{q}_1 + 2\lambda_1 \dot{q}_1 + \omega_0^2 q_1 = b_1 \dot{q}_2 \\ \ddot{q}_2 + 2\lambda_2 \dot{q}_2 + \omega_0^2 q_2 = b_2 \dot{q}_1 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

Such that: $\boldsymbol{b_1 \dot{q}_2}$ and $\boldsymbol{b_2 \dot{q}_1}$ are the coupling term, b_1 and b_2 are Constants.

4.2.3 Inertial coupling



The corresponding differential equations are:

$$\begin{cases} \ddot{q}_1 + 2\lambda_1 \dot{q}_1 + \omega_0^2 q_1 = c_1 \ddot{q}_2 \\ \ddot{q}_2 + 2\lambda_2 \dot{q}_2 + \omega_0^2 q_2 = c_2 \ddot{q}_1 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.4)$$

Such that: $\boldsymbol{c_1 \ddot{q}_2}$ and $\boldsymbol{c_2 \ddot{q}_1}$ are the coupling terms, $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ are Constants.

4.3 Examples of systems with 2 degrees of freedom

4.3.1 Translational mass-spring system

We neglect all friction

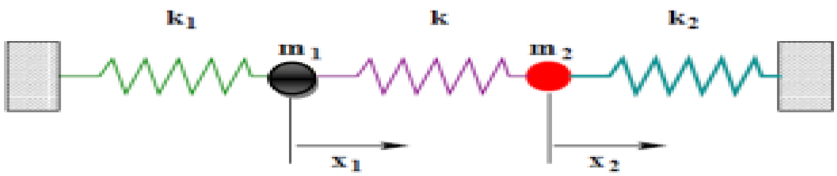


Figure 1: Systems with 2 degrees of freedom

4.3.2 Differential equations of movement

Consider two springs of stiffness constants k_1, k_2 which are coupled by a horizontal spring of stiffness constant k, shown in the figure above.

These can be obtained from the Lagrange equations for each coordinate $x_1(t)$ and $x_2(t)$.

The kinetic energy is $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

The potential energy is $U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k(x_1 - x_2)^2$

$$U = \frac{1}{2}(k_1 + k)x_1^2 + \frac{1}{2}(k_2 + k)x_2^2 - kx_1x_2$$

The Lagrangian $L = T - U$ is written as

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - \frac{1}{2}(k_1 + k)x_1^2 - \frac{1}{2}(k_2 + k)x_2^2 + kx_1x_2$$

The Lagrange equations can be written as:

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \left(\frac{\partial L}{\partial x_1}\right) = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \left(\frac{\partial L}{\partial x_2}\right) = 0 \end{cases}$$

Hence the system of differential equations of motion is

$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k)x_1 - kx_2 = 0 \\ m_2\ddot{x}_2 + (k_2 + k)x_2 - kx_1 = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (4.5)$$

Hence

$$\begin{cases} \ddot{x}_1 + \left(\frac{k_1+k}{m_1}\right)x_1 - \left(\frac{k}{m_1}\right)x_2 = 0 \\ \ddot{x}_2 + \left(\frac{k_2+k}{m_2}\right)x_2 - \left(\frac{k}{m_2}\right)x_1 = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (4.6)$$

The terms $-\boldsymbol{kx_2}$ and $-\boldsymbol{kx_1}$which appear in the first and second equations are called coupling terms, and the two differential equations are said to be coupled, and the two differential equations are said to be coupled.

4.3.3 Solutions of differential equations

The two solutions to these two differential equations are periodic functions and are composed of two harmonic functions with different pulsations and different amplitudes. Suppose that one of these harmonic components is written as:

$$\begin{cases} x_1(t) = A_1 \cos(\omega t + \phi_1) \\ x_2(t) = A_2 \cos(\omega t + \phi_2) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots.. (4.7)$$

Where A_1, A_2et $\phi_1, \phi_2$are constants, depend on the initial conditions and ω is one of the pulsations of the system.

To find the solution, we use the complex representation as before and are written as

$$\begin{cases} x_1(t) = A_1 \cos(\omega t + \phi_1) \;\longrightarrow\; \underline{x}_1(t) = \underline{A}_1 e^{j\omega t} \\ x_2(t) = A_2 \cos(\omega t + \phi_2) \;\longrightarrow\; \underline{x}_2(t) = \underline{A}_2 e^{j\omega t} \end{cases} \ldots\ldots\ldots (4-7)$$

With: $\begin{cases} \underline{\dot{x}}_1(t) = \underline{A}_1\, j\omega\, e^{j\omega t} \\ \underline{\dot{x}}_2(t) = \underline{A}_2\, j\omega\, e^{j\omega t} \end{cases}$; $\begin{matrix} \underline{\ddot{x}}_1(t) = -\underline{A}_1\, \omega^2\, e^{j\omega t} \\ \underline{\ddot{x}}_2(t) = -\underline{A}_2\, \omega^2\, e^{j\omega t} \end{matrix}$ And $\begin{cases} \underline{A}_1 = A_1 e^{j\phi_1} \\ \underline{A}_2 = A_2 e^{j\phi_2} \end{cases}$

By injecting in (4.5)

4.3.4 The proper modes of oscillation (Eigenmodes)

The mode is the state in which the dynamic elements of the system perform a harmonic oscillation with the same pulsation corresponding to one of its two pulsations.

- **Calculation of proper pulsations**

$$\omega_i \quad i = 1,2 \ldots\ldots$$

By replacing the solutions in the differential system, we obtain the following symmetrical linear system, which can be written as:

$$\begin{cases} (-m_1\omega^2 + k_1 + k)\underline{A}_1 - k\underline{A}_2 = 0 \\ -k\underline{A}_1 + (-m_2\omega^2 + k_2 + k)\underline{A}_2 = 0 \end{cases} \ldots\ldots\ldots\ldots.. (4.9)$$

Or

$$\begin{cases} \left(-\omega^2+\dfrac{k_1+k}{m_1}\right)\underline{A}_1-\left(\dfrac{k}{m_1}\right)\underline{A}_2=0 \\ -\left(\dfrac{k}{m_2}\right)\underline{A}_1+\left(-\omega^2+\dfrac{k_2+k}{m_2}\right)\underline{A}_2=0 \end{cases}$$

The equivalent matrix equation (4.9) is as follows:

$$\begin{bmatrix} \left(-\omega^2+\frac{k_1+k}{m_1}\right) & \left(-\frac{k}{m_1}\right) \\ \left(-\frac{k}{m_2}\right) & \left(-\omega^2+\frac{k_2+k}{m_2}\right) \end{bmatrix}\begin{pmatrix}\underline{A}_1\\ \underline{A}_2\end{pmatrix}=0 \;\ldots\ldots\ldots\ldots\; (4.10)$$

This system admits a non-identically zero solution (non-trivial solution) only if the determinant of the coefficients of $\underline{A}_1$ and $\underline{A}_2$is equal to zero

$$\Delta(\omega)=0$$

$$\Delta(\omega)=\begin{vmatrix}(k_1+k-m_1\omega^2) & -k \\ -k & (k_2+k-m_2\omega^2)\end{vmatrix}=0$$

Or other formula

$$\Delta(\omega)=\begin{vmatrix}\left(-\omega^2+\dfrac{k_1+k}{m_1}\right) & -\left(\dfrac{k}{m_1}\right) \\ -\left(\dfrac{k}{m_2}\right) & \left(-\omega^2+\dfrac{k_2+k}{m_2}\right)\end{vmatrix}=0$$

This gives us the **characteristic** equation or **proper pulsation** equation (we obtain the following **parametric** equation):

$$\left(-\omega^2+\frac{k_1+k}{m_1}\right)\left(-\omega^2+\frac{k_2+k}{m_2}\right)-\left(\frac{k}{m_2}\right)\left(\frac{k}{m_1}\right)=0$$

$$(-m_1\omega^2+k_1+k)(-m_2\omega^2+k_2+k)-\left(\frac{k}{m_2}\right)\left(\frac{k}{m_1}\right)=0$$

Or encoure:

$$\omega^4-\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)\omega^2+\frac{k_1k_2+k_1k+k_2k}{m_1m_2}=0 \;\ldots\ldots\ldots\ldots\ldots\; (4.11)$$

This equation is a quadratic equation in **ω** which admits two positive real solutions **ω_1** and **ω_2,** called as the system's proper pulsations. Each of the coordinates, $x_1(t)$ and $x_1(t)$, has two harmonic components with pulsations $\boldsymbol{\omega_1}$and $\boldsymbol{\omega_2}$

After calculation, we obtain the two proper pulsations:

$$\begin{cases} \omega_{1P}^2 = \frac{1}{2}\times\left[\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)+\sqrt{\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)^2-4\left(\left(\frac{k_1+k}{m_1}\right)\left(\frac{k_2+k}{m_2}\right)-\left(\frac{k}{m_1}\right)\left(\frac{k}{m_2}\right)\right)}\right] \\ \omega_{2P}^2 = \frac{1}{2}\times\left[\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)-\sqrt{\left(\frac{k_1+k}{m_1}+\frac{k_2+k}{m_2}\right)^2-4\left(\left(\frac{k_1+k}{m_1}\right)\left(\frac{k_2+k}{m_2}\right)-\left(\frac{k}{m_1}\right)\left(\frac{k}{m_2}\right)\right)}\right] \end{cases} \ldots.(4.12)$$

The general solution of the system is a combination of two modes written as a superposition of the two eigenmodes (proper modes) as follows:

$$\begin{cases} x_1(t) = A_{11}\cos(\omega_1 t+\varphi_1)+A_{12}\cos(\omega_2 t+\varphi_2) \\ x_2(t) = A_{21}\cos(\omega_1 t+\varphi_1)+A_{22}\cos(\omega_2 t+\varphi_2) \end{cases} \ldots\ldots\ldots\ldots (4.13)$$

The constants $A_{11}, A_{12}, A_{21}, A_{22}, \varphi_1, \varphi_2$ and are determined from the initial conditions.

We have four conditions for six unknowns. The amplitudes in each mode are linked by relationships that we call amplitude ratios.

Remark:

1- The lowest frequency term corresponding to the pulsation $\boldsymbol{\omega_1}$is called the fundamental. The other term, of pulsation $\boldsymbol{\omega_2}$, is called the harmonic.

2- The first index refers to the coordinate and the second to the pulsation. For example, $A_{12}$is the amplitude of $x_1(t)$ at pulsation ω_2.

<u>**Mode 1**</u> $(\omega = \omega_1 = \omega_{1P})$

- ☞ When $A_{12} = A_{22} = 0$, $x_1(t)$ and $x_2(t)$corresponding to the first particular solution are sinusoidal functions, **in phase**, of pulsation ω_1, the system is said to oscillate in the first mode. In this case the solution is written:

$$\begin{cases} x_1(t) = A_{11}\cos(\omega_1 t+\varphi_1) \\ x_2(t) = A_{21}\cos(\omega_1 t+\varphi_1) \end{cases} \ldots\ldots\ldots (4.14)$$

$x_1(t)$ and $x_2(t)$They satisfy the equations of motion or must satisfy the system of differential equations, which gives :

$$\begin{cases} (-m_1\omega_1^2 + k_1 + k)A_{11} - kA_{21} = 0 \\ -kA_{11} + (-m_2\omega_1^2 + k_2 + k)A_{21} = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.15)$$

These two equations give the ratio of the amplitudes in the first or fundamental mode, and we replace $\omega_1^2$by its expression to find the first ratio:

$$\mu_1 = \frac{A_{21}}{A_{11}} = \frac{-m_1\omega_1^2 + k_1 + k}{k} = \frac{k}{-m_2\omega_1^2 + k_2 + k} > 0$$

<u>Mode 2</u> $(\omega = \omega_2 = \omega_{2P})$

☞ When $A_{11} = A_{21} = 0$, $x_1(t)$ and $x_2(t)$ correspond to the second particular solution and are sinusoidal functions, in phase opposition, of pulsation ω_2, the system is said to oscillate in the second mode. In this case the solution is written :

$$\begin{cases} x_1(t) = A_{12}\cos(\omega_2 t + \varphi_2) \\ x_2(t) = A_{22}\cos(\omega_2 t + \varphi_2) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.16)$$

$x_1(t)$ et $x_2(t)$ They satisfy the equations of motion or must satisfy the system of differential equations, which gives :

$$\begin{cases} (-m_1\omega_2^2 + k_1 + k)A_{12} - kA_{22} = 0 \\ -kA_{12} + (-m_2\omega_2^2 + k_2 + k)A_{22} = 0 \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.17)$$

These two equations give the ratio of the amplitudes in the first or fundamental mode, and we replace $\omega_2^2$by its expression to find the first ratio:

$$\mu_2 = \frac{A_{22}}{A_{12}} = \frac{-m_1\omega_2^2 + k_1 + k}{k} = \frac{k}{-m_2\omega_2^2 + k_2 + k} < 0$$

These ratios allow us to write the general solution $x_1(t), x_2(t)$ is a linear combination of these two particular solutions. $x_1(t), x_2(t)$ are written as follows:

$$\begin{cases} x_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\cos(\omega_2 t + \varphi_2) \\ x_2(t) = \mu_1 A_{11}\cos(\omega_1 t + \varphi_1) + \mu_2 A_{12}\cos(\omega_2 t + \varphi_2) \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.18)$$

The constants $A_{11}, A_{12}, \varphi_1, \varphi_2$ are integration constants whose values are fixed by the initial conditions.

4.3.5 Special case of two identical oscillators

- **Calculation of proper pulsations and integration**

We suppose that: $m_1 = m_2 = m$ and $k_1 = k_2 = k_0$

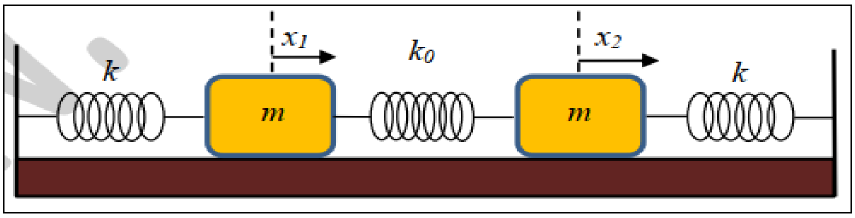


Figure 2: System with 2 degrees of equimassic freedom.

We have (4.10)

$$\begin{bmatrix} \left(-\omega^2 + \frac{k_0 + k}{m}\right) & \left(-\frac{k}{m}\right) \\ \left(-\frac{k}{m}\right) & \left(-\omega^2 + \frac{k_0 + k}{m}\right) \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\left(-\omega^2 + \frac{k_0 + k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

In this case the eigen pulsations are respectively equal to

$$\begin{cases} \omega_1 = \sqrt{\frac{k_0}{m}} \\ \omega_2 = \sqrt{\frac{k_0+2k}{m}} = \omega_1\sqrt{1 + \frac{2k}{k_0}} \end{cases} \quad \dots\dots\dots\dots (4.19)$$

We replace ω_1 and $\omega_2$by their expressions, to find the amplitude ratios:

$$\begin{cases} \mu_1 = \frac{A_{21}}{A_{11}} = 1 \\ \mu_2 = \frac{A_{22}}{A_{12}} = -1 \end{cases} \quad \dots\dots\dots\dots (4.20)$$

From the initial conditions we can determine the constants $A_{11}, A_{12}, \varphi_1, \varphi_2$equation (4.18).

Let x_{10}, x_{20}, $\dot{x}_{10}$ and $\dot{x}_{20}$be the respective initial values of x_1, x_2, $\dot{x}_1$ and $\dot{x}_2$

$$\text{If} \quad t = 0s \rightarrow \begin{cases} x_1 = x_{10} \\ x_2 = x_{20} \end{cases} \rightarrow \begin{cases} \dot{x}_1 = \dot{x}_{10} \\ \dot{x}_1 = \dot{x}_{10} \end{cases} \dots\dots\dots (4.21)$$

We obtain the following system of equations after the derivative and replacing:

$$\begin{cases} A_{11}\cos(\varphi_1) + A_{12}\cos(\varphi_2) = x_{10} \\ A_{11}\cos(\varphi_1) - A_{12}\cos(\varphi_2) = x_{20} \end{cases} \ldots\ldots\ldots (4.22)$$

And

$$\begin{cases} -\omega_1 A_{11}\sin(\varphi_1) - \omega_2 A_{12}\sin(\varphi_2) = \dot{x}_{10} \\ -\omega_1 A_{11}\sin(\varphi_1) + \omega_2 A_{12}\sin(\varphi_2) = \dot{x}_{20} \end{cases} \ldots\ldots\ldots.. (4.23)$$

The solutions to this system of equations are

$$\begin{cases} A_{11} = \frac{x_{10}+x_{20}}{2\cos(\varphi_1)} \\ A_{12} = \frac{x_{10}-x_{20}}{2\cos(\varphi_2)} \end{cases} \ldots\ldots\ldots.. (4.24)$$

And

$$\begin{cases} A_{11} = \frac{\dot{x}_{10}+\dot{x}_{20}}{2\sin(\varphi_1)} \\ A_{12} = \frac{\dot{x}_{20}-\dot{x}_{10}}{2\sin(\varphi_2)} \end{cases} \ldots\ldots\ldots.. (4.25)$$

Consider the following special case:

Case 1: $(x_{10} = x_{20} = x_0$ and $\dot{x}_{10} = \dot{x}_{20} = 0)$

In this case

$$\begin{cases} A_{11} = x_0 \\ A_{12} = 0 \end{cases} \quad et \quad \begin{cases} \varphi_1 = 0 \\ \varphi_2 = 0 \end{cases} \ldots\ldots\ldots (4.26)$$

These conditions give:

$$x_1(t) = x_2(t) = x_0\cos(\omega_1 t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (4.27)$$

The two masses oscillate in phase at the same pulsation ω_1. The system is said to oscillate in the first mode (Figure 3).

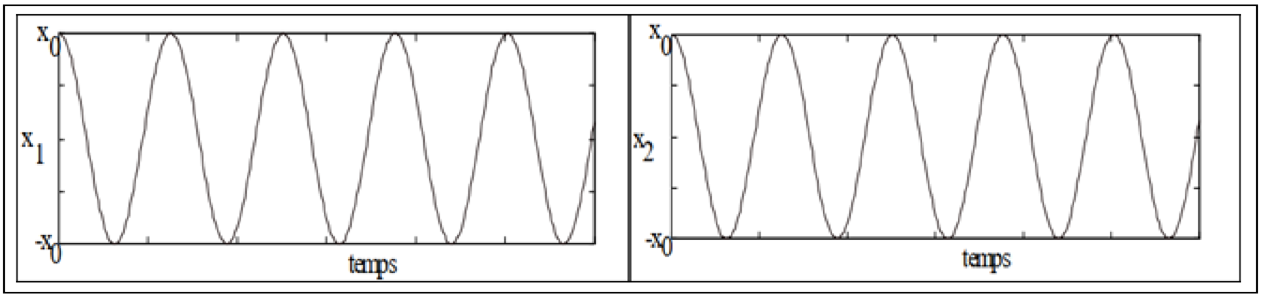


Figure 3: Oscillations in the fundamental mode

Case 2: ($x_{10} = -x_{20} = x_0$ and $\dot{x}_{10} = \dot{x}_{20} = 0$)

In this case

$$\begin{cases} A_{11} = 0 \\ A_{12} = x_0 \end{cases} \; et \; \begin{cases} \varphi_1 = 0 \\ \varphi_2 = 0 \end{cases} \quad \text{..........} \; (4.28)$$

These conditions give:

$$\begin{cases} x_1(t) = x_0 \cos(\omega_2 t) \\ x_2(t) = -x_0 \cos(\omega_2 t) \end{cases} \quad \text{.....................} \; (4.29)$$

The system oscillates in the second mode because the two masses oscillate in phase opposition with the same pulsation ω_2(figure 4).

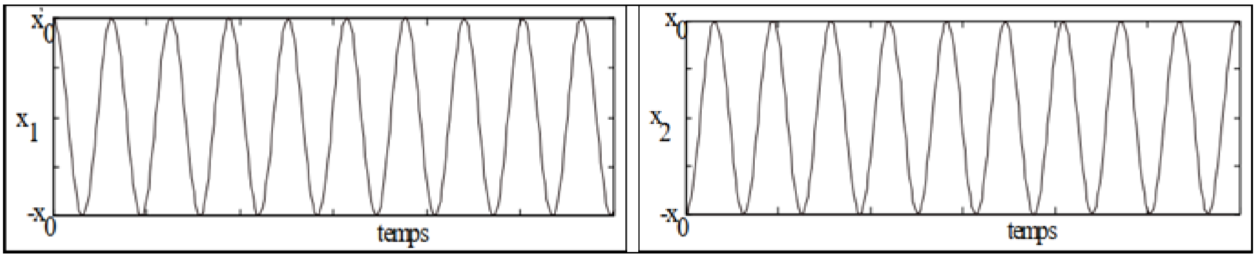


Figure 4: Oscillations in the harmonic mode

Case 3: ($x_{10} = x_0, x_{20} = 0$ and $\dot{x}_{10} = \dot{x}_{20} = 0$)

In this case

$$\begin{cases} A_{11} = {x_0}/{2} \\ A_{12} = {x_0}/{2} \end{cases} \; et \; \begin{cases} \varphi_1 = 0 \\ \varphi_2 = 0 \end{cases} \quad \text{..........} \; (4.30)$$

These conditions give solutions which can be written as:

$$\begin{cases} x_1(t) = \frac{x_0}{2}\cos(\omega_1 t) + \frac{x_0}{2}\cos(\omega_2 t) \\ x_2(t) = \frac{x_0}{2}\cos(\omega_1 t) - \frac{x_0}{2}\cos(\omega_2 t) \end{cases} \quad \text{...............} \; (4.31)$$

The solutions are no longer purely sinusoidal functions of time but linear combinations of two sinusoidal functions of respective pulsations ω_1 and ω_2. $x_1(t)$ and $x_2(t)$ can be written as :

$$\begin{cases} x_1(t) = x_0 \cos\left[\left(\frac{\omega_2 - 1}{2}\right) t\right] \cos\left[\left(\frac{\omega_1 + \omega_2}{2}\right) t\right] \\ x_2(t) = x_0 \sin\left[\left(\frac{\omega_2 - 1}{2}\right) t\right] \sin\left[\left(\frac{\omega_1 + \omega_2}{2}\right) t\right] \end{cases} \quad \text{..........} \; (4.32)$$

We call this mode of oscillation the beating mode, which we obtain in the case where $\omega_1$is very different from ω_2 (i.e. if $K \gg k_0$) (Figure 5).

