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Theme

On some classes of operators in a Hilbert space and a semi-Hilbertian space

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List of Symbols

$\mathcal{H}$ Complex Hilbert space.
$\mathcal{B}(\mathcal{H})$ The space of all bounded linear operators on $\mathcal{H}$.
$\mathcal{B}(\mathcal{H})^+$ Cone of all positive operators on $\mathcal{H}$.
$\mathcal{B}^A(\mathcal{H})$ Space of all A-bounded operators $T \in \mathcal{B}(\mathcal{H})$.
$\mathcal{B}_A(\mathcal{H})$ Set of A-bounded operators admitting an A-adjoint.
$\langle \cdot | \cdot \rangle$ Inner product on $\mathcal{H}$.
$\| \cdot \|$ Norm induced by $\langle \cdot | \cdot \rangle$.
$\langle \cdot | \cdot \rangle_A$ Semi-inner product associated with a positive operator A.
$\| \cdot \|_A$ Seminorm induced by $\langle \cdot | \cdot \rangle_A$.
T^* Adjoint operator of T (Hilbert spaces).
$T^\sharp$ A-adjoint operator of T (semi-Hilbertian spaces).
$T^\dagger$ pseudoinverse of T.
T^{-1} Inverse operator of T.
$A^{\frac{1}{2}}$ square root of a positive operator A.
I Identity operator.
U Unitary operator.
$|T|^2$ equal to $T^\star T$
$|T|_A^2$ equal to $T^\sharp T$
$\mathcal{N}(T)$ Null space (kernel) of T.
$\mathcal{R}(T)$ Range space of T.
$\overline{\mathcal{R}(T)}$ Closure of the range of T.
$M^\perp$ Orthogonal complement of a set M.
$\overline{M}$ Closure of a set M.
$P_\mathcal{M}$ Orthogonal projection onto a closed linear subspace $\mathcal{M}$.
$\mathbb{N}$ Set of non-negative integers.
$\mathbb{R}$ Set of real numbers.
$\mathbb{C}$ Set of complex numbers.
$\mathbb{C}^*$ $\mathbb{C} \setminus \{0\}$.
$T \otimes S$ Tensor product of operators.
$\mathcal{H} \otimes \mathcal{H}$ Algebraic tensor product of $\mathcal{H}$ with itself.
$\mathcal{H} \overline{\otimes} \mathcal{H}$ Hilbertian tensor product of $\mathcal{H}$ with itself.
$T_1T_2 = T_2T_1 \wedge T_1^\star T_2 = T_2 T_1^\star$ T_2 doubly commutes with T_1 .

introduction

The theory of operators in Hilbert spaces was developed in the early 20th century by David Hilbert in the context of integral equations.Subsequently,several researchers rigorously organized the theory,integrated it with quantum mechanics,and established the modern definition of a Hilbert space [8,22].This field has undergone remarkable development in recent decades,driven by the interest of many researchers and its necessity in various fields,including quantum mechanics,physics,and mathematics.This evolution has led to the expansion of the basic concepts and ideas of the theory,which are of great importance to the development of our contemporary world.In recent years,a new structure,considered a generalization of Hilbert space,has emerged and is called semi-Hilbert space[15,16].

In this thesis,we have studied specific classes of operators that are of personal interest and are considered developments and extensions of classes previously investigated by several authors.The most important of these are the normal,hyponormal,and Class A operators (see [1,4,10]).We have studied these classes in both Hilbert and semi-Hilbert spaces,examining some of their important properties and the extent to which algebraic properties are preserved.

Among the most important classes discussed and developed by researchers are normal operators,studied by C.R.Putnam and J.B.Conway.An operator T is defined as normal if it satisfies $TT^\star = T^\star T$,meaning it commutes with its adjoint (see [1]).J.G.Stampfli expanded on this in 1962,introducing hyponormal operators.An operator T is defined as hyponormal if it satisfies $TT^\star \geq T^\star T$,and a set of important properties was presented (see [4]).Decades later,T.Furuta and M.Ito introduced Class A,defined by the condition $T^2T^{\star 2} \geq (T^\star T)^2$ (see [10,12]).

In this work,we will present and study new classes that are considered generalizations of those previously mentioned,in both Hilbert space and semi-Hilbert space.

This thesis consists of five chapters:

The first chapter revisits fundamental properties of linear operators on a complex Hilbert space $\mathcal{H}$,which is an important foundation for this field.We also introduce the different classes of operators in semi-Hilbert space,a framework based on the seminorm derived from the positive operator A.Many results derived from this theory,as studied in a remarkable analytical work by Arias ([12,16]),are revealed.One of the most important theories we discuss in this chapter is Douglas's theorem,which allows us to define an adjoint operator for the semi-inner product.This is particularly important because the conditions for its existence are met in this context.This theorem facilitates understanding of its existence,and we will examine some of its basic properties that are necessary for our study.We will also define some concepts and terms of great importance.Based on this,we present and summarize some of the most important classes related to the theory.In this section,we focus on positive operators A,from which we can later deduce some classes.We researched and presented all the important results and concepts,and then studied A-normal operators,A-self-adjoint operators,A-unitary operators,A-hyponormal operators,and Class $A^\sharp$.This chapter also contains some definitions and results that will be useful for A-bounded linear operators and the tensor product.

In Chapter 2,we introduced a new class of operators in semi-Hilbert space called Class $A_k^\sharp$ operators.This class is defined by the condition

$$T^{\sharp k}T^k - (TT^\sharp)^k \geq_A 0$$

We studied and explored this class in depth and derived several structural properties.We provided numerous examples to facilitate understanding and control of these properties.In addition,we discussed the sum and product of two operators from the same class and gave the optimal conditions for preserving these algebraic operations,as well as presenting the tensor product for this class.

Chapter 3 focuses on the Quasi Class $A_k^\sharp$.We provide a description of this class and examine some properties previously studied and discussed by several authors for their respective classes.We also provide some observations and cases in which this class resembles previously defined classes.We will prove the stability of this class under the direct sum of two factors,a study conducted in the context of semi-Hilbert spaces.

Chapter 4 is devoted to studying the same class as in Chapter 2,but this time in Hilbert space.We present some examples and observations,along with some basic results.Our focus is on preserving algebraic properties and comparing them with several important classes that have been discussed by some authors in recent years,as this class is considered a generalization of many of these.In addition,we examined its stability under the tensor product for operators of this class.

The fifth and final chapter examines the quasi-totally Class $A_k^\star$ in Hilbert space.In this chapter,we prove that many results obtained for other classes remain valid for this class,given certain conditions on the operators belonging to it.We also present simple yet important examples to demonstrate its stability under direct sum and tensor product.We further prove its stability under the tensor product.

Chapter 1

Preliminaries

1.1 Bounded Linear Operators on Hilbert Space

this section serves to fix notation,present basic definitions,and assemble key mathematical tools that will be utilized throughout our work.The foundational concepts discussed here primarily build upon existing results in the field ([3,8]).

1.1.1 Norm Space

Definition 1.1.1 *Let $\mathcal{V}$ be a Vector Space over $\mathbb{R}$ or $\mathbb{C}$.A norm on $\mathcal{V}$ is a map from*

$$\|.\| : \mathcal{V} \to [0; +\infty($$

that satisfies the following axioms for all $v, v' \in \mathcal{V}$ and $\rho \in \mathbb{R} \vee \mathbb{C}$

1. $\|v\| \geq 0.$

2 $\|v\| = 0 \Leftrightarrow v = 0.$

3 $\|\rho v\| = |\rho|\|v\|.$

4 $\|v + v'\| \leq \|v\| + \|v'\|.$

The vector space endowed with a norm denoted by $(\mathcal{V}; \|.\|)$ is called a Norm space.