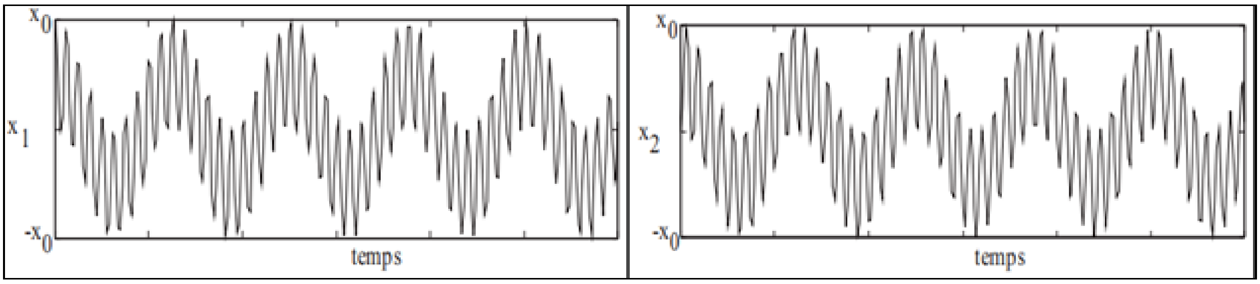


Figure 5: Oscillations in beat mode

If ω_1 is not very different from ω_2 (i.e. if $K \ll k_0$), we observe a beating phenomenon (see Figure 6 below).

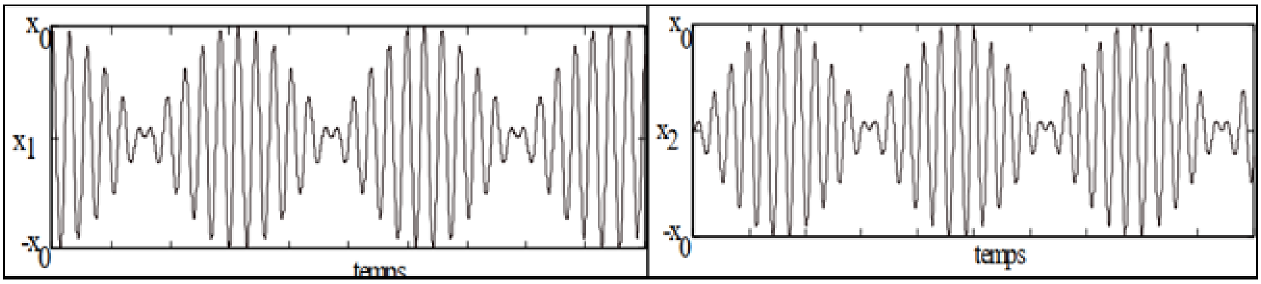


Figure 6: Beat (Beating) phenomenon.

4.4 Coupled pendulums

Let's consider the case of two identical simple pendulums coupled by a spring of stiffness T and which perform oscillations of low amplitude marked by the angles θ_1 and θ_2. Rods of lengths $l_1$and l_2 ($l = l_1 = l_2$) are negligible masses.

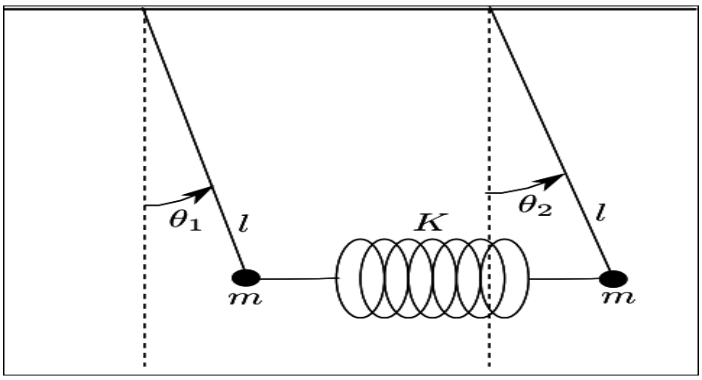


Figure 7: Coupled pendulums.

4.4.1 Differential equations of the system

Let's establish the differential equations of motion for small-amplitude oscillations. Kinetic energy and potential energy can be written as:

The kinetic energy: $T=\frac{1}{2}j_{/\Delta_1}\dot{\theta}_1^2+\frac{1}{2}j_{/\Delta_2}\dot{\theta}_2^2$

$$T=\frac{1}{2}ml^2\dot{\theta}_1^2+\frac{1}{2}ml^2\dot{\theta}_2^2 \text{ ……….. (4.33)}$$

Avec : $\begin{cases}x_1=l\theta_1\\x_2=l\theta_2\end{cases}\rightarrow\begin{cases}\dot{x}_1=l\dot{\theta}_1\\\dot{x}_2=l\dot{\theta}_2\end{cases}$ et $\cos\theta=1-\frac{\theta^2}{2}=-\frac{\theta^2}{2}$

The potential energy: $U=-mgl\cos\theta_1-mgl\cos\theta_2+\frac{1}{2}K(x_1-x_2)^2$

$$U=-mgl\left(-\frac{\theta_1^2}{2}\right)-mgl\left(-\frac{\theta_2^2}{2}\right)+\frac{1}{2}K(l\theta_1-l\theta_2)^2$$

$$U=\frac{1}{2}(Kl^2+mgl)\theta_1^2+\frac{1}{2}(Kl^2+mgl)\theta_2^2-Kl^2\theta_1\theta_2 \text{ …………….. (4.34)}$$

The term $-Kl^2\theta_1\theta_2$ is the coupling term.

The Lagrangian $L=T-U$ is then written

$$L=\frac{1}{2}ml^2\dot{\theta}_1^2+\frac{1}{2}ml^2\dot{\theta}_2^2-\frac{1}{2}(Kl^2+mgl)\theta_1^2+\frac{1}{2}(Kl^2+mgl)\theta_2^2-Kl^2\theta_1\theta_2\ldots \text{ (4.35)}$$

The Lagrange equations can be written as:

$$\begin{cases}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\left(\frac{\partial L}{\partial\theta_1}\right)=0\\\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\left(\frac{\partial L}{\partial\theta_2}\right)=0\end{cases}$$

Hence the system of differential equations of motion is:

$$\begin{cases}ml^2\ddot{\theta}_1^2+(Kl^2+mgl)\theta_1-Kl^2\theta_2=0\\ml^2\ddot{\theta}_2^2-Kl^2\theta_1+(Kl^2+mgl)\theta_2=0\end{cases} \text{ ………. (4.36)}$$

4.4.2 Solution the differential equations

The solution to this system of differential equations is of the form:

$$\begin{cases}\theta_1(t)=A_1\cos(\omega t+\phi_2)\\\theta_2(t)=A_2\cos(\omega t+\phi_2)\end{cases} \text{ …….. (4.37)}$$

By replacing the solutions in the differential system, we obtain a system which can be written as:

$$\begin{cases} [(Kl^2 + mgl - ml^2\omega^2)]A_1 - Kl^2 A_2 = 0 \\ -Kl^2 A_1 + [(Kl^2 + mgl - ml^2\omega^2)]A_2 = 0 \end{cases} \quad \dots\dots\dots\dots\dots\dots (4.38)$$

This system of equations has non-zero solutions only if ω is a solution to the frequency equation

$$(Kl^2 + mgl - ml^2\omega^2)^2 - (Kl^2)^2 = 0$$

From this we can calculate the expression for the Eigen pulsations $\omega_1$and ω_2:

$$\omega_1 = \sqrt{\frac{g}{l}} \quad \text{And} \quad \omega_1 = \sqrt{\frac{g}{l} + \frac{2K}{m}}$$

The solutions to the system of differential equations are therefore

$$\begin{cases} \theta_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\cos(\omega_2 t + \varphi_2) \\ \theta_2(t) = A_{21}\cos(\omega_1 t + \varphi_1) + A_{22}\cos(\omega_2 t + \varphi_2) \end{cases} \quad \dots\dots\dots\dots\dots\dots (4.39)$$

To calculate the ratios of the amplitudes in the modes, we assume that the system oscillates in either the first mode or the second mode. In the first mode, we obtain the system

$$\begin{cases} (Kl^2 + mgl - ml^2\omega_1^2) - Kl^2\mu_1 = 0 \\ -Kl^2 + (Kl^2 + mgl - ml^2\omega_1^2)\mu_1 = 0 \end{cases}$$

In the second mode, we obtain

$$\begin{cases} (Kl^2 + mgl - ml^2\omega_2^2) - Kl^2\mu_2 = 0 \\ -Kl^2 + (Kl^2 + mgl - ml^2\omega_2^2)\mu_2 = 0 \end{cases}$$

The solutions of the system of differential equations, for values of the ratio of the amplitudes in the modes $\mu_1 = 1$ and $\mu_2 = -1$ are then written:

$$\begin{cases} \theta_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\cos(\omega_2 t + \varphi_2) \\ \theta_2(t) = A_{11}\cos(\omega_1 t + \varphi_1) - A_{12}\cos(\omega_2 t + \varphi_2) \end{cases} \quad \dots\dots\dots\dots\dots\dots (4.40)$$

Part II: Waves

Chapter V

Generalities on propagation phenomena

5. One-dimensional propagation phenomena

Introduction

In the vibratory phenomena discussed in the previous chapters, we were interested in phenomena or physical quantities that depended on a single variable, time. We are now going to look at a whole series of phenomena that are described by a function that depends on both time $\boldsymbol{t}$ and a space variable, $\boldsymbol{r}$ for example.

5.1 Generalities and Basic Definitions

5.1.1. Definition of a Wave: A wave is the propagation of a temporary local disturbance that travels through a material medium with energy transport but without matter transport (there is no matter transport).

A Wave: is the propagation of a disturbance that produces a reversible variation in the local physical properties of the medium as it passes.

A Wave: is characterized by its frequency f, wavelength λ, and amplitude $A(r\ ,t)$ (defined at every point in space-time with the three spatial coordinates and the time coordinate).

Example: Energy transfer without matter transport: Figure 1 illustrates these properties. As the wave passes, the boat rises, its potential energy increases, but the boat remains at the same position.

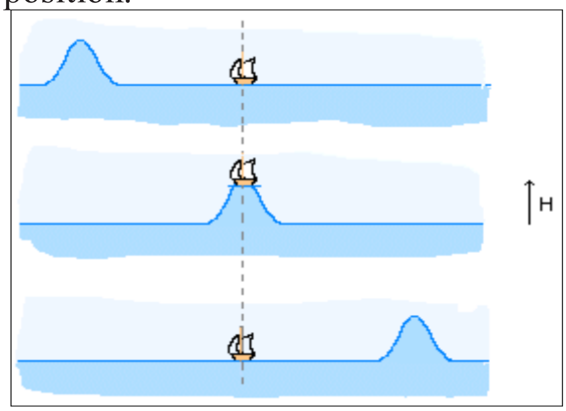


Figure 1: Wave movement (Motion)

5.1.2 Types of Waves: There are different types of waves

5.1.2.1 Mechanical Waves: These propagate in a solid, liquid, or gaseous material medium.

Examples: Sound propagation, water waves, elastic waves in a rod, piano string, spring, etc.

- ❖ **One dimensional mechanical wave**

Examples: A disturbance moving along a string or a spring is a one-dimensional mechanical wave (Figure 2).

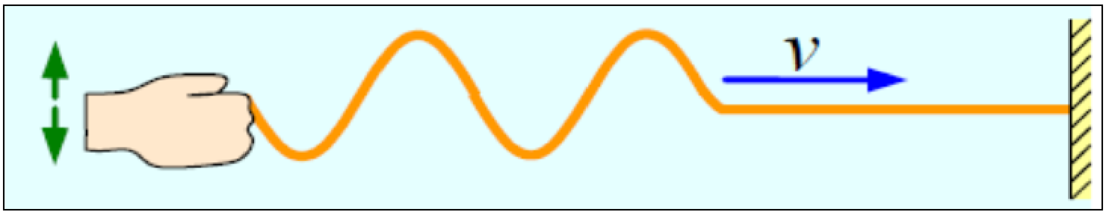


Figure 2: Motion of a disturbance along a string

- ❖ **Two dimensional mechanical wave**

Example: On the surface of water

Waves on the surface of water constitute two-dimensional mechanical waves. A series of concentric dark and light circles is observed (Figure 3).

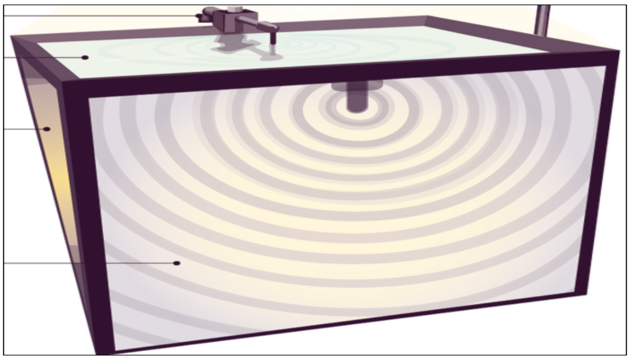


Figure 3: Wave tank for studying wave propagation.

- ❖ **Three dimensional mechanical wave**

Sound is a three-dimensional wave. Sound waves are associated with a frequency between 20 Hz and 20,000 Hz. To visualize sound propagation, microphones connected to computerized data processing systems are used.

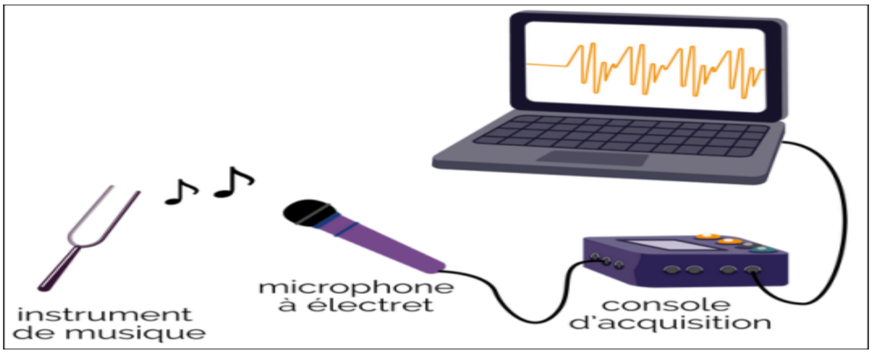


Figure 4: Sound Wave

5.1.2.2 Electromagnetic Waves

It is propagate in a variable electromagnetic field and does not require a material medium for propagation.

Exemples: Radio waves, micro waves, light, etc.

5.1.3 Modes of Propagation

5.1.3.1 Transverse waves

A wave is **transverse** when the direction of deformation (or disturbance) and the direction of wave propagation are perpendicular.

Example: Waves on the surface of water and waves propagating along an elastic string.

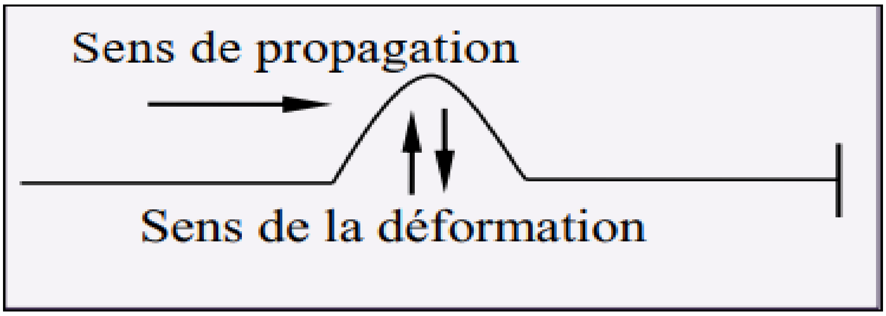


Figure 5: Movement along a string and a spring

5.1.3.2 Longitudinal mechanical waves

A wave is longitudinal when the direction of deformation (or disturbance) and the direction of wave propagation are the same.

Example: A wave propagating along a spring when we compress a few coils at its end.

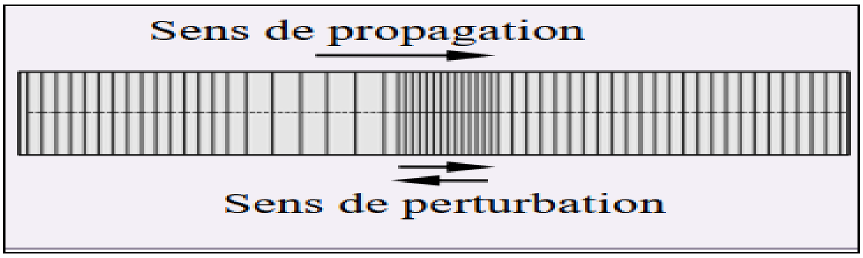


Figure 6: Propagation of a disturbance along a spring

5.1.3.3 Progressive waves: The disturbed quantity propagates in one direction like the incident wave. It is a sustained succession of mechanical signals propagating in an assumed infinite medium. We can generate a progressive mechanical wave by the sustained vibration of a source. (There are no waves in the opposite direction).

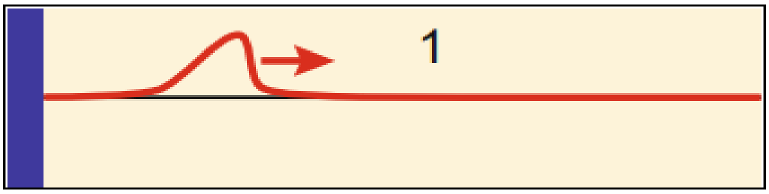


Figure 7: Incident Wave

5.1.3.4 Standing waves

This is the superposition of an *incident* wave and a *reflected* wave. The disturbed physical quantity does not propagate.

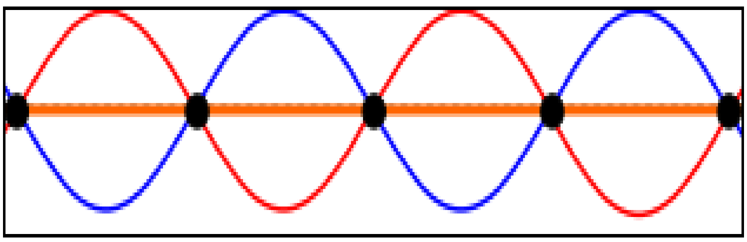


Figure 8: Standing Wave

5.2 One-dimensional propagation

5.2.1 Wave equation

The previous phenomena are governed by a partial differential equation, called the d'Alembert equation or wave equation $\psi(x,t)$, which depends on both time $\boldsymbol{t}$ and a spatial variable $\boldsymbol{x}$, described as follows:

$$\frac{\partial^2\psi}{\partial t^2} = V^2\frac{\partial^2\psi}{\partial x^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.1)$$

Where

$\psi(x,t)$: is the wave propagating in the direction (ox).

V: Is the propagation speed constant (Uniform).