Remark 1.1.1

1. If Norm Space $(\mathcal{V}; \|.\|)$ is complete then it is Banach Space.

2 If condition (2) of Definition 1.1.1 is possible that $\|v\| = 0$ then $v \neq 0$,we called $\|.\|$ is a semi-norm.

Definition 1.1.2 *Let $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspace of a Banach Space $(\mathcal{V}, \|.\|)$.*

1. a Banach Space $(\mathcal{V}, \|.\|)$ over $\mathbb{R} \vee \mathbb{C}$ is said to be the direct sum of subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ written $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$,i.e:
Each $v \in \mathcal{V}$ can be uniquely decomposed as $v = v_1 + v_2$ with $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$

2 A Banach Space $(\mathcal{V}, \|.\|)$ over $\mathbb{F}$ is called complemented,if there is a subspace $\mathcal{V}_1$ and closed subspace $\mathcal{V}_2$ such that $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$.

1.1.2 Hilbert Space

Definition 1.1.3 *Let $\mathcal{V}$ be a Vector space over $\mathbb{R}$ or $\mathbb{C}$.An inner product on $\mathcal{V}$ is a function*

$$\langle .|. \rangle : \mathcal{V} \to \mathbb{F}$$

if that satisfies the following axioms for all $v_1, v_2, v_3 \in \mathcal{V}$ and $\rho, \rho' \in \mathbb{R} \vee \mathbb{C}$

1. $\langle v_1|v_2 \rangle \geq 0.$

2 $\langle v_1|v_2 \rangle = \overline{\langle v_2|v_1 \rangle}.$

3 $\langle \rho v_1 + \rho' v_2|v_3 \rangle = \rho\langle v_1|v_3 \rangle + \rho'\langle v_2|v_3 \rangle.$

4 $\langle v_1|v_1 \rangle = 0$ *if and only if* $v_1 = 0.$

A Vector Space $\mathcal{V}$ with an inner product $(\mathcal{V}; \langle .|. \rangle)$ is an inner product space.If complete,it is a Hilbert space,also denoted by $(\mathcal{H}; \langle .|. \rangle)$(see[8]).

Remark 1.1.2 *If condition (4)of Definition 1.1.3 is not satisfied,then $\langle .|. \rangle$ is called semi-inner product.*

Example 1.1.1 *Let $\mathcal{V} = \mathbb{C}^n$,then the map*

$$\langle .|. \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$
$$(u|v) \mapsto \langle u|v \rangle = \sum_{i=0}^{n} u_i \overline{v_i}$$

is an inner product on $\mathbb{C}^n$.

In the following proposition is considered important in this field.It gives a relationships between inner product and norm.

Proposition 1.1.1 *Let $(\mathcal{V}, \langle .|. \rangle)$ be an inner product space over $\mathbb{F}$.Then,the map*

$$\|.\| : \mathcal{V} \to \mathbb{C}$$
$$v \mapsto \|v\| = \langle v|v \rangle^{\frac{1}{2}}$$

is a norm on $\mathcal{V}$.

Corollary 1.1.1 *(Cauchy-Schwarz inequality)[50] For any two vectors u, v in an inner product space,it states:*

$$\langle u|v \rangle \leq \langle u|u \rangle^{\frac{1}{2}} \langle v|v \rangle^{\frac{1}{2}}$$

Example 1.1.2 (1) *Consider the following Hilbert space*

$$l^2_{\mathbb{N}^\star}(\mathbb{C}) = \{v = (v_i)_{i \in \mathbb{N}^\star} \in \mathbb{C} | \exists M \in \mathbb{R}^+, \sum_{i=1}^{\infty} |v_i|^2 = M\}$$