5.2.2 Solution of the wave equation

- **D'Alembert's method**

To solve the one-dimensional wave equation, we use the following change of variables:

$$\begin{cases} g = t + \dfrac{x}{V} \\ h = t - \dfrac{x}{V} \end{cases}$$

Calculating the partial derivatives with respect to $\boldsymbol{t}$ and $\boldsymbol{x}$, we obtain:

$$\begin{cases} \dfrac{\partial g}{\partial t} = 1 \\ \dfrac{\partial h}{\partial t} = 1 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial g}{\partial x} = \dfrac{1}{V} \\ \dfrac{\partial h}{\partial x} = -\dfrac{1}{V} \end{cases}$$

We calculate the quantities $\frac{\partial\psi}{\partial t}, \frac{\partial^2\psi}{\partial t^2}, \frac{\partial\psi}{\partial x}, \frac{\partial^2\psi}{\partial x^2}$ in terms of the new variables g, h and $\psi(x,t)$:

- For the first term of equation (5.1), we have:

1) $\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial g}\frac{\partial g}{\partial t} + \frac{\partial\psi}{\partial h}\frac{\partial h}{\partial t}$

$$\frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial g} + \frac{\partial\psi}{\partial h} \ldots\ldots\ldots (5.2)$$

Because: $\frac{\partial g}{\partial t} = 1 \quad and \quad \frac{\partial h}{\partial t} = 1$

2)

$$\frac{\partial^2\psi}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)=\frac{\partial}{\partial g}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)\frac{\partial g}{\partial t}+\frac{\partial}{\partial h}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)\frac{\partial h}{\partial t}$$

$$\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial g^2}+\frac{\partial^2\psi}{\partial h^2}+2\ \frac{\partial^2\psi}{\partial g\partial h}\ \ldots\ldots\ldots\ldots\ldots\ldots\ (5.3)$$

- For the second term of equation (5.1), we have:

3)

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial g}\frac{\partial g}{\partial x}+\frac{\partial\psi}{\partial h}\frac{\partial h}{\partial x}=\frac{1}{V}\left(\frac{\partial\psi}{\partial g}-\frac{\partial\psi}{\partial h}\right)$$

Hence

$$\frac{\partial\psi}{\partial x}=\frac{1}{V}\left(\frac{\partial\psi}{\partial g}-\frac{\partial\psi}{\partial h}\right)\ \ldots\ldots\ldots\ (5.4)$$

Because: $\frac{\partial g}{\partial x}=\frac{1}{V}$ *and* $\frac{\partial h}{\partial t}=-\frac{1}{V}$

4)

$$\frac{\partial^2\psi}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)=\frac{\partial}{\partial g}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)\frac{\partial g}{\partial x}+\frac{\partial}{\partial h}\left(\frac{\partial\psi}{\partial g}+\frac{\partial\psi}{\partial h}\right)\frac{\partial h}{\partial x}$$

Taking these results into account and knowing that

$$\frac{\partial^2\psi}{\partial g\partial h}=\frac{\partial^2\psi}{\partial h\partial g}$$

Hence

$$\frac{\partial^2\psi}{\partial x^2}=\frac{1}{V^2}\left(\frac{\partial^2\psi}{\partial g^2}+\frac{\partial^2\psi}{\partial h^2}-2\ \frac{\partial^2\psi}{\partial g\partial h}\right)\ \ldots\ldots\ldots\ldots\ldots\ldots\ (5.5)$$

By substituting relations (5.3) and (5.5) into relation (5.1), we obtain:

$$\frac{\partial^2\psi}{\partial t^2}=V^2\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{\partial^2\psi}{\partial g^2}+\frac{\partial^2\psi}{\partial h^2}+2\ \frac{\partial^2\psi}{\partial g\partial h}=V^2\left[\frac{1}{V^2}\left(\frac{\partial^2\psi}{\partial g^2}+\frac{\partial^2\psi}{\partial h^2}-2\ \frac{\partial^2\psi}{\partial g\partial h}\right)\right]$$

On the other hand:

$$\frac{\partial^2\psi}{\partial g\partial h}=0 \Rightarrow \begin{cases}\frac{\partial}{\partial g}\left(\frac{\partial\psi}{\partial h}\right)=0\\ \frac{\partial}{\partial h}\left(\frac{\partial\psi}{\partial g}\right)=0\end{cases} \Rightarrow \begin{cases}\frac{\partial\psi}{\partial h}=\acute{f}(h)\\ \frac{\partial\psi}{\partial g}=\acute{f}(g)\end{cases}$$

This expression indicates that:

$\frac{\partial\psi}{\partial h}$: does not depend on h.

$\frac{\partial\psi}{\partial g}$: does not depend on g.

This implies that:

$$\begin{cases}\frac{\partial\psi}{\partial h}=\acute{f}(h)\\ \frac{\partial\psi}{\partial g}=\acute{f}(g)\end{cases} \Rightarrow \begin{cases}\psi_1(h)=\int\frac{\partial\psi}{\partial h}=f(h)+C_1(g)\\ \psi_2(g)=\int\frac{\partial\psi}{\partial g}=f(g)+C_1(h)\end{cases}$$

The integration constants cancel out at points $x=0$ and $t=0$.

Thus, the general (total) solution is written in the form:

$$\psi(x,t)=\psi_1(h)+\psi_2(g)$$

$$\psi(x,t)=F\left(t-\frac{x}{V}\right)+L\left(t+\frac{x}{V}\right) \text{ .... (5.6)}$$

Or

$$\psi(x,t)=F(Vt-x)+L(Vt+x)$$

The functions $\mathrm{F}\left(\mathrm{t}-\frac{\mathrm{x}}{\mathrm{V}}\right)$ and $\mathrm{L}\left(\mathrm{t}+\frac{\mathrm{x}}{\mathrm{V}}\right)$ are functions whose nature is determined by the boundary conditions (initial conditions) imposed on the solution $\psi(x,t)$.

Remark:

In the sinusoidal regime, the solution is written as follows:

$$\psi(x,t)=A\cos\left[\omega\left(t-\frac{x}{V}\right)\right]+B\cos\left[\omega\left(t+\frac{x}{V}\right)\right] \text{ .... (5.7)}$$

5.2.3 Properties of particular solutions $\psi_1\left(t-\frac{x}{V}\right)$ *and* $\psi_2\left(t+\frac{x}{V}\right)$

a) Properties of $\psi_1\left(t-\frac{x}{V}\right)$

If we consider the function $F\left(t-\frac{x}{V}\right)$, the disturbance at position x_1 at time t_1 is the same as at position x_2 at time $t_2 > t_1$. For this, we assume that the boundary conditions are such that $F\left(t-\frac{x}{V}\right)$ is constantly zero, i.e., this problem is formulated by the following equality:

$$\psi(x_1, t_1) = \psi(x_2, t_2)$$

This translates to

$$F\left(t_1-\frac{x_1}{V}\right) = F\left(t_2+\frac{x_2}{V}\right)$$

This equation is satisfied if

$$t_1-\frac{x_1}{V} = t_2+\frac{x_2}{V} \implies x_2 - x_1 = V(t_2 - t_1) > 0$$

Hence, since $t_2 + t_1 > 0 \implies x_2 > x_1$, these two points are distant by:

$$x_2 = x_1 + V(t_2 - t_1) > 0$$

This function $F\left(t_1-\frac{x_1}{V}\right)$ is called a progressive wave, corresponding to a disturbance propagating in the positive xx direction (see Figure 9).

With

$$V = \frac{x_2 - x_1}{t_2 - t_1} > 0$$

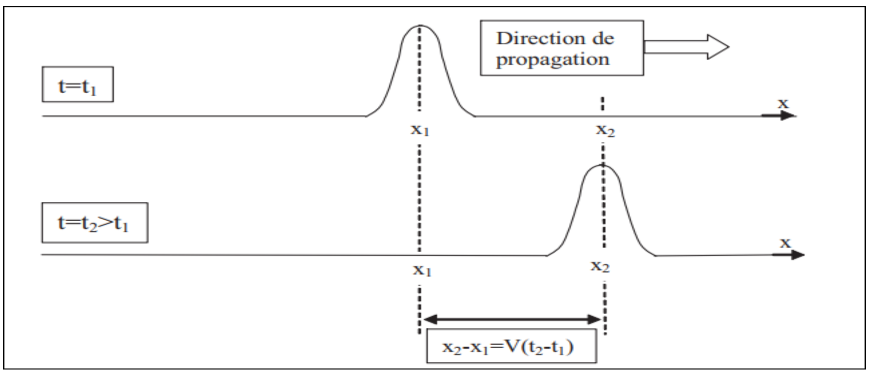


Figure 9: Progressive Wave in the Direction of Increasing xx (Positive)

b) Properties of $\psi_2\left(t+\frac{x}{V}\right)$

If we consider the function $L\left(t+\frac{x}{V}\right)$, the disturbance at position x_1 at time t_1 is the same as at position x_2 at time $t_2 > t_1$. For this, we assume that the boundary conditions are such that $L\left(t+\frac{x}{V}\right)$ is constantly zero, i.e., this problem is formulated by the following equality:

$$\psi(x_1, t_1) = \psi(x_2, t_2)$$

This translates to

$$L\left(t_1+\frac{x_1}{V}\right) = L\left(t_2+\frac{x_2}{V}\right)$$

This equation is satisfied if

$$t_1+\frac{x_1}{V} = t_2+\frac{x_2}{V} \implies x_2 - x_1 = -V(t_2 - t_1) < 0$$

Hence, since $t_2 + t_1 > 0 \implies x_2 < x_1$ (x_2 is less than x_1), these two points are distant by:

$$x_1 = x_2 + V(t_2 - t_1) > 0$$

This function $L\left(\mathrm{t}+\frac{x}{V}\right)$ is called a retrograde progressive wave, corresponding to a signal moving in the negative x direction.

This function $L\left(\mathrm{t}+\frac{x}{V}\right)$ corresponds to a progressive wave propagating in the decreasing xx direction (see Figure 10).

With $V = \frac{x_1 - x_2}{t_2 - t_1} < 0$

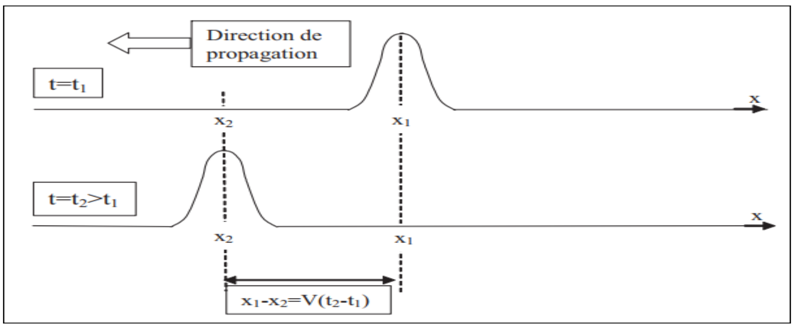


Figure 10: Progressive Wave in the Direction of Decreasing x (Negative).

In general, the displacement of the string is a superposition of progressive and reflected (retrograde) waves.

5.2.4 Notions of functions F and L

The function F is produced by the incident wave at a distance $x = 0$ (the source), so the function F represents a wave called the "incident wave" (Figure 11).

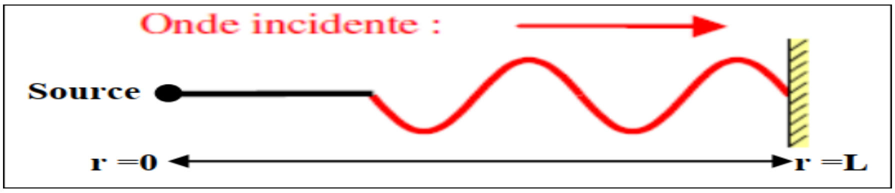


Figure 11: Incident Wave

The function L is produced by the reflection at a distance $x = L$ (the obstacle), so the function L represents a wave called the "reflected wave" (Figure 12).

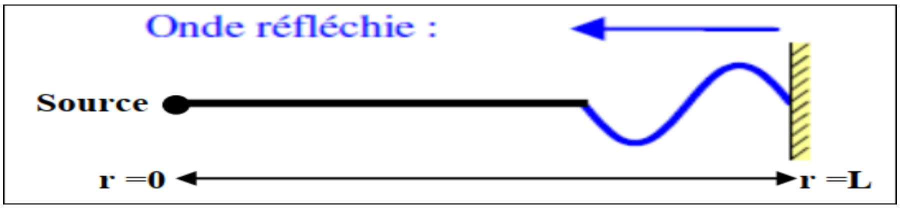


Figure 12: Reflected Wave

5.2.5 Sinusoidal progressive wave

A sinusoidal function is a special case of a periodic function. In physics, we speak of a sinusoidal wave if the disturbance of the medium evolves, in time and space, in a sinusoidal manner. The figure below then defines the period, wave length, and amplitude of a sinusoidal wave:

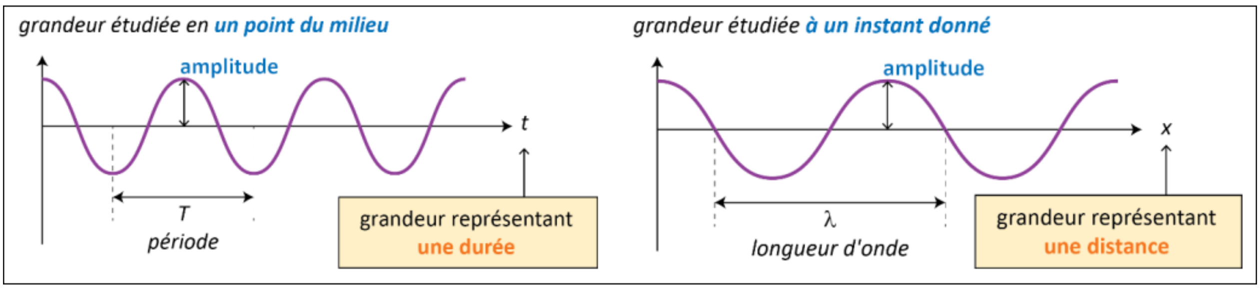


Figure 13: These two graphs are similar but do not show the same thing: the first shows a temporal evolution, while the other shows a spatial evolution.

Consider a progressive wave propagating in the direction of the $\boldsymbol{x}$ axis, when point A at position $x = 0$ is subjected to a sinusoidal vibration of the form (source movement):

$$y(x = 0, t) = A\cos(\omega t) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5.8)$$

The point (M) at position $x > 0$ will have the same vibration as that at point $x = 0$ but with a delay equal to $\frac{x}{V}$, (Figure 14):

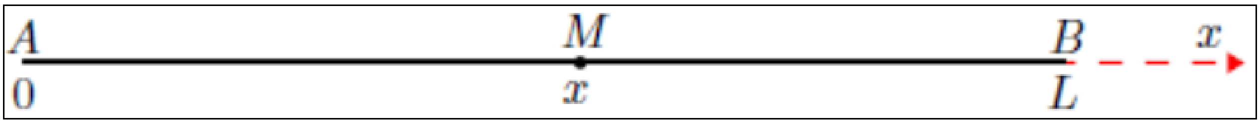


Figure 14: M material point of a homogeneous string (AB) of length L, located at a distance from point A (source).

$$y_M(x,t) = A\cos\left(\omega\left(t - \frac{x}{V}\right)\right) = A\cos\left(\omega t - \omega\frac{x}{V}\right)$$

$$y_M(x,t) = \left(\omega t - \emptyset(x)\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\; (5.9)$$

Where $\emptyset(x) = \omega\frac{x}{V}$: Represents the phase shift related to the propagation time. The sinusoidal progressive wave is written in a form that highlights the double periodicity (in time and space):

$$y_M(x,t) = A\cos\left(\omega t - \omega\frac{x}{V}\right)$$

We obtain

$$y_M(x,t) = A\cos\left(\frac{2\pi}{T}t - \frac{2\pi}{T}\frac{x}{V}\right)$$

5.2.5 .1 Wave length, wave number and wave vector

Given by

$$\begin{cases} \lambda = VT \\ k = \frac{2\pi}{\lambda} = \frac{\omega}{V} \\ \omega = \frac{2\pi}{T} = 2\pi f \end{cases} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5.10)$$

Where

T: The temporal period,

f: is the frequency,

λ: is the wave length,

k: is called the modulus of the wave vector $\vec{k}$ which is expressed in m^{-1}, $\left(\vec{k} = \frac{2\pi}{\lambda}\,\vec{n}\right)$

Hence, the **harmonic plane progressive wave** is written as

$$y_M(x,t) = A\cos(\omega t - kx) \quad \text{……………..} \ (5.11)$$

We use the complex representation (notation) of a sinusoidal progressive wave

$$S_M(x,t) = A\cos(\omega t - kx)$$

$$\begin{cases} S_M(x,t) = Ae^{j(\omega t - kx)} \\ S_M(x,t) = \underline{A}e^{j(\omega t)} \end{cases} \quad \text{……………………} \ (5.12)$$

Where

$$\underline{A} = Ae^{-jkx}$$

Remarks:

- ✓ The quantity $\frac{\omega}{V}$ is perpendicular to the wave plane and is designated by the vector $\vec{k}$ carried by the normal vector to the plane $\vec{n}$ (Figure 15)

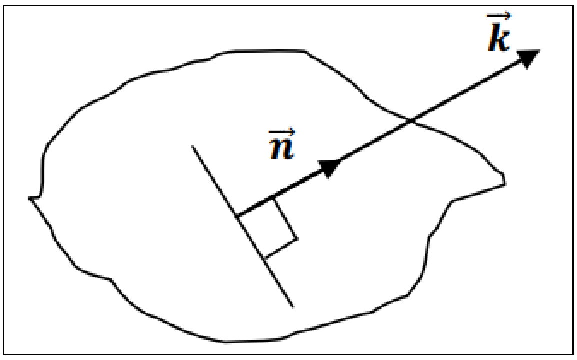


Figure 15: Vector $\vec{k}$ perpendicular to the wave plane.

- ✓ The harmonic plane progressive wave can be written in the form:

$$y_M(x,t) = A\cos(\omega t - kx + \varphi) \quad \text{…………….} \ (5.13)$$

- This expression characterizes a monochromatic (HPPW) or harmonic plane progressive wave,
- **Plane:** because the wave propagates in a single direction (ox).
- **Monochromatic:** the wave propagates with a single frequency.
- It is characterized by its angular frequency ω and its wave vector $\vec{k} = k\vec{e}_x$, it has two periods, a temporal period T and a spatial period (wavelength) λ.

☞ **There are two types of media**
Dispersive medium and Non-dispersive medium

We also define the wave number $\boldsymbol{\sigma}$, the inverse of the wave length, given by:

$$\sigma = 1/k \quad \text{................ (5.14)}$$

5.2.6 Superposition of two sinusoidal progressive waves

a) Case of Two Waves of the Same Frequency Propagating in the Same Direction

Consider two waves of the same frequency and the same direction of propagation, with respective amplitudes A_1 and A_2, and respective phases φ_1 and φ_2. The resulting wave will then be:

$$y(x,t) = A_1 e^{j(\omega t - kx + \varphi_1)} + A_2 e^{j(\omega t - kx + \varphi_2)} = A e^{j(\omega t - kx + \emptyset)} = A\cos(\omega t - kx + \emptyset)$$

Hence, we obtain

$$y(x,t) = A\cos(\omega t - kx + \emptyset) \quad \text{............. (5.15)}$$

With

$$\begin{cases} A = \sqrt{A_1^2 + A_2^2 + 2(A_1.A_2)\cos(\varphi_1 - \varphi_2)} \\ \emptyset = Arc\ \tan\left(\frac{A_1 \sin\varphi_1 + A_2 \sin\varphi_2}{A_1 \cos\varphi_1 + A_1 \cos\varphi_2}\right) \end{cases} \quad \text{............. (5.16)}$$

The superposition of two harmonic waves of the same frequency, propagating in the same direction, results in another harmonic progressive wave of the same frequency, with amplitude A and phase $\emptyset$.

b) Case of Two Waves of the Same Frequency Propagating in Opposite Directions

The superposition $y(x,\ t)$ of two harmonic progressive waves y_1 and y_2 of the same frequency, same amplitude $A_0 = A_1 = A_2$, the result is quite different

$$y(x,t) = A_1 e^{j(\omega t - kx + \varphi_1)} + A_2 e^{j(\omega t - kx - \varphi_2)}$$
$$= A_0 e^{j\omega t}.e^{-jkx}.e^{j\varphi_1} + A_0 e^{j\omega t}.e^{-jkx}.e^{-j\varphi_2}$$

$$y(x,t) = 2A_0 \cos\left(kx + \frac{\varphi_1 - \varphi_2}{2}\right) e^{j\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right)} \ \ldots\ldots\ldots\ldots\ (5.17)$$

And therefore, in real notation:

$$y(x,t) = 2A_0 \cos\left(\omega t + \frac{\varphi_1 - \varphi_2}{2}\right) \cos\left(kx + \frac{\varphi_1 - \varphi_2}{2}\right) \ \ldots\ldots\ldots\ldots\ (5.16)$$

If $\varphi_1 = \varphi_2 = 0$ equation (5.16) becomes

$$y(x,t) = 2A_0 \cos(\omega t)\cos(kx) \ \ldots\ldots\ldots\ldots\ (5.17)$$

This mode of vibration is very different from a progressive wave since all points xx of the string vibrate in phase with different amplitudes. There exists a series of points:

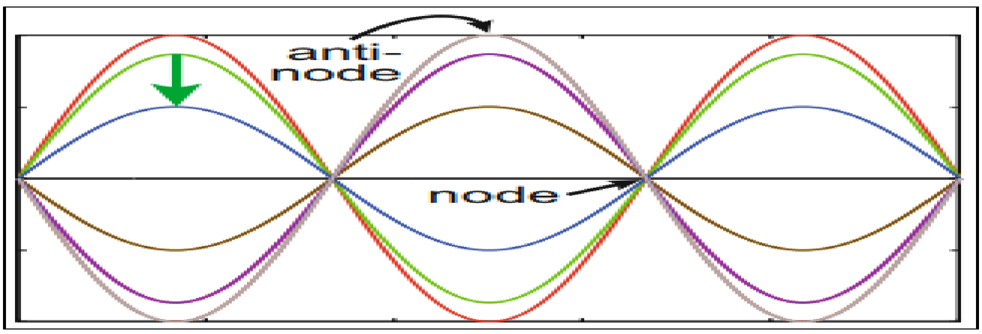
$$x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad \text{Of} \quad \varphi_1 = \varphi_2 = 0, \text{ with } \ n = 0,\ \pm 1,\ \pm 2,\ \ldots\ldots$$

- ❖ **Node:** A point where the **amplitude** of the medium's oscillation is **zero**. It is determined by:

$$\cos(kx) = 0 \ \Rightarrow \ x_n = \left(n + \frac{1}{2}\right)\frac{\lambda}{2} \quad or \quad x_n = \left(n + \frac{1}{2}\right)\frac{\pi}{k}$$

- ❖ **Antinode:** A point where the **amplitude** of the medium's oscillation is **maximum** and equal to $2A_0$. Between each pair of nodes exists an antinode, determined by:

$$\cos(kx) = \pm 1 \ \Rightarrow \ x_n = n\frac{\lambda}{2} = \quad ou \quad x_n = n\frac{\pi}{k}$$



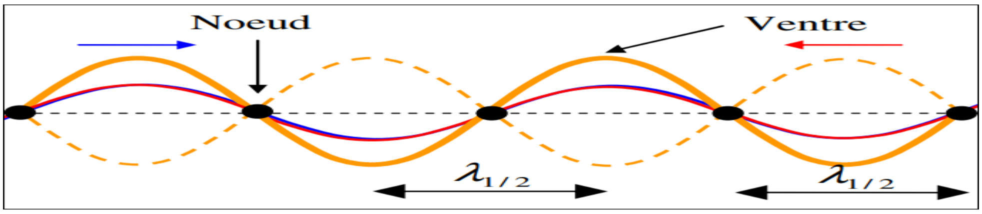


Figure 16: Anti-nod and node in the standing wave.