Define by the following inner product

$$\langle .|. \rangle : l^2_{\mathbb{N}^*} \times l^2_{\mathbb{N}^*} \to \mathbb{C}$$
$$(u;v) \mapsto \langle u|v \rangle = \sum_{i=1}^{\infty} u_i \overline{v_i}$$

therefore,the vector $v = (v_i)_{i \in \mathbb{N}^\star}$ *in* $l^2_{\mathbb{N}^\star}(\mathbb{C})$ *if there is* $M \in \mathbb{R}^+$,$\|v\|^2_{l^2} = \sum_{i=1}^{\infty} |v_i|^2 = M$
(2) *We shall also deal with the Hilbert space* $l^2_{\mathcal{Z}}(\mathbb{C})$ *which is two-sided version of* $l^2_{\mathbb{N}^\star}(\mathbb{C})$ *it consists of all doubly infinite sequences* $(x_n)_{n \in \mathbb{Z}} = (\ldots, x_{-1}, x_0, x_1, \ldots)$ *of complex numbers that are square summable,that is,*$\sum_{i=-\infty}^{\infty} |x_i|^2 < \infty$.

Now,we recall the following definition.For more details,one may consult Note that every closed subspace of a Hilbert space is complemented as it is shown in the next theorem.

Theorem 1.1.1 *Let* $\mathcal{M}$*be a closed subspace of a Hilbert Space.Then,each vector admits a unique decomposition into* $\mathcal{M}$ *and* $\mathcal{M}^\perp$*.Formally:*

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$$

where the orthogonal complement $\mathcal{M}^\perp$ *is defined as:*

$$\mathcal{M}^\perp = \{v \in \mathcal{H} : \langle v, u \rangle = 0 \quad \text{for all } u \in \mathcal{M}\}$$

Remark 1.1.3 *It is crucial to emphasize that the converse of theorem 1.1.1 also holds.Specifically,if* $(\mathcal{V}; \|.\|)$*is a Banach space where every closed subspace is complemented,then the norm* $\|.\|$ *necessarily arises from a scalar product,there by making* $(\mathcal{V}; \langle .|. \rangle)$ *a Hilbert space.However,this property does not extend to infinite-dimensional Banach Space over the field* $\mathbb{F}$*.In such spaces,closed subspaces may lack complements.*

1.1.3 Bounded operators

Definition 1.1.4 *Let* $(\mathcal{H}_1|\langle .|. \rangle_{\mathcal{H}_1})$ *and* $(\mathcal{H}_2|\langle .|. \rangle_{\mathcal{H}_2})$ *are Hilbert Spaces over* $\mathbb{F}$*.An operator* T *from* $\mathcal{H}_1$ *to* $\mathcal{H}_2$ *is a linear mapping from a* $\mathcal{H}_1$ *to* $\mathcal{H}_2$*.That is*

$$T(\alpha v) = \alpha T(v) \quad \text{and} \quad T(v_1 + v_2) = T(v_1) + T(v_2)$$

for all $v_1, v_2 \in \mathcal{H}_1$ *and* $\alpha \in \mathbb{C}$.
When $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$*,we shall simply say that* T *is an operator on* $\mathcal{H}$.

For convenience,we write Tv instead of $T(v)$.Clearly $T(0) = 0$.

From now,all bounded linear operators will be considered to be defined on the whole of $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ with $\mathcal{H}$ is assumed to be a non trivial complex Hilbert space with inner product $\langle .|. \rangle$ and associated norm $\|.\|$.As usual we will normally write just $\mathcal{H}$ instead of $(\mathcal{H}; \langle .|. \rangle)$.Notice that the space of all bounded linear operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$.

Definition 1.1.5 *Let A linear map* $T : \mathcal{H}_1 \to \mathcal{H}_2$ *is said to be bounded if there exists a constant* $\rho > 0$ *such that*

$$\|Tv\|_{\mathcal{H}_2} \leq \rho \|v\|_{\mathcal{H}_1}$$

for all $v \in \mathcal{H}_1$.