5.2.7 Phase velocity

We define the phase velocity $V_{\phi} = \frac{\Delta x}{\Delta t}$, which is expressed in terms of ω and k as:

$$V_{\phi} = \frac{\omega}{k} \ldots\ldots\ldots\ldots\ldots\ldots (5.18)$$

If the phase velocity does not depend on ω, the **medium** is said to be non-dispersive. Otherwise, it is said to be dispersive.

5.2.8 Group velocity

Definition: In signal propagation, the point of maximum amplitude, i.e., where the harmonic waves group to form the physical wave, moves at the group velocity V_G.

The group velocity is defined by:

$$V_G = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \ldots\ldots\ldots\ldots\ldots\ldots (5.19)$$

The amplitude of the beat propagates at a speed that is the group velocity, while the sinusoid contained within the beat propagates at the phase velocity (see Figure 17):

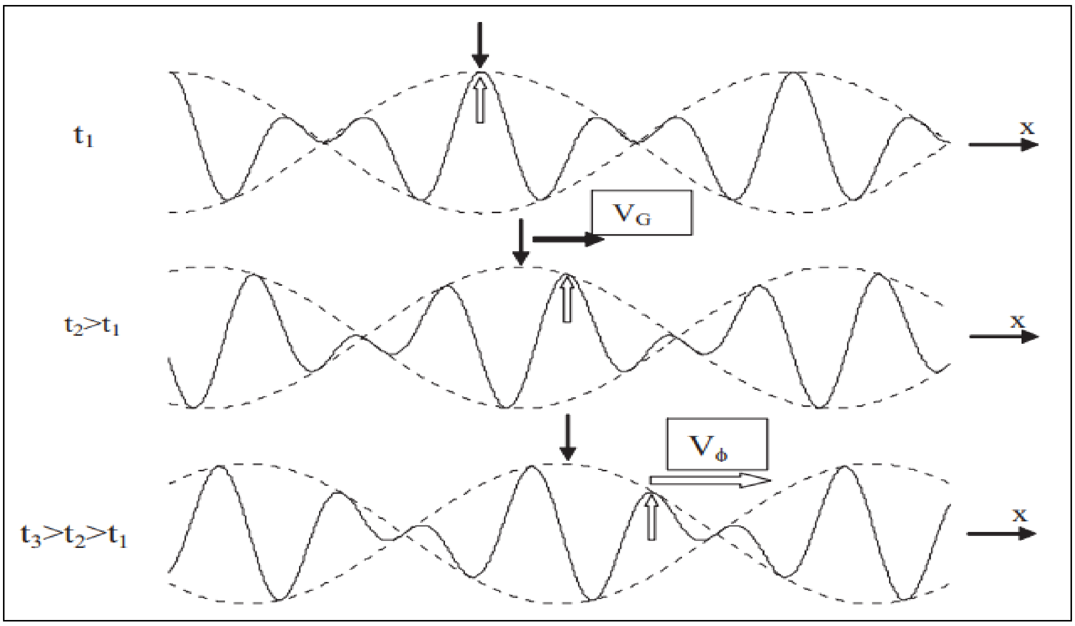


Figure 17: The black vertical arrows correspond to the maximum of the beats propagating at the group velocity. The white vertical arrows correspond to the maximum of the vibrations propagating at the phase velocity.

Remark:

When the phase velocity is independent of ω, then $V_G = V_\phi$.

5.3 Linear Chain Model

5.3.1 Wave equation

We consider a chain of atoms $\boldsymbol{A_n}$, each of mass $\boldsymbol{m}$, positioned at $\boldsymbol{X_n = na}$, where $\boldsymbol{a}$ is the distance between adjacent atoms. The total length of the chain is $\boldsymbol{L = (n+1)a}$. Each atom vibrates around its equilibrium position, (the interatomic forces are modeled as springs with a spring constant $\boldsymbol{k}$). Refer to Figure 18 for a visual representation.

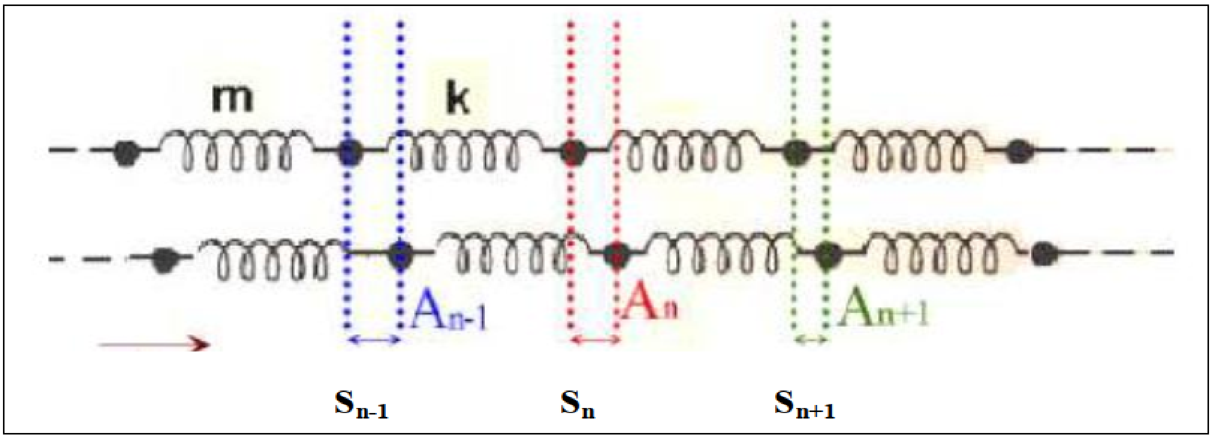


Figure 18: Chain of atoms $\boldsymbol{A_n}$.

We apply the Lagrangian formalism to the atom $\boldsymbol{A_n}$, assuming the displacement of the chain occurs horizontally.

Kinetic Energy: $T_n = \frac{1}{2} m \dot{S}_n^2$

Potential Energy: $U_n = T_0 \Delta l$

$$\Delta l_{n,n-1} = \frac{1}{2a}(S_n - S_{n-1})^2$$

$$\Delta l_{n,n+1} = \frac{1}{2a}(S_{n+1} - S_n)^2$$

The potential energy for the $\boldsymbol{n}$ mass is:

$$U_n = T_0 \frac{1}{2a}\left[\Delta l_{n,n-1} + \Delta l_{n,n+1}\right]$$

$$U_n = \frac{T_0}{2a}\ [(S_n - S_{n-1})^2 + (S_{n+1} - S_n)^2]$$

Lagrangian is written as:

$$L_n = T_n - U_n = \frac{1}{2} m\dot{S}_n^2 - \frac{1}{2}\frac{T_0}{a}[(S_n - S_{n-1})^2 + (S_{n+1} - S_n)^2]$$

The Lagrangian for the entire system is:

$$L = \sum_1^N L_n = \frac{1}{2}\left[\sum_1^N \left(m\dot{S}_n^2\right) - \frac{T_0}{a}\sum_1^N [(S_n - S_{n-1})^2 + (S_{n+1} - S_n)^2]\right]$$

Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{S_n}}\right) - \left(\frac{\partial L}{\partial S_n}\right) = 0$$

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{S_n}}\right) = m\ddot{S}_n \quad \dots\dots\dots (1) \\ \dfrac{\partial L}{\partial S_n} = -\dfrac{1}{2}\dfrac{T_0}{a}[2S_n - 2S_{n-1} + 2S_n - 2S_{n+1}] \ \dots\dots.(2) \end{cases}$$

$$(1) - (2) = 0$$

$m\ddot{S}_n + \frac{1}{2}\frac{T_0}{a}[2S_n - 2S_{n-1} + 2S_n - 2S_{n+1}]$ Dividing by m, gives:

$\ddot{S}_n + \frac{T_0}{ma}(2S_n - S_{n-1} - S_{n+1}) = 0 \ \Leftrightarrow\ \ddot{S}_n + \omega_0^2(2S_n - S_{n-1} - S_{n+1}) = 0$ … (5.20)

This is the equation of movement for the **n**-th mass,

Where

$$\omega_0^2 = \frac{k}{m} = \frac{T_0}{ma}$$

For $a = \delta x \Longrightarrow \ddot{S}_n = \frac{T_0}{m}\left(\frac{S_{n+1}-S_n}{\delta x} - \frac{S_n - S_{n-1}}{\delta x}\right) = \frac{T_0}{m}\left[\left(\frac{\partial S}{\partial x}\right)_{n+1} - \left(\frac{\partial S}{\partial x}\right)_n\right] = \frac{T_0}{m}\frac{d^2S}{dx^2}dx$

$$\ddot{S}_n = \frac{T_0}{m}\frac{d^2S}{dx^2}dx = \frac{T_0}{\mu}\frac{d^2S}{dx^2}$$

This gives us the wave propagation equation:

$$\frac{d^2S}{dt^2} = \frac{T_0}{\mu}\frac{d^2S}{dx^2} = V^2\frac{d^2S}{dx^2} \quad \ldots\ldots\ldots\ldots. (5.21)$$

Where

μ : is the linear mass density,

$V = \sqrt{\frac{T_0}{\mu}}$: is the wave propagation speed in the chain.

5.3.2 Relationship between amplitudes

To solve the wave propagation equation, we use the following complex representation:

$$\begin{cases} S_n = A_r e^{jwt} \\ S_{n-1} = A_{r-1} e^{jwt} \\ S_{n+1} = A_{r+1} e^{jwt} \end{cases}$$

Substituting into Equation (5.20), we obtain:

$$(2\omega_0^2 - \omega^2)A_r - \omega_0^2(A_{r+1} + A_{r-1}) = 0$$

Dividing by ω_0^2, we arrive at **the amplitude equation**:

$$\left(2 - \omega^2\frac{ma}{T_0}\right)A_r - A_{r-1} + A_{r+1} = 0 \quad \ldots\ldots\ldots\ldots\ldots (5.22)$$

$$\frac{A_{n+1}+A_{n-1}}{A_n} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2} \quad \ldots\ldots\ldots\ldots (5.23)$$

For $\boldsymbol{n}$ masses, we obtain $\boldsymbol{n}$ equations, leading to $\boldsymbol{n}$ values of ω, corresponding to $\boldsymbol{n}$ proper modes.

5.3.3 Derivation of the relationship between frequency and dispersion

We assume:

$$\begin{cases} A_n = A\,sin(nka) \\ A_{n-1} = A\,sin\big((n-1)ka\big) \\ A_{n+1} = A\,sin\big((n+1)ka\big) \end{cases}$$

$$\frac{2\omega_0^2-\omega^2}{\omega_0^2} = \frac{A_{r+1}+A_{r-1}}{A_r} = \frac{A\,sin\big((n+1)ka\big)+A\,sin\big((n-1)ka\big)}{A\,sin(nka)}$$

$$\frac{2\omega_0^2-\omega^2}{\omega_0^2} = 2\cos(ka) \quad \Longrightarrow \omega^2 = 2\omega_0^2(1-cos(ka)\,)$$

✍ Hence the proper pulsations of movement for $\boldsymbol{n}$ identical masses are given by:

$$\omega_k^2 = 4\omega_0^2\sin^2\left(\frac{ka}{2}\right) = 4\omega_0^2\,sin^2\left(\frac{k\pi}{2(n+1)}\right) \quad ,n = 1.2\,\ldots\ldots.n \;\ldots\ldots\ldots\ldots (5.24)$$

✍ The amplitude of the $\boldsymbol{n}$-th mass in the $\boldsymbol{k}$-th normal mode is:

$$A_{n,k} = C_n\sin\left(\frac{nk\pi}{n+1}\right) \quad , \; k = 1,2,\ldots\ldots n \;\ldots\ldots\ldots (5.25)$$

Examples:

- For one mass

$$n = 1 \Longrightarrow k = 1$$

$$\begin{cases} \omega^2 = 4\omega_0^2\,sin^2\left(\dfrac{\pi}{4}\right) = 2\omega_0^2 = 2\dfrac{T_0}{ma} \\ A = C_n\,sin\left(\dfrac{\pi}{n+1}\right) = C \end{cases}$$

- For two masses

$$n = 2 \Longrightarrow k = 1,2$$

$$\begin{cases} k = 1 \;\Longrightarrow\; \omega_1^2 = 4\omega_0^2\,sin^2\left(\dfrac{\pi}{6}\right) = \omega_0^2 = \dfrac{T_0}{ma} \\ k = 2 \;\Longrightarrow\; \omega_2^2 = 4\omega_0^2\,sin^2\left(\dfrac{2\pi}{6}\right) = 3\omega_0^2 = 3\dfrac{T_0}{ma} \end{cases}$$

Problem

Show that the function $y(x,t) = 12\,cos\,(10\pi(t+5x))$ is a solution to the wave equation: $\frac{\partial^2 y}{\partial t^2} = 0.04\frac{\partial^2 y}{\partial x^2}$

Solution

$y(x,t)$ Can be written as a complex function: $\mathrm{y(x,t)} = 12\mathrm{e}^{\mathrm{j}(10\pi t+50\pi x)}$ after replacement we find:

$$\begin{cases} \frac{\partial S}{\partial t} = 120\pi j e^{j(10\pi t+50\pi x)} \Longrightarrow \frac{\partial^2 S}{\partial t^2} = -1200\pi^2 e^{j(10\pi t+50\pi x)} \\ \frac{\partial S}{\partial x} = 600\pi j e^{j(10\pi t+50\pi x)} \Longrightarrow \frac{\partial^2 S}{\partial x^2} = -310^4\pi^2 e^{j(10\pi t+50\pi x)} \end{cases}$$

$$\Longrightarrow \left[-1200\pi^2 e^{j(10\pi t+50\pi x)}\right] = 0.04\left[-310^4\pi^2 e^{j(10\pi t+50\pi x)}\right]$$

$$\Longrightarrow 1 = 1$$

And that's it.

Chapter VI

6. Acoustic waves in fluids

Introduction

Acoustic waves are elastic waves, similar to mechanical waves, that propagate through a fluid (gas or liquid). In a perfect fluid (without viscosity, there is no absorption), the acoustic wave is longitudinal. Consider an acoustic wave of small amplitude propagating in the $\boldsymbol{x}$-direction. This wave is accompanied by infinitesimal variations in pressure $p(x,t)$ and density $\rho(x,t)$.

In the following, we will use the following symbols to study the acoustic wave propagating along the $\boldsymbol{x}$-axis:

x: Equilibrium coordinate of a particle in the medium.

u: Displacement of the particle relative to its equilibrium position.

ρ_0: Equilibrium density of the fluid.

P: Instantaneous pressure at any point.

P_0: Equilibrium pressure (at rest).

$p = P - P_0$: Acoustic pressure or overpressure.

C: Speed (velocity) of wave propagation

Examples:

In air at $20°c, C \approx 343\ m/s$. In water,$C \approx 1480\ m/s$.

To derive the wave propagation equation, we will neglect gravitational effects so that P_0 and ρ_0 are uniform throughout the medium. We also assume that the medium is homogeneous, isotropic, and perfectly elastic, meaning it is non-dissipative.

The fluid is characterized by three fields: Velocity V, pressure P, and density volimic μ.

6.1 Equation for acoustic wave propagation in fluids

- Consider a slice of fluid of thickness Δx located at positions $\boldsymbol{x}$ and $(x + dx)$. Let $P(x)$ and $P(x + dx)$ be the pressures acting on the planes at x and $(x + dx)$, respectively, which generate the movement of the slice, as shown in figure 1.
- Let $U(x, t)$ and $U(x + dx, t)$ be the displacements at time $\boldsymbol{t}$ of the planes at positions $\boldsymbol{x}$ and $(x + dx)$, respectively.

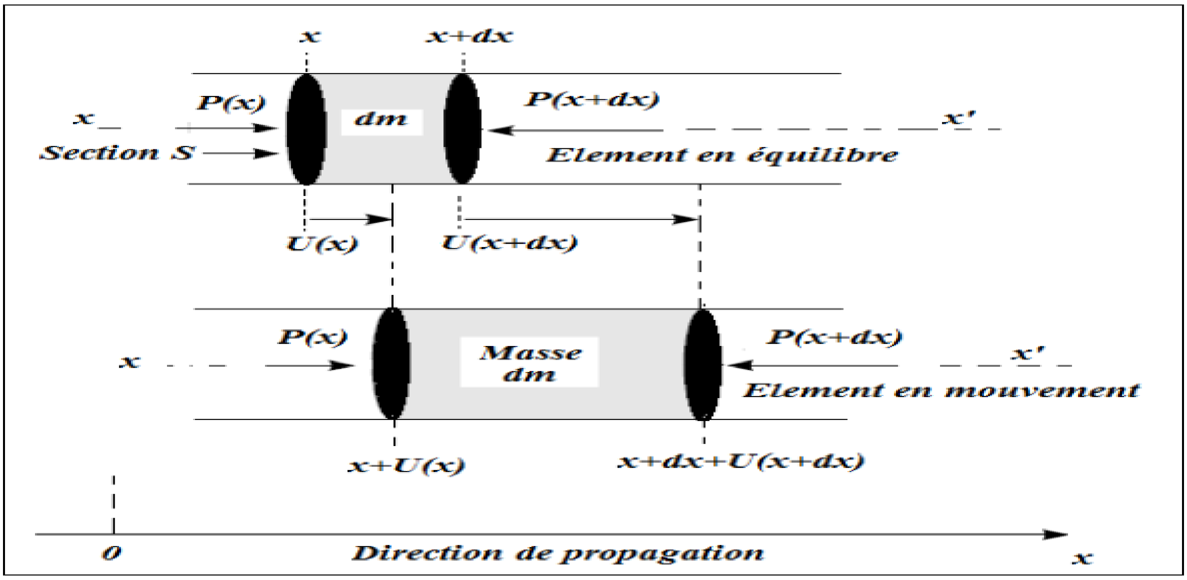


Figure 1: Acoustic wave propagation in a

- Applying Newton's second law of motion:

$$F(x + dx, t) - F(x, t) = dm \frac{\partial U^2}{\partial t^2} \ \ldots\ldots.\ (6.1)$$

Where $F(x, t)$ and $F(x + dx, t)$ are the forces applied to the planes at $\boldsymbol{x}$ and $(x + dx)$, respectively.

The resultant force is:

$$\big(F(x + dx, t) - F(x, t)\big) = -S\big(P(x + dx, t) - P(x, t)\big) \ \ldots\ldots\ldots\ (6.2)$$

Where S: pipe surface area

The equation of movement then written:

$$-S\big(P(x+dx,t)-P(x,t)\big)=dm\frac{\partial U^2}{\partial t^2} \text{ ....} (6.3)$$

By developing $P(x,t)$ in Taylor series to first order, we obtain:

$$P(x+dx,t)-P(x,t)=dP=\frac{\partial P}{\partial x}dx \text{ ....} (6.4)$$

Thus:

$$dm\frac{\partial U^2}{\partial t^2}=-S\frac{\partial P}{\partial x}dx \text{ ....} (6.5)$$

Since $P = P0 + p$, the resultant force is expressed as:

$$dm\frac{\partial U^2}{\partial t^2}=-S\frac{\partial P}{\partial x}dx=-S\frac{\partial p}{\partial x}dx \text{ ....} (6.6)$$

The overpressure p in a compressible fluid is defined as:

$$p=-\frac{1}{\chi}\frac{\partial U}{\partial x} \text{ ....} (6.7)$$

Where $\boldsymbol{\chi}$ is the compressibility coefficient.

Injecting the overpressure value into the equation of motion (6.5) we obtain:

$$\rho_0 S dx\frac{\partial U^2}{\partial t^2}=-S\frac{\partial}{\partial x}\left(-\frac{1}{\chi}\frac{\partial U}{\partial x}\right)dx$$

Hence the equation for the propagation of sound waves in the fluid is written:

$$\frac{\partial U^2}{\partial t^2}=\frac{1}{\chi\rho_0}\frac{\partial^2 U}{\partial x^2} \quad \Leftrightarrow \quad \frac{\partial U^2}{\partial t^2}=V^2\frac{\partial^2 U}{\partial x^2} \text{ ....} (6.8)$$

Where

$V=\sqrt{\frac{1}{\chi\rho_0}}$: This is called the celerity of the sound wave or the speed of propagation in the fluid.

6.1.1 Speed of sound

We found that the sound speed in air or water was given (using C instead of V) as:

$$C = \sqrt{\frac{K}{\rho_0}}$$

Where

C: Speed of sound.

$K = \frac{1}{\chi}$: is the modulus of compressibility and ρ_0 is the mass density of the fluid.

Changes in pressure are therefore adiabatic and the change in volume $\mathcal{V}$ is related to the change in pressure P as follows:

$$P\mathcal{V}^{\gamma} = constante \quad \text{................ (6.9)}$$

Where

$\gamma = \frac{C_p}{C_v}$: is the quotient (ratio) of specific heat capacity at constant pressure C_p, and $C_v$is the specific heat capacity at constant volume.

Calculating the differential of the gas law gives, we obtain:

$$\mathcal{V}^{\gamma} dP + \gamma P \mathcal{V}^{\gamma-1} d\mathcal{V} = 0 \quad \text{....................... (6.10)}$$

If we consider that dP represents the pressure variation in the vicinity of the equilibrium pressure P_0, we obtain

$$\mathcal{V}^{\gamma} p + P_0 \gamma \mathcal{V}^{\gamma-1} \Delta\mathcal{V} = 0 \implies p = -\gamma P_0 \frac{\Delta\mathcal{V}}{\mathcal{V}} \quad \text{....... (6.11)}$$

Taking into account the definition of the modulus of compressibility χ, we obtain :

$$p = -\frac{1}{\chi}\frac{\partial U}{\partial x} \quad and \quad p = -\gamma P_0 \frac{\Delta\mathcal{V}}{\mathcal{V}}$$

By conformity identification we obtain:

$$V = \sqrt{\frac{\gamma P_0}{\rho_0}} \quad \text{.................... (6.12)}$$

In the case of perfect fluids:

$$V = \sqrt{\frac{\gamma R T_0}{M}} \quad \text{.................... (6.13)}$$

Where:

γ Is the ratio of specific heats (massic), R is the molar constant of perfect (ideal) gases, $T_0$the temperature of the gas at rest and M the molar mass.

6.2 Sinusoidal progressive wave

- **Definition**: A sinusoidal acoustic wave is written as:

$$p(x,t) = p_0 \cos\left[\omega\left(t - \frac{x}{V}\right)\right]$$

The wave number k is defined as:

$$k = \frac{\omega}{V}$$

Where:

$$p(x,t) = p_0 \cos(\omega t - kx) \;\ldots\ldots\ldots\ldots (6.14)$$

Using complex representation, the sinusoidal progressive wave is written as:

$$p(x,t) = p_0 e^{j(\omega t - kx)} \ldots\ldots\ldots\ldots (6.15)$$

The relationship between acoustic pressure and compressibility χ is:

$$p = -\frac{1}{\chi}\frac{\partial U}{\partial x}$$

Allows you to write

$$\frac{\partial U}{\partial x} = -\chi p \quad \Longrightarrow \quad U(x,t) = -\chi \int p(x,t)dx$$

$$U(x,t) = -\chi \int p_0 e^{j(\omega t - kx)}\, dx = \frac{\chi p_0}{jk} \int e^{j(\omega t - kx)}\, dx$$

Where

$$U(x,t) = \frac{p_0}{j\omega\rho_0 V} \int e^{j(\omega t - kx)}\, dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.16)$$

The speed of particles:

$$\dot{U}(x,t) = \frac{\partial U(x,t)}{\partial t} = \frac{p_0}{\rho_0 V}\int e^{j(\omega t - kx)}\,dx \;\ldots\ldots\ldots\ldots\ldots\ldots\;(6.17)$$

It can be seen that for a progressive wave, the speed of particles is **in phase** with the acoustic pressure.

6.2.1 Acoustic Pressure (sound pressure)

Is defined by:

$$p(x,t) = p_0\cos(\omega t - kx) = p_0 e^{j(\omega t - kx)} \;\ldots\ldots\ldots\ldots\ldots\;(6.18)$$

6.2.2 Acoustic impedance

- The acoustic impedance at a point is defined as the quotient (ratio) of the complex amplitude of the pressure $p(x,t)$ to the complex amplitude of the particle speed $\dot{U}(x,t)$, is given as follows:

$$Z(x,t) = \frac{p(x,t)}{\dot{U}(x,t)} = \rho_0 V \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(6.19)$$

- The product $\rho_0 V$ is defined as the characteristic impedance of the fluid.

where ρ_0 is the mass density of the medium (kg/m3), and c is the speed (m/s) of sound in this medium. Z depends on the material and its units are (Ns/m^3) or $(Pa\ s/m)$

6.2.3 Acoustic energy

- **The kinetic energy :** Is defined as

$$E_c = \frac{1}{2}\rho_0 v_0 \dot{u}^2 \;\ldots\ldots\ldots\;(6.20)$$

- The kinetic **energy per unit volume**, or kinetic energy density, is:

$$\epsilon_c = \frac{E_c}{v_0} = \frac{1}{2}\rho_0 \dot{u}^2 \;\ldots\ldots\ldots\;(6.21)$$

Where v_0 is the volume element.

- **Potential energy density**

The potential energy stored is equal to the work done by the pressure to compress or expand the volume v_0:

$$E_p = \int -p dv$$

Given that:

$$p = -\frac{1}{\chi}\frac{v - v_0}{v_0} \quad \Longrightarrow \quad dv = -\chi v_0 dp$$

$$E_p = \frac{\chi v_0}{2} p^2 \text{ ………. (6.22)}$$

We deduce the potential energy density:

$$\epsilon_c = \frac{E_p}{v_0} = \frac{1}{2}\chi p^2$$

$$\epsilon = \epsilon_c + \epsilon_c$$

$$\epsilon = \frac{1}{2}\rho_0 \dot{u}^2 + \frac{1}{2}\chi p^2 \text{ ………. (6.23)}$$

In the particular case of a sinusoidal plane (travelling) wave:

$$\epsilon_c = \epsilon_p = \frac{1}{2\rho_0 V^2} p_0^2 \cos^2(\omega t - kx) \text{ ………. (6.24)}$$

And

$$\epsilon = \frac{1}{\rho_0 V^2} p_0^2 \cos^2(\omega t - kx) \text{ ………. (6.25)}$$

6.2.4 Acoustic intensity

We call the intensity of an acoustic wave the power that passes through, per unit time, a unit surface perpendicular to the direction of propagation.

We calculate the energy that passes through a surface $\boldsymbol{S}$ perpendicular to the direction of propagation $\boldsymbol{x}$ during a time interval $\boldsymbol{t}$.

This energy dE is equal to the energy contained in a volume $SVdt$.

$$dE = \epsilon\, S\, V dt$$

The power $\mathcal{P}$ passing through this surface is given by:

$$\mathcal{P} = \frac{dE}{dt} = \epsilon\, S\, V$$

Hence the expression for the intensity of the acoustic wave is:

$$I(t) = \frac{1}{S}\mathcal{P} = \frac{1}{S}\,\epsilon\, S\, V$$

$$I(t) = \epsilon\, V$$

$$I(t) = \frac{1}{\rho_0 V} p_0^2 \cos^2(\omega t - kx)$$

We call the intensity of the acoustic wave the mean value

$$I = \frac{1}{T}\int_t^{t+T} I(t) = \frac{1}{\rho_0 V} p_0^2$$

$$I = \frac{p_0^2}{2Z_c} \quad \text{.........} \ (6.26)$$

6.3 Reflection and transmission of acoustic waves at normal incidence

Consider two semi-infinite fluid media separated by a plane surface (separation interface). When an acoustic wave originating from $-\infty$, propagating in the former in the direction of the x axis, arrives at the separating surface, it gives rise to two waves :

- ✓ A reflected wave traveling in the direction $x < 0$ back into the first medium.
- ✓ A transmitted wave traveling in the direction $x > 0$ into the second medium.

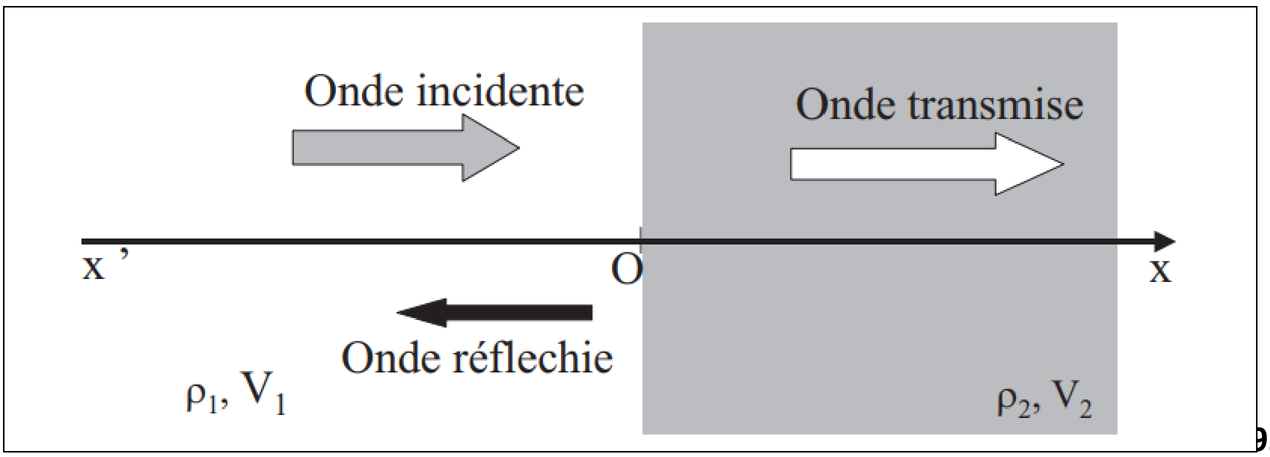


Figure 2: Reflection and transmission at a fluid-fluid interface.

☞ The types of acoustic waves are presented in the first medium and the second medium respectively is given as follows:

$$\begin{cases} P_1(x,t) = P_i(x,t) + P_r(x,t) \\ P_i(x,t) = p_{0i}e^{j(\omega t - k_1 x)} \quad Onde \ incidente \\ P_r(x,t) = p_{0r}e^{j(\omega t + k_1 x)} \quad Onde \ réfléchie \\ \dot{u}_1(x,t) = \frac{1}{Z_1}\left(P_i(x,t) - P_r(x,t)\right) \\ P_2(x,t) = P_t(x,t) \\ P_t(x,t) = p_{0t}e^{j(\omega t - k_2 x)} \quad Onde \ transmise \\ \dot{u}_2(x,t) = \frac{1}{Z_2}p_{0t}e^{j(\omega t - k_2 x)} \end{cases} \quad \dots\dots\dots\dots (6.27)$$

☞ The continuity relationships at the interface are:

$$\begin{cases} P_1(x=0,t) = P_2(x=0,t) \\ \dot{u}_1(x=0,t) = \dot{u}_2(x=0,t) \end{cases} \dots\dots\dots\dots (6.28)$$

We deduce the continuities equations at the interface as a function of the characteristics:

$$\begin{cases} P_i(x,t) - P_r(x,t) = P_t(x,t) \\ \frac{1}{Z_1}\left(P_i(x,t) - P_r(x,t)\right) = \frac{1}{Z_2}P_t(x,t) \end{cases} \dots\dots\dots\dots (6.29)$$

Or even

$$\begin{cases} 1 + \frac{p_r}{p_t} = \frac{p_t}{p_i} \\ 1 - \frac{p_r}{p_i} = \frac{Z_1}{Z_2}\frac{p_t}{p_i} \end{cases}$$

We define:

✍ The pressure reflection coefficient:

$$R_P = \frac{P_r}{P_i} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (6.30)$$

✍ The pressure transmission coefficient:

$$T_P = \frac{P_t}{P_i} \qquad (6.31)$$

The two continuity equations can then be written as:

$$\begin{cases} 1 + R_P = T_P \\ 1 - R_P = \frac{Z_1}{Z_2} T_P \end{cases} \qquad (6.32)$$

- We deduce the reflection and transmission:

$$\begin{cases} R_P = \frac{Z_2 - Z_1}{Z_2 + Z_1} \\ T_P = \frac{2Z_2}{Z_2 + Z_1} \end{cases} \qquad (6.33)$$

- Taking into account the relations $p_i = Z_1 \dot{u}_i,\ p_r = -Z_1 \dot{u}_r\ and\ p_t = Z_2 \dot{u}_t$, we can calculate the reflection and transmission coefficients for the particle speed and for the particle displacement:

$$\begin{cases} R_{\dot{u}} = R_u = \frac{Z_1 - Z_2}{Z_1 + Z_2} \\ T_{\dot{u}} = T_u = \frac{2Z_1}{Z_1 + Z_2} \end{cases} \qquad (6.34)$$

- Taking into account the relations $I_i = P_i^2/2Z_1, I_r = P_r^2/2Z_1\ \ and\ \ I_t = P_t^2/2Z_2$, we can calculate the reflection and transmission coefficients for the acoustic intensity:

$$\begin{cases} \beta_R = \frac{I_r}{I_i} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \\ \beta_T = \frac{I_t}{I_i} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} \end{cases} \qquad (6.35)$$

Chapter VII

Principles and laws of geometrical optics

7. Introduction to the fundamentals concepts of light

In this chapter, light is considered as rectilinear rays radiated in all directions from a point at an object. The rays are refracted at air/glass interfaces according to the laws of refraction. We show that all rays emerging from a point object entering a spherical interface will collect (and use equal time) to a focal point to a good approximation for common conditions.

Optics studies luminous phenomena, i.e. mainly phenomena perceived by the eye. The cause of these phenomena is light, because in order to be visible, an object must send light to the eye.

Geometrical optics is a branch of optics based on the notion of light rays. This simple approach makes it possible to construct geometric images, and to explain the formation of the images produced.

7.1 The Nature of Light

Until the sixteenth century, most physicists considered light as a continuous flow of matter particles emitted by luminous sources. Many, including Galileo, attempted to measure the speed of light C, but all their efforts were unsuccessful. The theory of light evolved, especially in the 19th century, leading to the wave theory of light, represented by the works of Young, Maxwell, Huygens, and Hertz. Light can be described as an electromagnetic wave consisting of an electric field and a magnetic field oscillating in phase, perpendicular to each other and perpendicular to the direction of propagation (Figure 1). Light waves propagate in a vacuum.

7.2 The double property of light

There are different models for define the behavior of light:

- **The corpuscular model**: This model considers light as material particles subject to universal gravitation.
- **The wave model**: This model states that light propagates like a mechanical wave (similar to how sound propagates in air).
- A photon has an energy given by:

$$E = h\nu \text{ .... } (7.1)$$

Where $h = 6..62 \, . \, 10^{-34} j.s$ (Planck's constant), $\nu = \frac{1}{T}$ is the wave frequency, and T is the period.

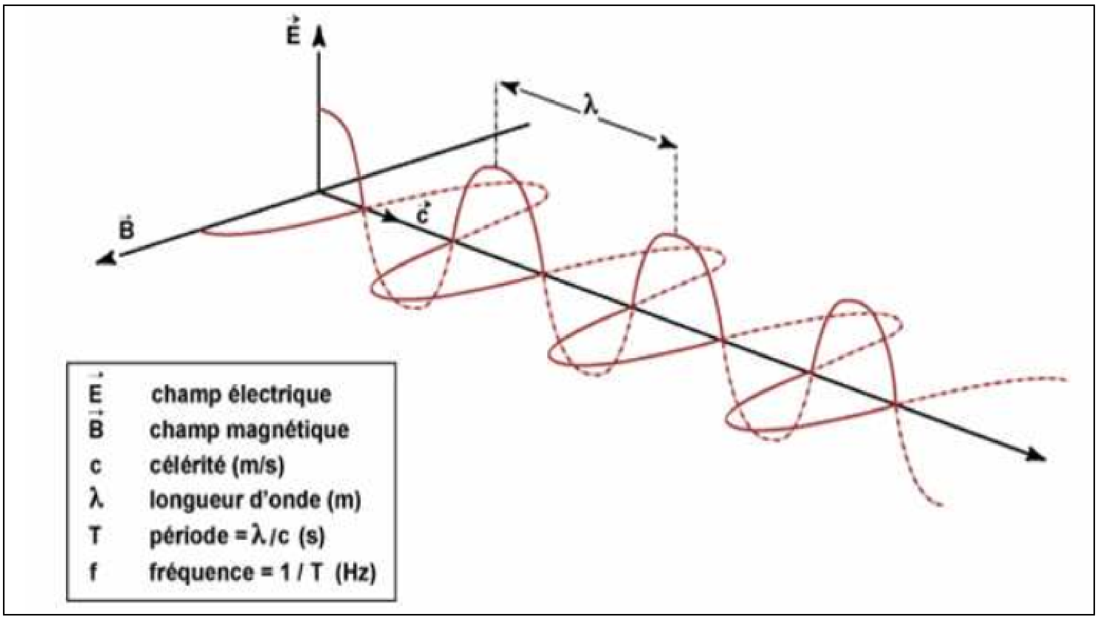


Figure 1: Nature and Propagation of an Electromagnetic Wave.

- The ensemble of wavelengths visible to the human eye is called the visible spectrum (from 400 to 780 nm), as shown in Figure 2.

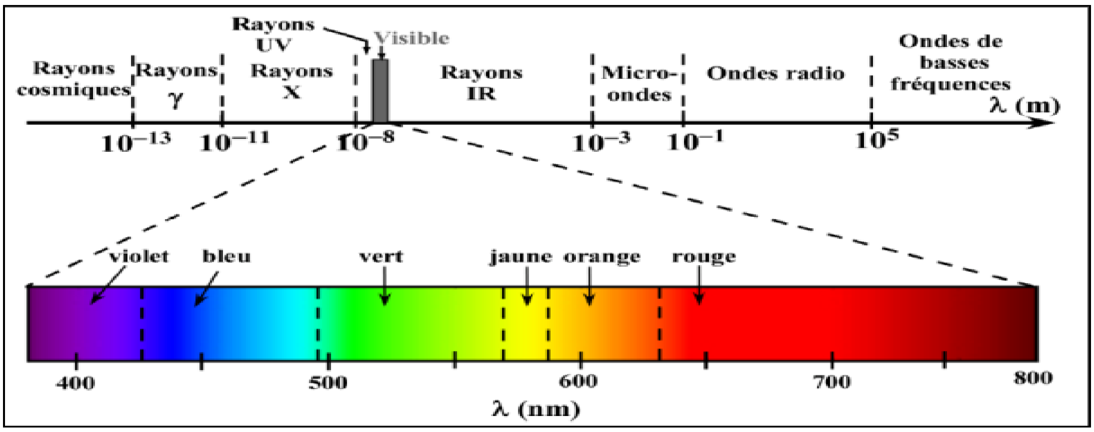


Figure 2: Visible spectrum

7.3 Light Beams

A **light beam** consists of a collection of rays. It can be:

- **Parallel** if the rays remain parallel.
- **Convergent** if the rays converge toward a single point.
- **Divergent** if the rays appear to originate from a single point.

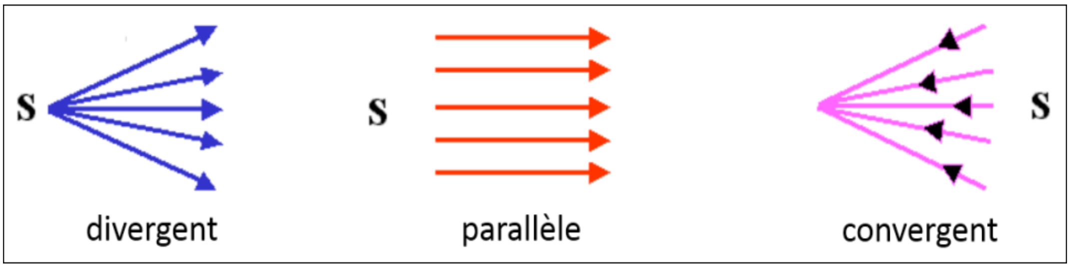


Figure 3: Ensemble of Light Rays.

- In a vacuum, light propagates in a straight line with a speed $\boldsymbol{C}$ that is independent of direction:

$$C = \lambda_0 \nu \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(7.2)$$

ν: Is the frequency,

λ_0 : Is the wavelength.

7.4 Characteristics of an optical medium

7.4.1 Transparent, homogeneous, isotropic media

- ✓ **Transparent**: Allows light to pass through (opposite of an opaque medium).
- ✓ **Homogeneous**: Optical properties do not vary with position.
- ✓ **Isotropic**: Optical properties are independent of the direction in which the light ray propagates.

7.4.2 Refractive Index

The refractive index $\boldsymbol{n}$ is an intrinsic property of a medium, defined as:

$$n = \frac{C}{V} \qquad n \geq 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (7.3)$$

Where:

$C = 3.10^8\ m.s^{-1}$ Is the speed of light in a vacuum.