Definition 1.1.6 *Let $T \in \mathcal{B}(\mathcal{H})$.Then,*

1. *We define The range $\mathcal{R}(T)$ as the subspace $\mathcal{H}$ consisting of all Tv where $v \in \mathcal{H}$.*

2 *We define The kernel of T is the subspace $\mathcal{H}$ consisting of all v where $Tv = 0$.*

1.1.4 Adjoint of a bounded operator

The following definition introduces the adjoint of a bounded linear operator a fundamental concept in operator theory.Notably,studying an operator's adjoint often provides valuable insights into the original operator's properties.This approach proves particularly useful as adjoints frequently exhibit more tractable behavior than the operators themselves.

Definition 1.1.7 *Let $(\mathcal{H}; \langle .|. \rangle)$ be a Hilbert Space and $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear.The adjoint $T^\star$ is defined as the unique operator for which:*

$$\langle Tv_1|v_2\rangle = \langle v_1|T^\star v_2\rangle, \quad \forall v_1, v_2 \in \mathcal{H}$$

The adjoint $T^\star$ exists and is unique for every bounded linear operator T on a Hilbert space.

Example 1.1.3

1. *For a matrix $M = (a_{ij})_{n\geq i|j\geq 1}$ (viewed as a linear operator on $\mathbb{C}^n$ or $\mathbb{R}^n$,the adjoint operator corresponds to the conjugate transpose $M^* = (\overline{a_{ji}})_{n\geq i|j\geq 1}$.*

In the following proposition we collect some basic properties of $T^\star$.

Proposition 1.1.2 *Let T_1, T_2 be bounded linear operators.We obtain,*

1. ***Linearity:*** $(\rho T_1 + T_2)^\star = \bar{\rho} T_1^\star + T_2^\star$.

2 ***Composition:*** $(T_1T_2)^\star = T_2^\star T_1^\star$ *and* $(T_1^k)^\star = (T_1^\star)^k$ *for all* $k \in \mathbb{N}^\star$.

3 ***Involution:*** $[T_1^\star]^\star = T_1$

The kernel and range of an operator are two fundamental concepts that describe the behavior of linear transformations or operators.Here' s a detailed look at these concepts and their relationship.

Theorem 1.1.2 *If T is a linear operator on Hilbert spaces,and $T^\star$ is its adjoint,there is an important relationship between the kernel and range of T and $T^\star$.*

1. *The range of T is the kernel of the adjoint operator,i.e:*

$$\mathcal{N}(T) = \mathcal{R}(T^\star)^\perp \quad \textit{and} \quad \mathcal{N}(T^\star) = \mathcal{R}(T)^\perp.$$

2 *The range of T(range of $T^\star$) content of the kernel of the adjoint operator(kernel of T),i.e:*

$$\mathcal{R}(T) \subseteq \mathcal{N}(T^\star)^\perp \quad \textit{and} \quad \mathcal{R}(T^\star) \subseteq \mathcal{N}(T)^\perp.$$

Corollary 1.1.2 *Let $T \in \mathcal{B}(\mathcal{H})$.Then,*

1.

$$\overline{\mathcal{R}(T)} = \mathcal{N}(\mathcal{T}^\star)^\perp.$$

2

$$\overline{\mathcal{R}(T^\star)} = \mathcal{N}(T)^\perp.$$

Remark 1.1.4 *Let $T \in \mathcal{B}(\mathcal{H})$. Then,*

1. T has closed range $\mathcal{R}(T)$ if and only if $T^\star$ has closed range $\mathcal{R}(T^\star)$.

2 If $\mathcal{R}(T)$ is closed. Then, $\mathcal{R}(T) = \mathcal{N}(T)^\perp$ and $\mathcal{R}(T^\star) = \mathcal{N}(T)^\perp$.