V : is the speed of propagation in the medium considered.

Table 1 gives the values of the indices n for some material media

Matériels	Air	Eau	Verre
n	1	1.33	1.51

Tableau 1 : Some refractive index

7.5 Light Sources

7.5.1 Primary Sources

A primary light source is a body that creates and emits light in all directions (e.g., the Sun, stars, and candles).

7.5.2 Secondary Sources

A secondary light source is a body that reflects received light in all directions (e.g., the Moon, planets, a cinema screen).

7.5.2 Sources secondaires

Une source de lumière secondaire est un corps qui renvoie la lumière reçue dans toutes les directions. (La Lune, les planètes, un écran de cinéma,...).

7.6 Snell-Descartes Laws

a. Reflection Law

Consider a light ray SI incident on the interface of two distinct media. The light reflects in a single direction, forming the reflected ray IR. The angle of incidence i_1 and the angle of reflection i'_1 are shown in Figure 4.

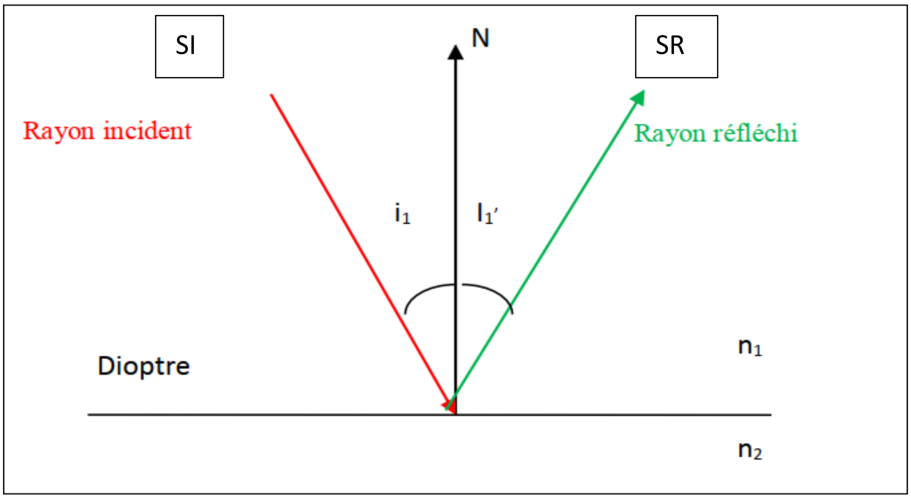


Figure 4 : Reflection.

- The **first law of Descartes** states that the angle of incidence $\mathbf{i_1}$ is equal to the angle of reflection $\mathbf{i'_1}$:

$$i_1 = i'_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7.4)$$

b. Refraction Law

When a ray of light falls on the separating surface between two media, part of the light is reflected according to the **law of reflection** in medium 1 of index n_1, and part penetrates medium 2 of index n_2. The light ray changes its direction of propagation when passing from medium 1 to medium 2, it is refracted. This process is called "refraction of light".

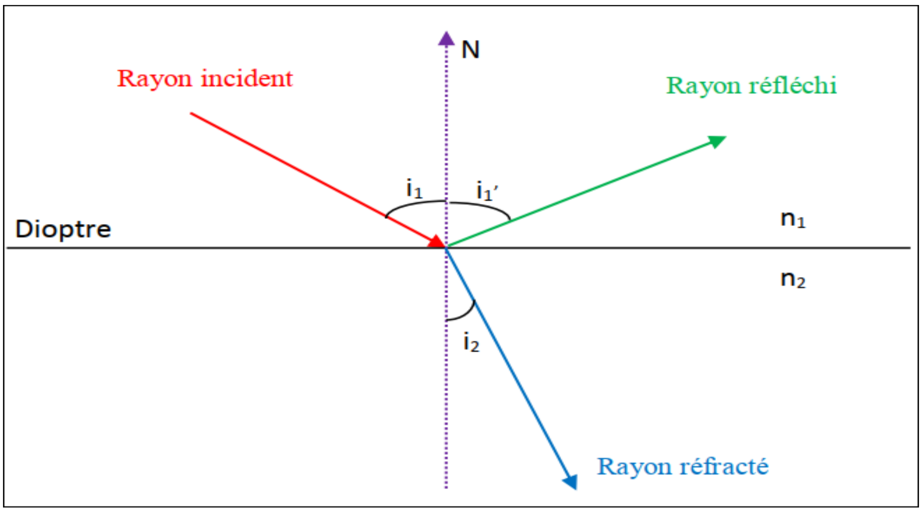


Figure 5: Refraction.

- **First Law**: The incident ray, the refracted ray and the normal to the surface are in the same plane (plane of incidence).
- **Second Law of Snell-Descartes**: the angle of incidence $\mathbf{i_1}$ and the angle of refraction $\mathbf{i_2}$ are related by:

$$n_1 \sin \boldsymbol{i_1} = n_2 \sin \boldsymbol{i_2} \text{ .... (7.5)}$$

Example: On the effect of refraction when light passes between water and air as shown in the figure below.

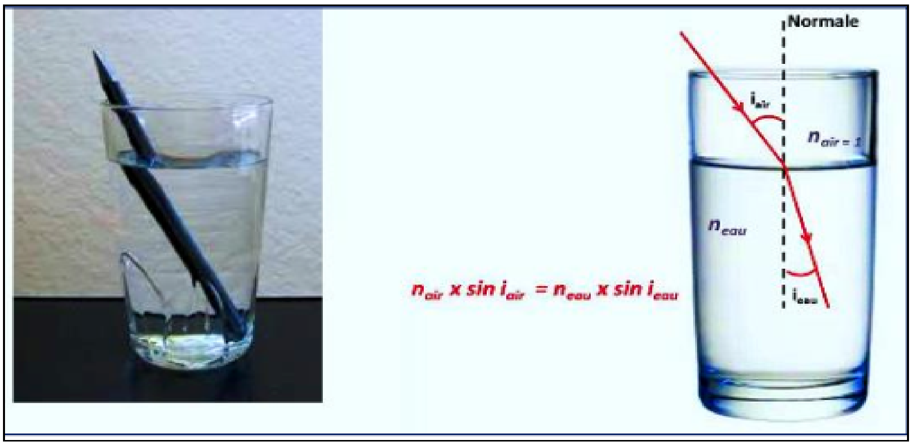


Figure 6: Refraction effects between air and water.

Exercise 1:

A light ray in air falls on the surface of a liquid; it makes an angle α = 56° with the horizontal plane. The deviation between the incident ray and the refracted ray is θ = 13.5°. What is the index n of liquid n_2?

Solution: we have

$$n_1 \sin \boldsymbol{i_1} = n_2 \sin \boldsymbol{i_2} \quad Avec \;\; n_1 = 1$$

Hence

$$n_2 = \frac{\sin i_1}{sin\, i_2} = \frac{sin\, \alpha}{sin\, \theta} = \frac{sin\, 56}{sin\, 13.5} = 1.6$$

Exercise 2 :

A plane diopter separates air of index $n_0 = 1$ from a medium of index n. For what value of the angle of incidence $\boldsymbol{i_1}$ is the reflected ray perpendicular to the refracted ray.

Solution

The IT ray must be perpendicular to the IR ray: We have:

$$i2 + \pi\, 2 + r \;=\; \pi \qquad with \qquad r \;=\; -i1$$

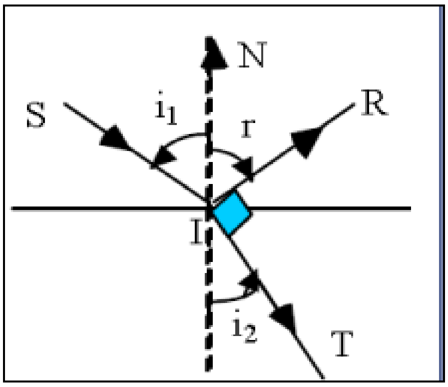
Either:

$$i1 + i2 = \pi\, 2 \qquad et \qquad i2 \;=\; \pi\, 2 \;-\; i1$$

Applying Descartes' 2nd law we find:

$$sin\; i1 = n\; sin\; i2 = n\; sin\; (\pi 2 \;-\; i1) = n\; cos\; i1 \Longrightarrow\; tgi1 = n$$

The angle of incidence must have a value such that its tangent is equal to n.



c. Refraction Limit-Total Reflection

❖ Refraction Limit in a More Refractive Medium $\boldsymbol{n_1 < n_2}$

If a ray propagates towards a more refractive medium after crossing a separation surface (Figure 7), the angles of incidence $\boldsymbol{i_1}$ and refraction $\boldsymbol{i_2}$ are linked by the relation:

$$n_1 \sin \boldsymbol{i_1} = n_2 \sin \boldsymbol{i_2} \;\; \text{With} \;\; n = \frac{n_2}{n_1} > 1 \;\ldots\ldots\ldots\ldots\ldots\ldots (7.6)$$

The result is that $\sin i_2 < \sin i_1$, the angles $\boldsymbol{i_1}$and $\boldsymbol{i_2}$being between 0^0 and $\pi/2$, i.e. $\boldsymbol{i_2} < \boldsymbol{i_1}$. The refracted ray therefore approaches the normal.

A normal incident ray $(S_1\mathrm{I})$, for which $\boldsymbol{i_1} = 0$, enters without deflection $\boldsymbol{i_2} = 0$. As $\boldsymbol{i_1}$ increases, $\boldsymbol{i_2}$ also increases but remains less than $\boldsymbol{i_1}$, ($\boldsymbol{i_2} < \boldsymbol{i_1}$).

At grazing incidence $\boldsymbol{i_1} = \pi/2$, the refraction angle is maximum (limit refraction angle noted $\boldsymbol{i}_{lim}$) and is worth:

$$\sin \boldsymbol{i}_{lim} = \frac{n_1}{n_2} \text{ ………………… (7.7)}$$

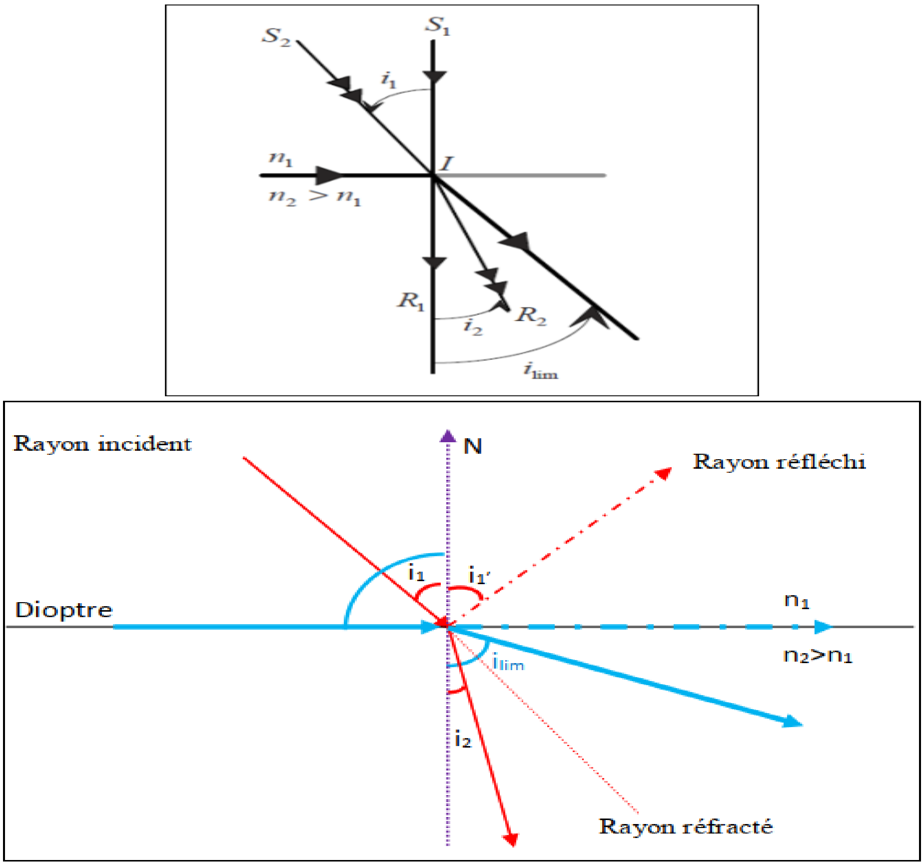


Figure 7: Propagation towards a more refractive medium.

❖ **Refraction Limit in a Less Refractive Medium $\boldsymbol{n_1} > \boldsymbol{n_2}$**

If a ray crosses a separating surface and propagates towards a less refractive medium (Figure 8), the angles of incidence i1 and refraction i2 are always linked by the following relation:

$$n_1 \sin \boldsymbol{i_1} = n_2 \sin \boldsymbol{i_2} \quad \text{Avec} \quad n = \frac{n_2}{n_1} < 1 \ \ldots\ldots\ldots\ldots\ldots\ (7.8)$$

✍ **The third Snell-Descartes** law then implies that:

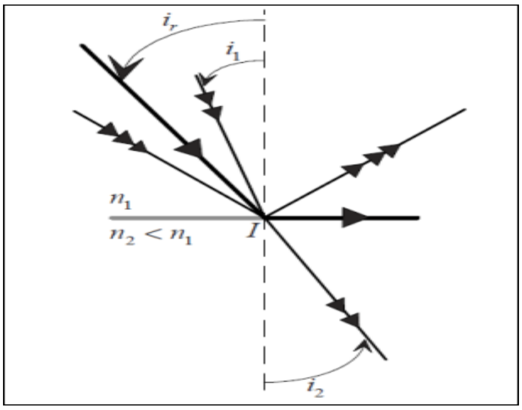
$$\boldsymbol{i_1} < \boldsymbol{i_2} \ \ldots\ldots\ldots\ldots\ldots\ (7.9)$$

The refracted ray deviates from the normal and the angle of refraction is maximal ($i_2 = \pi/2$) for a limiting angle of incidence $\boldsymbol{i_r}$ such that:

$$\sin \boldsymbol{i_r} = sin\, \boldsymbol{i_{lim}} = \frac{n_2}{n_1} \ \ldots\ldots\ldots\ldots\ldots\ldots\ (7.10)$$

We are then in a situation of total reflection, there is no longer any refracted ray.

So if the angle of incidence is greater than $\boldsymbol{i_r}$, there is no longer any refracted ray (in fact, we then have $sin\, \boldsymbol{i_2} > \boldsymbol{i_1}$ is therefore no longer defined), the incident ray is totally reflected: we speak of total reflection. The diopter behaves like a mirror.



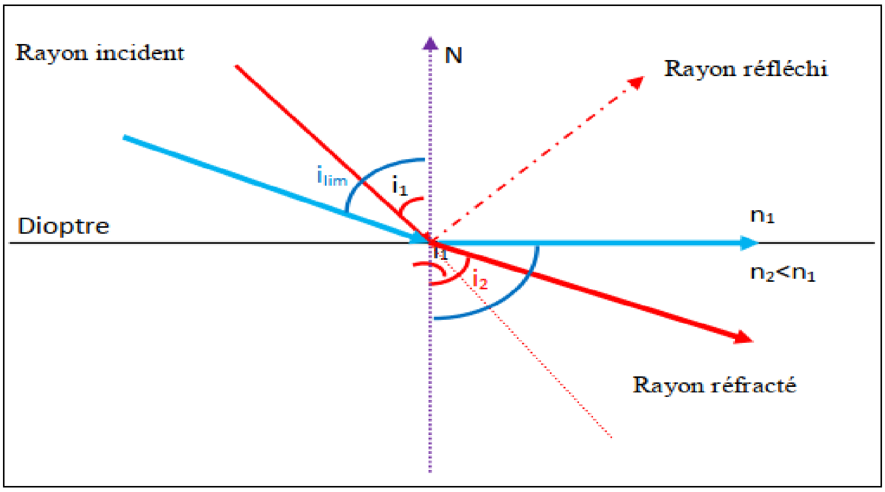


Figure 8: Propagation towards a less refractive medium.

7.7 Prism

Definition

A prism is a transparent, homogeneous, isotropic and refractive medium containing two non-parallel flat faces (diopters planes) which form the edge of the prism. It is made of glass.

The intersection of the two faces of the prism forms the edge of the prism, characterized by an angle $\boldsymbol{A}$. The base of the prism is the third face, whose edges are generally parallel to the edge and by its refractive index $\boldsymbol{n}$.

The prism is used either to analyze polychromatic light grace (due) to its dispersive properties, or to change the direction of propagation of a light ray following refractions or reflections, (Figure 9).

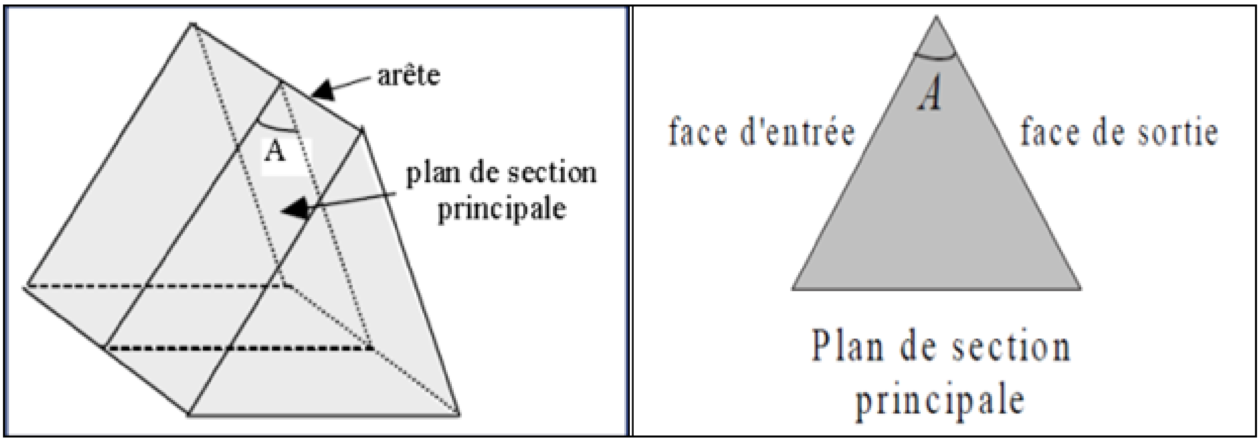


Figure 9: The prism.

7.7.1 Path of a light ray passing through a prism

Let **SI** be any incident ray which strikes at **I** the entry face AB of the prism; coming from a medium less refracting than that of the prism, this ray undergoes in **I** the phenomenon of refraction while respecting the two laws of Descartes.

If $\boldsymbol{n}$ is the index of the prism immersed in the index area taken to be 1. Applying the Snell-Descartes relationship to the input and output of the prism at **I** and **I'**, we obtain the following relationships see (Figure 9):

$$\begin{cases} \sin \boldsymbol{i} = n \sin \boldsymbol{r} \\ \sin \boldsymbol{i'} = n \sin \boldsymbol{r'} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.11)$$

The deflection angle $\boldsymbol{D}$ is by definition the angle by which the incident ray $\boldsymbol{SI}$ must be rotated to bring it in the direction of the emerging ray $\boldsymbol{I'R}$. This deflection is therefore the sum of two successive deflections which take place in the same direction, one at the entrance, and the other at the exit of the prism, either:

$$\boldsymbol{D} = (\boldsymbol{i} - \boldsymbol{r}) + (\boldsymbol{i}' - \boldsymbol{r}') \quad \ldots\ldots\ldots\ldots\ldots\ldots (7.12)$$

On the other hand, in the quadrilateral $AIHI'$, we have:

$$A + \frac{\pi}{2} + H + \frac{\pi}{2} \gg H = \pi - A$$

In the triangle IHI', we have:

$$r + H + r^{'} = \pi \gg H = \pi - (r + r')$$

Either

$$r + r' = A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (7.13)$$

In the triangle IKI', we have:

$$(i - r) + k + (i^{'} - r^{'}) = \pi$$

As, $k = \pi - D$ then

$$(i + i') - (r + r') + \pi - D = \pi$$

Either

$$D = (i + i') - A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.14)$$

The prism formulas are therefore:

$$\begin{cases} \sin \boldsymbol{i} = n \sin \boldsymbol{r} \\ \sin \boldsymbol{i'} = n \sin \boldsymbol{r'} \\ \boldsymbol{r} + \boldsymbol{r'} = \boldsymbol{A} \\ \boldsymbol{D} = (\boldsymbol{i} + \boldsymbol{i'}) - \boldsymbol{A} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots (7.15)$$

The refractive index $\boldsymbol{n}$ of the prism of angle $\boldsymbol{A}$ can be expressed as a function of $\boldsymbol{D}$ according to the following formula:

$$n = \frac{\sin\left(\frac{A+D}{2}\right)}{\sin\left(\frac{A}{2}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.16)$$

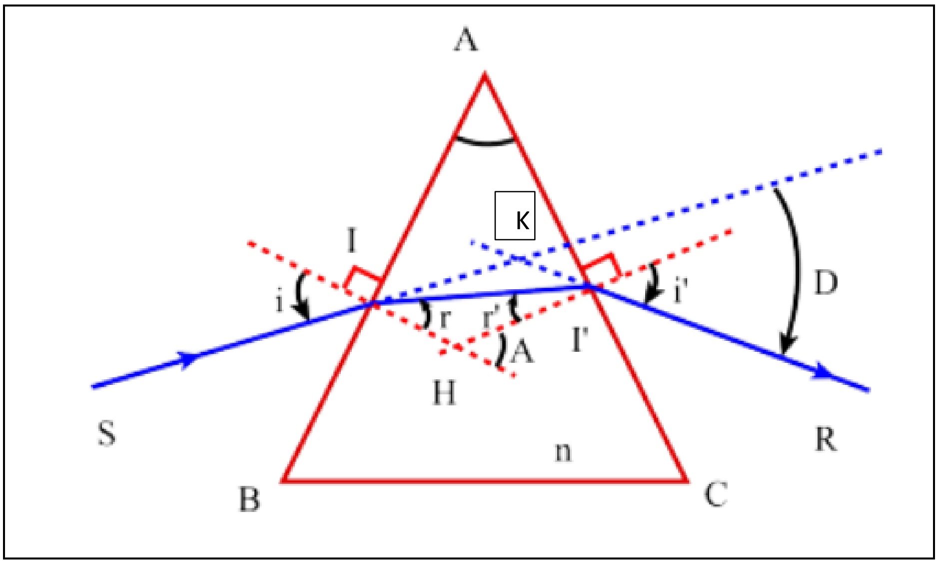


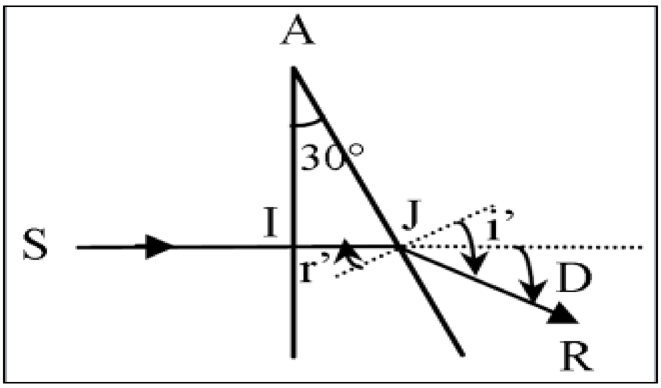
Figure 10: The prism.

Exercise 3

A prism of angle $\boldsymbol{A}$ and index $n = 1.5$ is illuminated by an incident ray perpendicular to the entrance face of the prism. Trace the path of the light ray and calculate the deviation $\boldsymbol{D}$ in the case $A = 30°$.

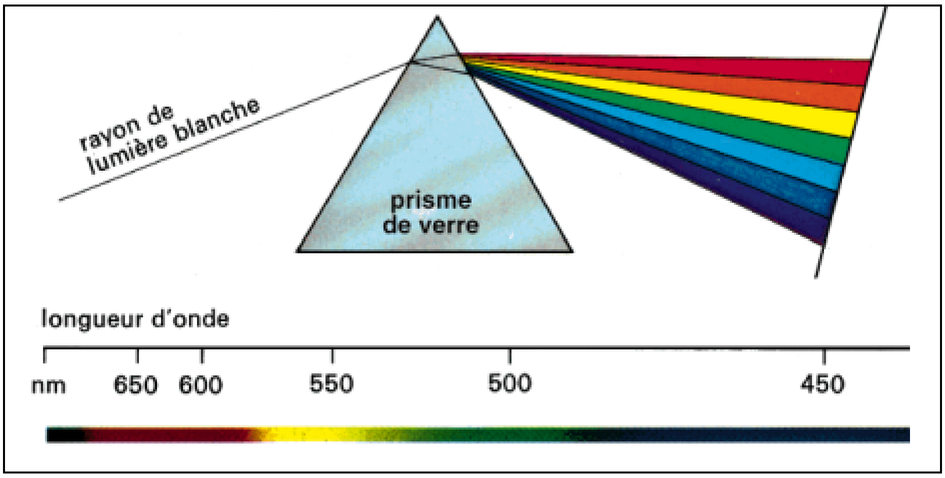
Solution

- ✓ Since the angle of incidence i is zero, the angle $\boldsymbol{r}$ is also zero. We therefore have: $\boldsymbol{r' = A \rightarrow A = r' = 30°}$
- ✓ The angle $\boldsymbol{i'}$ is then given by: $sin\, \boldsymbol{i'} = n\, sin\, \boldsymbol{r'} = n\, sin\, \boldsymbol{A} = 0{,}75$.
- ✓ Either $\boldsymbol{i' = 48°.6}$.
- ✓ The deflection is : $D = i' - r' = \boldsymbol{18°.6}$



7.7.2 Dispersion of light in a prism

We saw in this chapter that the refractive index depends on the wavelength (color) of visible light. This is called dispersion. Because of this phenomenon, a prism disperses (breaks down) white light into its different components. All of these components together make up the spectrum of white light (seven dominant colors are generally listed: red, orange, yellow, green, blue, indigo, and violet).



Chapter VIII

8. Construction of images

Introduction

Optics studies light phenomena, primarily those perceived by the eye. The cause of these phenomena is light, because to be visible, an object must transmit light to the eye. Geometric optics is a branch of science based on the concept of the light ray. This simple approach allows for geometric image constructions and explains the formation of the images produced by them.

8.1 Stigmatism

An optical system is of good quality if it gives a point image of a point source: this is the stigmatism condition.

a. **Rigorous stigmatism**
   An optical system is called rigorously stigmatic for a pair of points A and A', if any light ray passing through the object point A emerges from the optical system by passing through point A'. A' is then the image of A by the optical system; A and A' are also called conjugate with respect to the optical system. See Figure 1.

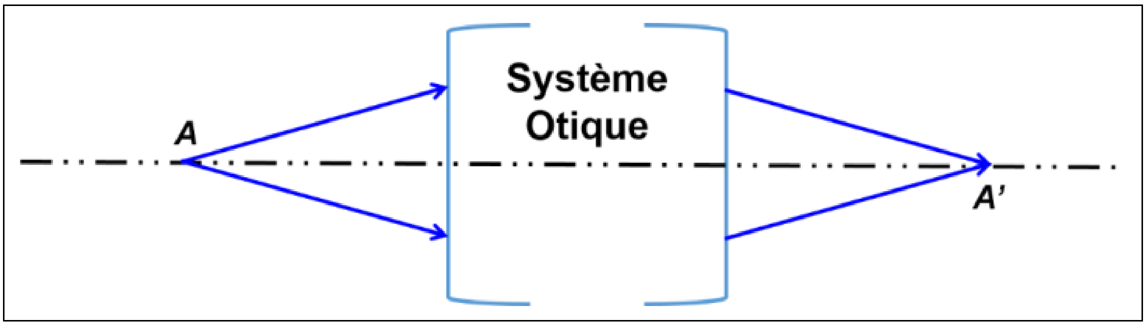


Figure 1: The conditions for rigorous stigmatism.

b. **Approximate stigmatism - Gauss approximation**
   We will only consider centred optical systems, i.e. systems for which there is an axis of symmetry of revolution called the optical axis. We then show that

such an optical instrument will give a good quality image of an object if the following two conditions, known as the Gaussian conditions, are satisfied:

- ✓ The objects are of small extent, located near the optical axis.
- ✓ The incident light rays form a small angle with the optical axis. This is called approximate stigmatism. Under these conditions, the image of a flat object perpendicular to the optical axis is flat and perpendicular to the optical axis (aplanatism).

8.2 Simple optical systems with flat faces

8.2.1 Mirror plane

A mirror is an object with a reflective surface (a metallized surface) that reflects light rays incident on it. The plane mirror is the only system that achieves rigorous stigmatism for any point in space. See Figure 2 below.

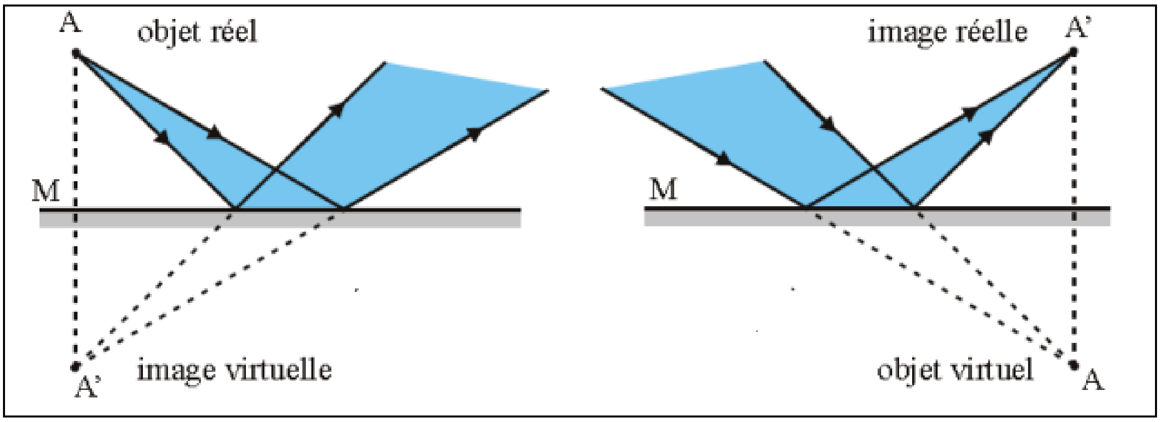


Figure 2: Mirror plane

a. Conjugation relation (Law of conjugate points)

Image A' is symmetrical to object A with respect to the mirror (M). The object and the image are of different natures:

- Real Object-Virtual Image.
- Virtual Object-Real Image.

The size of the image is equal to the size of the object. The angle of incidence is equal to the angle of reflection $(i = r)$.

b. Parallel mirrors

M1 and M2 are two plane mirrors with parallel reflecting surfaces facing each other as shown in Figure 3 below. We start with mirror M1, which gives the image of an object A, that is A_1 symmetrical to A with respect to M1, and so on, mirror M2 gives of A_1 an image $A_{1/2}$ whose image through M1 is $A_{12/1}$,.... and so on

And on the other hand, the mirror M2 gives the image of A is A_2 symmetrical to A with respect to M2; the images of A_2 given successively by M1 , M2 are $A_{2/1}$, $A_{21/2}$,… and so on.

From this we can conclude that the two mirrors give a double infinity of images aligned on x'Ax.

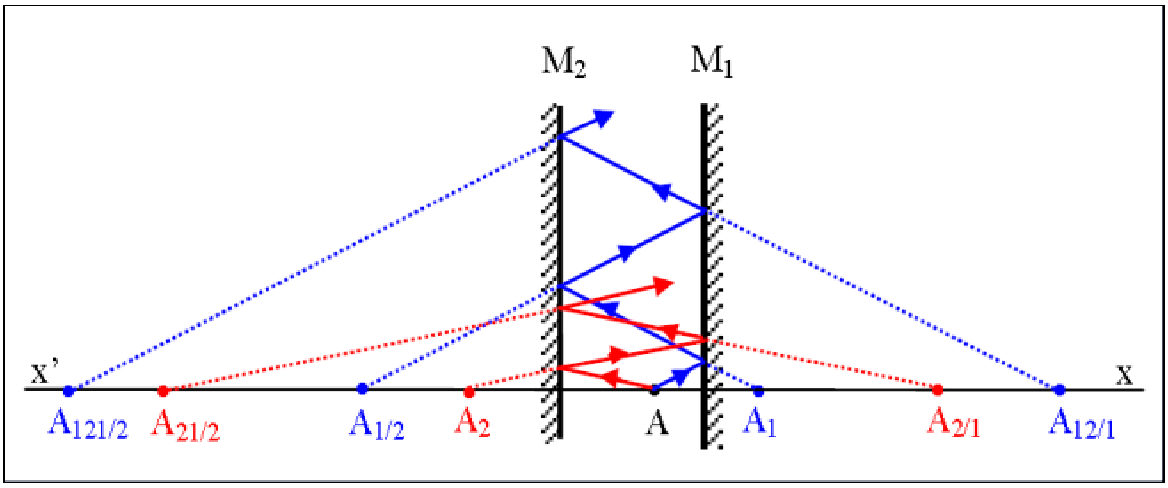


Figure 3: Parallel mirrors.

8.2.2 Plane diopter

A plane diopter is a plane surface that separates two homogeneous, transparent and isotropic media with different refractive indices, as mentioned earlier in chapter 7.

Examples:

1. Between a window pane and air.

2. Between seawater and air.

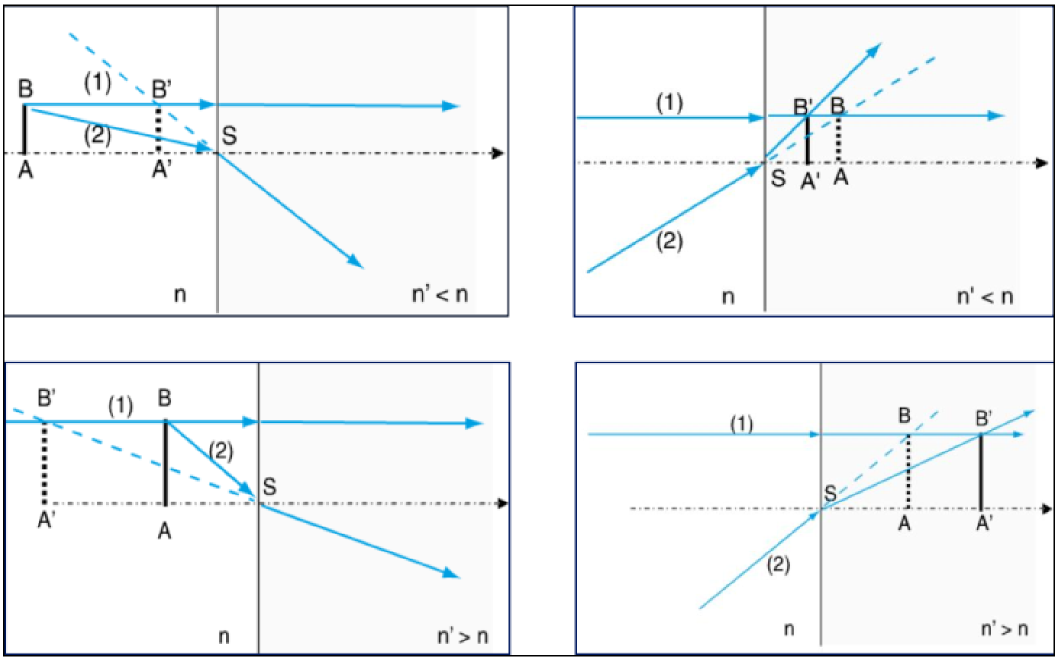
3. Two plane diopter diagrams for the two cases ($n' < n$ and $n' > n$) are shown in Figure 4 below.

Examples:

1. Entre le verre d'une vitre et l'air.

2. Entre l'eau de mer et l'air.

3. Deux schémas de dioptre plan pour les deux cas ($n' < n$ et $n' > n$) sont montrés par les Figures 4 ci-dessous.



Figures 4: Two flat diopters for both cases ($n' < n$ and $n' > n$).

The conjugation relation of the plane diopter relates the position of A on the principal axis to that of its conjugate image A' and is written as:

$$\frac{n'}{SA'} - \frac{n}{SA} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.1)$$

The size of the image equals the size of the object:

$$A'B' = AB \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.2)$$

8.3 Optical systems with spherical faces

8.3.1 Spherical mirror

A spherical mirror is a portion of a spherical surface with center C made reflective by a metallic deposit and it is characterized by a vertex S, radius $R = \overline{SC}$ and by the principal axis of the mirror CS.

There are two types of spherical mirrors:

- **Concave mirror**: the inner surface is reflective (Figure a).
- **Convex mirror**: the outer surface is reflective (Figure b).

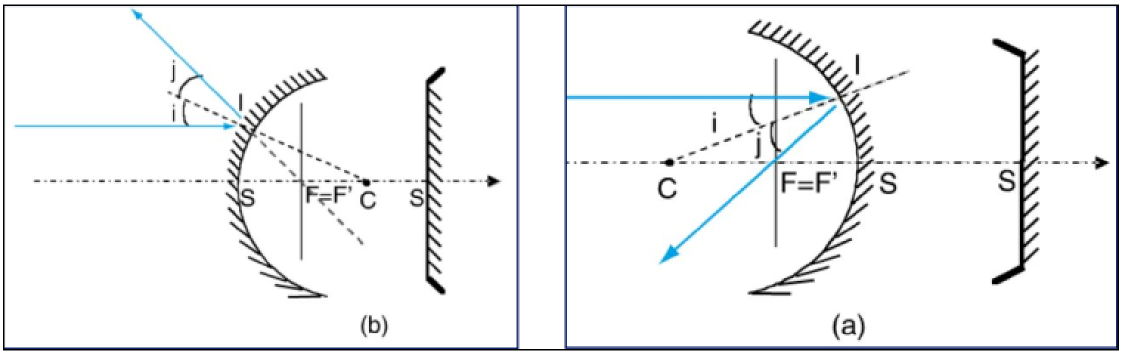


Figure (a), (b): Types of spherical mirror

The object focal (focus) F and image focal F' of a spherical mirror is located in the middle of CS i.e. $(F = F')$.

$$SF = \frac{\overline{SC}}{2} = \frac{[SC]}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.3)$$

a. Conjugation relationships (Law of conjugate points)

- **Relationship between the positions of the object and the image**
- **Origin at the center C**

The conjugation relation is written for the spherical mirror:

$$\frac{1}{\overline{CA'}} + \frac{1}{\overline{CA}} = \frac{2}{\overline{CS}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.4)$$

- **Origin at summit S**

The relationship is known as "**Formule de Descartes**"and is written as:

$$\frac{1}{\overline{SA'}} + \frac{1}{\overline{SA}} = \frac{2}{\overline{CS}} \dots\dots\dots\dots\dots\dots \quad (8.5)$$

8.3.2 Spherical diopter

A spherical diopter is a portion of a refracting spherical surface that separates two transparent, homogeneous media of different indices n_1 and n_2. It is characterized by its principal (optical) axis, its center C, its radius of curvature $r = \overline{SC}$ and its vertex S.

There are two types of spherical diopters: convex, and concave.

- Convex spherical diopters have a radius of curvature $r > 0$, and light hits the curved surface. See Figure 5

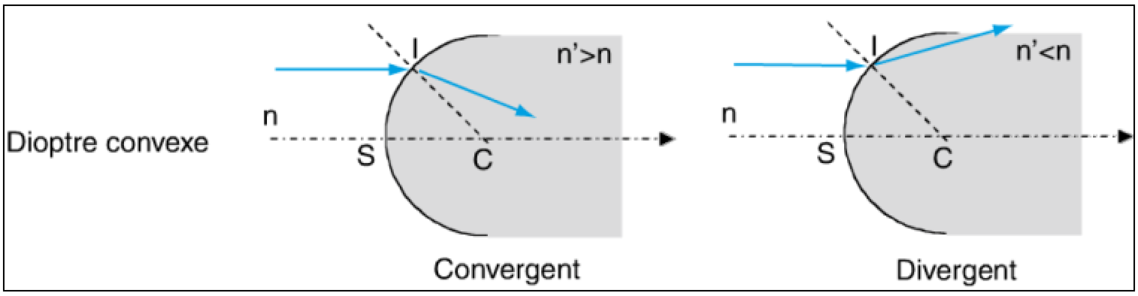


Figure 5: Convex spherical diopters.

- Concave spherical diopter, its radius of curvature $r < 0$ and the light meets the surface in a hollow. See Figure 6.

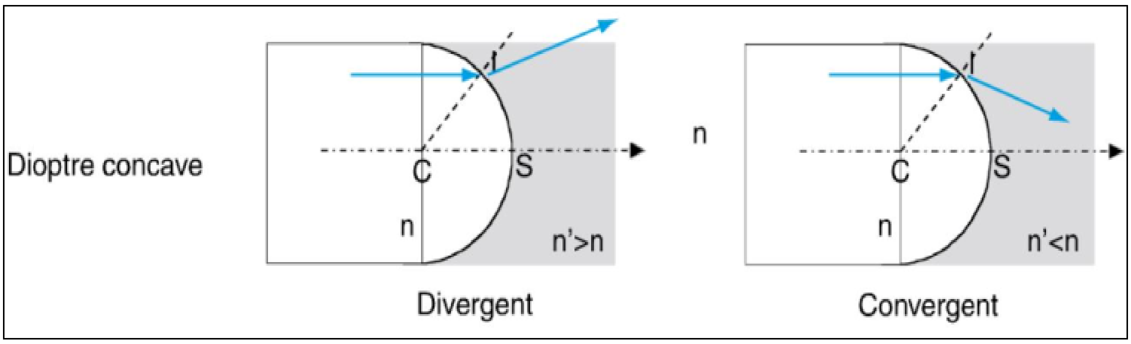


Figure 6 : Concave spherical diopter.

a. **Conjugation relations (Law of conjugate)**

- **Origin at center C**

The conjugation relations are written:

$$\frac{n_1}{\overline{CA'}} - \frac{n_2}{\overline{CA}} = \frac{n_1 - n_2}{\overline{CS}} \text{ ……………………. (8.6)}$$

- **Origin at summit S**

$$\frac{n_1}{\overline{SA}} - \frac{n_2}{\overline{SA'}} = \frac{n_1 - n_2}{\overline{SC}} \text{ ……………………… (8.7)}$$

8.3.3 Linear Magnification γ

If AB has image $A'B'$, the magnification γ is the algebraic ratio of the size of the image to that of the object:

$$\gamma = \frac{\overline{A'B'}}{AB} = -\frac{\overline{SA'}}{\overline{SA}} \text{ ………………………….. (8.8)}$$

- Magnification m is taken to be positive when the image has the same direction as the object, $\gamma < 0$ when the image is upside down.
- The minus sign is included only to indicate that the image is upside down in relation to the object.

Exercise:

Consider a concave spherical mirror with center C, vertex S, radius of curvature $R = \overline{SC} = -30cm$, and an object AB with a height of 1 cm.

1. Give the position of the focus F.

2. Determine the image $\overline{A'B'}$ of $\overline{AB}$ by specifying its position, nature, direction, and size in the following case: SA= $-60\ cm$.

3. Specify the nature of the object. Construct the image.

Solution

1. The focal (focus) F of the mirror is in the middle of the segment $[SC]$, and
$$SF = \frac{\overline{SC}}{2} = \frac{[SC]}{2} = -15 \text{ cm.}$$

2. The position of A' is obtained from the conjugation formula:

$$\frac{1}{\overline{SA'}}+\frac{1}{\overline{SA}}=\frac{2}{\overline{SC}}=\frac{1}{\overline{SF}} \quad \Longrightarrow \quad \overline{SA'}=\frac{\overline{SF}.\overline{SA}}{\overline{SA}-\overline{SF}}$$

The magnification is given by:

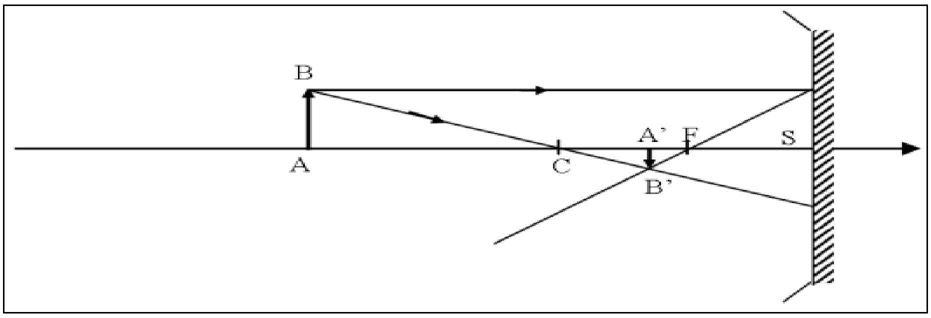
$$\gamma=\frac{\overline{A'B'}}{AB}=-\frac{\overline{SA'}}{\overline{SA}}$$

$$\overline{SA}=-60cm \quad \Longrightarrow \quad \overline{SA'}=-20\ cm$$

We find:

$$\gamma=\frac{1}{3} \ \ et\ \overline{A'B'}=0.33\ cm$$

3. The image and object are real. The image is inverted and three times smaller than the object.



8.4 Thin lenses

8.4.1 Definitions

- It is an association of two spherical diopters whose vertices are practically merged into a vertex S. The optical axis of the lens is the axis passing through the centers of the two spherical diopters. We note n the index of the medium constituting the lens ($\boldsymbol{n}>1$).
- A lens is a homogeneous transparent medium of index $\boldsymbol{n}$ limited by two diopters, at least one of which is spherical, the other being able to be, at the limit, plane. It is a centered system whose axis is the straight line which joins the two centers of the respective diopters.
- The thickness of a lens is the distance $S_1S_2$where S_1 and S_2 are the vertices of the two diopters. A lens is said to be thin or thick depending on whether its

thickness is or is not small compared to the radii of curvature of its two faces and compared to their difference if these are in the same direction.

There are six possible lens shapes see Figure below:

1. Biconvex lenses,
2. Plano-convex lenses,
3. Convex- concave,
4. Biconcave lenses,
5. Plano-concave lenses,
6. Thick-edged meniscus.

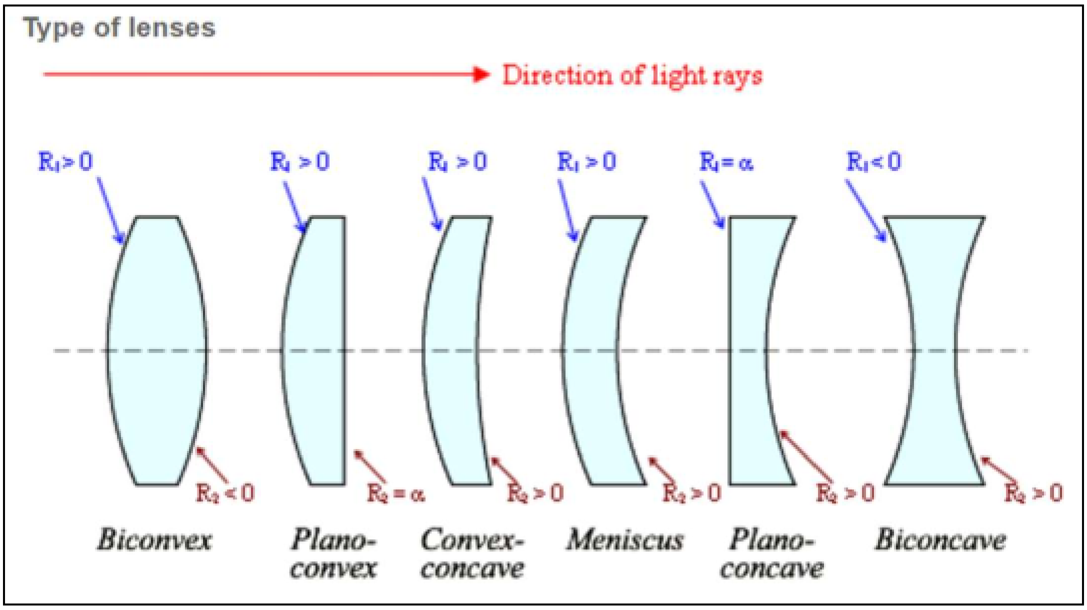


Figure 7: The possible shapes of lenses.

The first three are thin-edged, meaning that the edge of the lens is thinner than its center, and the last three are thick-edged.

A lens is characterized by:

- The vertices S_1 and S_2 of the diopters in the order in which the light meets them,
- The optical axis (Δ) oriented in the direction of propagation of the light,

- The centers C_1 and C_2 of the diopters which are carried by the optical axis,
- The radii of curvature $R_1 = \overline{S_1C_1}$and $R_2 = \overline{S_2C_2}$ of the diopters, one of which is infinite if one of the diopters is plane,
- The index n of the lens and those of the extreme media.

We will differentiate thin lenses with thin edges from thin lenses with thick edges by arrows of opposite directions placed at the ends of this line as shown in the Figure below.

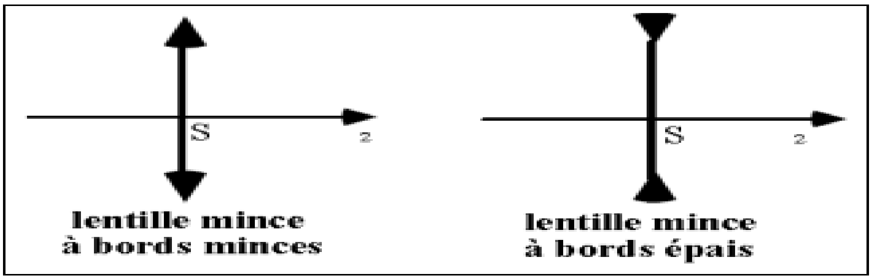


Figure 8: The different thin lenses.

8.4.2 Walk of a light ray

A ray A_1I_1 incident on the first dioptre gives a refracted ray I_1I_2A'' in the medium of index n. After refraction on the second diopter, this ray gives a ray I_2A_2 in the air.

A'' is the intermediate image of A_1 and A_2 is the final image of A_1 given by the lens, as shown in the Figure opposite.

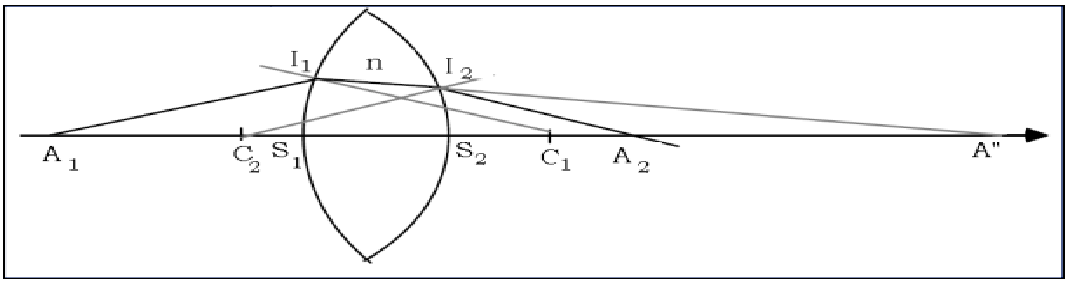


Figure 9: Trajectory of a light ray in the lens.

In the case of thin lenses for which S_1, S_2, and S are coincident, the conjugation relation with origin at the optical center S is written:

$$\frac{1}{\overline{SA_2}} - \frac{1}{\overline{SA_1}} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = V \quad \text{.............. (8.9)}$$

With $\quad SA_2 \equiv S_2A'$

Demonstration of formula (8.9)

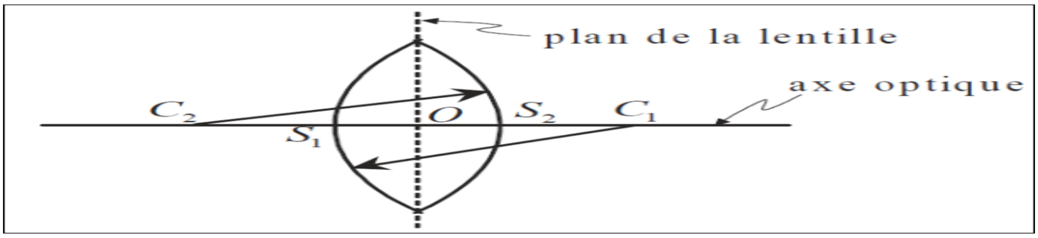
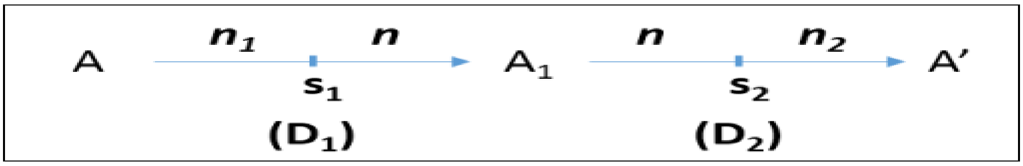


Figure 10: Thin lens.

☞ **Conjugation formula (Law of conjugate)**



- **Diopter (D1) :**

$$\frac{n}{\overline{S_1A_1}} - \frac{n_1}{\overline{S_1A}} = \frac{n-n_1}{\overline{S_1C_1}} = V_1 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8.10)$$

$$\gamma_1 = \frac{n_1}{n}\frac{\overline{S_1A_1}}{\overline{S_1A}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8.11)$$

- **Diopter (D2) :**

$$\frac{n_2}{\overline{S_2A'}} - \frac{n}{\overline{S_2A_1}} = \frac{n_2-n}{\overline{S_2C_2}} = V_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8.12)$$

$$\gamma_2 = \frac{n}{n_2}\frac{\overline{S_2A'}}{\overline{S_2A_1}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (8.13)$$

❖ **Thick lens:**

Summing equations (8.10) and (8.12), we find:

$$\left(\frac{n}{\overline{S_1A_1}} - \frac{n_1}{\overline{S_1A}}\right) + \left(\frac{n_2}{\overline{S_2A'}} - \frac{n}{\overline{S_2A_1}}\right) = \left(\frac{n-n_1}{\overline{S_1C_1}}\right) + \left(\frac{n_2-n}{\overline{S_2C_2}}\right) = V_1 + V_2 \;\ldots\ldots\; (8.14)$$

$$\gamma = \gamma_1 \times \gamma_2 = \frac{n_1}{n}\frac{\overline{S_1A_1}}{\overline{S_1A}} \times \frac{n}{n_2}\frac{\overline{S_2A'}}{\overline{S_2A_1}} \;\ldots\ldots\ldots\; (8.15)$$

❖ **Thin lens:** $S_1 \equiv S_2 \equiv S$

$$\left(\frac{n_2}{\overline{S_2A'}} - \frac{n_1}{\overline{S_1A}}\right) = \left(\frac{n-n_1}{\overline{S_1C_1}}\right) + \left(\frac{n_2-n}{\overline{S_2C_2}}\right) = V_1 + V_2$$

We obtain

$$\left(\frac{n_2}{\overline{SA'}} - \frac{n_1}{\overline{SA}}\right) = \left(\frac{n-n_1}{\overline{SC_1}}\right) + \left(\frac{n_2-n}{\overline{SC_2}}\right) = V \;\ldots\ldots\ldots\ldots\ldots\; (8.16)$$

$$\gamma = \frac{n_1}{n_2}\frac{\overline{SA'}}{\overline{SA}} \;\ldots\ldots\ldots\ldots\ldots\; (8.17)$$

With: V is the vergence or power of the diopter (unit: Diopter = m^{-1}).

❖ **Thin lens of index n, in two media n_1 and n_2**

- **Case where $n_1 = n_2$ (same medium)**

$$\left(\frac{n_1}{\overline{SA'}} - \frac{n_1}{\overline{SA}}\right) = \left(\frac{n - n_1}{\overline{SC_1}}\right) + \left(\frac{n_1 - n}{\overline{SC_2}}\right) = V$$

$$= (n - n_1)\left[\frac{1}{\overline{SC_1}} - \frac{1}{\overline{SC_2}}\right]$$

$$\frac{n_1}{\overline{SA'}} - \frac{n_1}{\overline{SA}} = (n - n_1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] \;\ldots\ldots\ldots\; (8.18)$$

$$\gamma = \frac{\overline{SA'}}{\overline{SA}} \;\ldots\ldots\ldots\ldots\ldots\; (8.19)$$

❖ **Thin lens of index n in air: $n_1 = n_2$**

$$\frac{1}{\overline{SA'}} - \frac{1}{\overline{SA}} = (n - 1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right] = V \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8.20)$$

$$\gamma = \frac{\overline{SA'}}{\overline{SA}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8.21)$$

Remark:

- If V > 0: Convergent diopter lens forming real images
- If V < 0: divergent dioptre lens forming virtual images
- $\boldsymbol{V = \frac{1}{F}}$ With: F = focal distance of the thin lens.

❖ **Focal Image F'**

$$\overline{SF'} = \frac{1}{V}$$

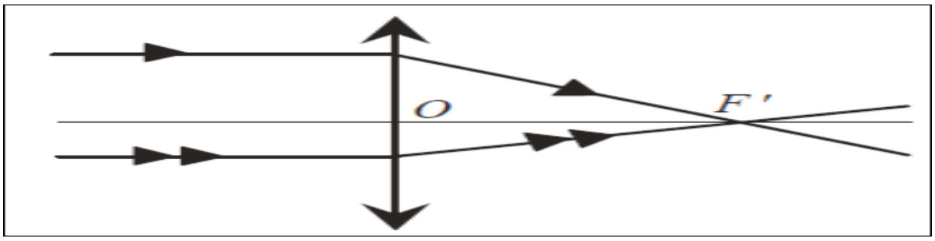


Figure 11 : Convergent lens.

❖ **Focal Image** ***F***

$$\overline{SF} = -\frac{1}{V}$$

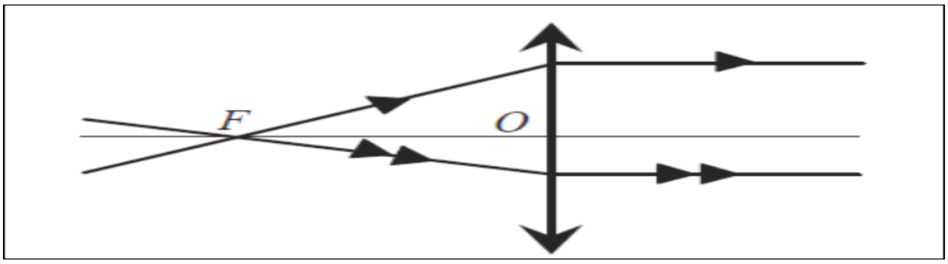


Figure 12: Divergent lens.

F and F' are symmetrical with respect to S.

For a converging lens, F and F' are real, whereas for a diverging lens, they are virtual.

8.4.3 Different types of lenses

8.4.3.1 Convergent lenses

Converging lenses transform a beam of light rays parallel to the optical axis into a converging beam. See Figure 11, 12 and Figure 13, which is verified:

$$V > 0,\ \overline{SF'} > 0 \quad et \quad \overline{SF} < 0$$

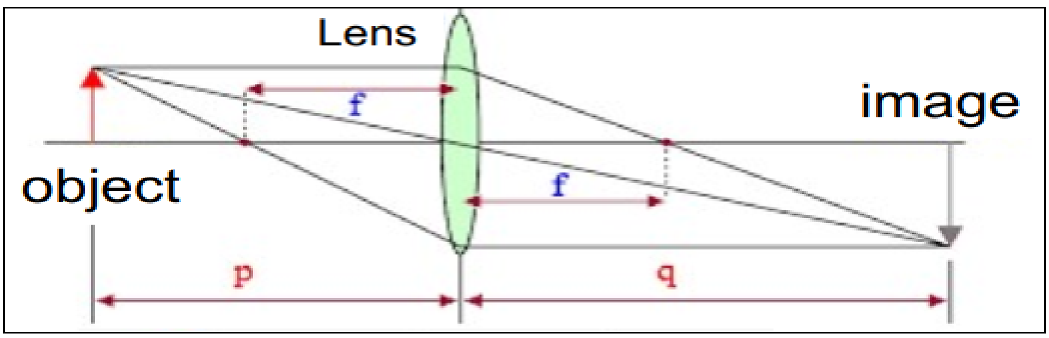


Figure 13: image formation by rays: convergent lens

8.4.3.2 Divergent lenses

Divergent lenses transform a beam of light rays parallel to the optical axis into a divergent beam. As in Figure 14

Any light ray passing through the optical centre of a thin lens does not undergo any deviation as it passes through it. Which is verified:

$$V < 0, \ \overline{SF'} < 0 \quad et \quad \overline{SF} > 0$$

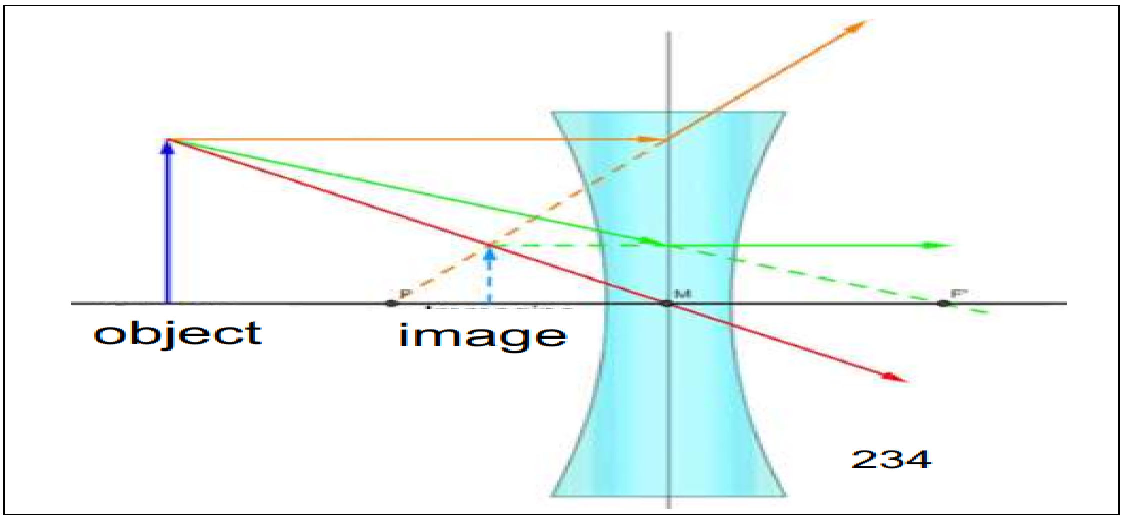


Figure 14 : image formation by rays: divergent lens.

Exercise:

I. Un objet réel est placé à 10 cm d'une lentille convergente (1) de 5 cm de distance focale. Une deuxième lentille convergente (2) de distance focale 15 cm est disposée à 34 cm de la première.

1. Déterminer la position de l'image donnée par ce système ainsi que l'agrandissement total.

II. On rapproche cette fois cette lentille (2) de la lentille (1) et on suppose les centres confondus.

2. Déterminer la distance focale.
3. Les caractéristiques de l'image obtenue pour le même objet.

Solution

1. $V_1 = 20$ Dioptrie, $S_1A_1 = 0.1m,\quad \gamma_1 = -1, \gamma_2 = -1.67, \gamma = \gamma_1 \times \gamma_2 = +1.6$
2. $S_2F_2' = 6.67$ Dioptrie, $S_2A_1 = -0.24\ m$, $S_2A_2 = +0.4\ m$, $S_1A_2 = -0.74\ m$.
3. Image réelle, droite et agrandie.

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