Theorem 1.1.3 *Let $T \in \mathcal{B}(\mathcal{H})$. Then,*

1. T is injective operator if and only if $\mathcal{R}(\mathcal{T}^\star)$ is dense in $\mathcal{H}$.

2 T is surjective operator if and only if $\mathcal{R}(\mathcal{T}^\star)$ is closed and $\mathcal{N}(T^\star) = \{0\}$.

1.1.5 Positive Operator in Hilbert Space

Positive operators acting on Hilbert spaces are a fundamental class of operators,they are used in many fields such that quantum mechanics.In the following we give some basic properties of this class which will be used in the sequel (see [28],[22]).

Definition 1.1.8 *The operator T is called positive if for all $v \in \mathcal{H}$. We have,*

$$\langle Tv|v\rangle \geq 0.$$

we often write: $T \geq 0$.

For the sequel,write $\mathcal{B}(\mathcal{H})^+$ all positive operators,i.e:

$$\mathcal{B}(\mathcal{H})^+ = \Big\{T \in \mathcal{B}(\mathcal{H}) | \langle Tv|v\rangle \geq 0, \forall v \in \mathcal{H}\Big\}.$$

Example 1.1.4

*1. **Identity Operator:** $I_\mathcal{H}$ is positive because*

$$\langle I_\mathcal{H} v|v\rangle = \langle v|v\rangle = \|v\|^2 \geq 0 \quad \forall v \in \mathcal{H}$$

*2 **projection Operators:** P is positive because*

$$\langle Pv|v\rangle = \|Pv\|^2 \geq 0.$$

Remark 1.1.5

1. The statement $T_1 \geq T_2$ is assumed to mean that $\langle (T_1 - T_2)v \mid v\rangle \geq 0$.

2 Every bounded operator T,satisfies $T^\star T$ and $TT^\star$ are positive operators.

Proposition 1.1.3

1. If T is a positive operator,there exists a unique positive operator S such that:

$$T = S^2$$

denoted by $T^{\frac{1}{2}}$.

2 If T is a positive operator and k is a positive integer,then T^k is also positive.i.e,

$$\langle T^k v|v\rangle \geq 0 \quad \forall v \in \mathcal{H}.$$

3 A positive operator T is necessarily self-adjoint,meaning

$$\langle Tv|u\rangle = \langle v|Tu\rangle \quad \forall u, v \in \mathcal{H}.$$

Theorem 1.1.4 *If an operator T commutes with A iff it also commutes with the square root of A.*

Lemma 1.1.1 *Let $T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$.Then,*

$$AT = TA \Rightarrow A^{\frac{1}{2}}T = TA^{\frac{1}{2}}. \tag{1.1}$$

Proof 1.1.1 *Let $T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})^+$*

$\Rightarrow$ $AT = TA$ in view of lemma 1.1.1,then $A^{\frac{1}{2}}T = TA^{\frac{1}{2}}$.

$\Leftarrow$ assume that $A^{\frac{1}{2}}T = TA^{\frac{1}{2}}$.Then,we have

$$\begin{aligned} A^{\frac{1}{2}}T = TA^{\frac{1}{2}} &\Rightarrow A^{\frac{1}{2}}(A^{\frac{1}{2}}T) = A^{\frac{1}{2}}(TA^{\frac{1}{2}}) \\ &\Rightarrow AT = TA^{\frac{1}{2}}A^{\frac{1}{2}} \\ &\Rightarrow AT = TA \end{aligned}$$

Hence is proved

Definition 1.1.9 *An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be*

1. *Self-adjoint: $T = T^\star$.*

2 *Isometry: $T^\star T = I$ (see[3]).*

3 *Normal: $T^\star T = TT^\star$ (see[1]).*

4 *Unitary: $T^\star T = TT^\star = I$ (see[3]).*

5 *Hyponormal : $T^\star T - TT^\star \geq 0$ (see[4]).*

6 *$\star$-Class A : $T^{2\star}T^2 - (TT^\star)^2 \geq 0$(see[10]).*

Proposition 1.1.4 *Let $\mathcal{H}$ be a complex Hilbert space.The following assertions hold:*

1. Every positive operator .it is self-adjoint.

2 If $\langle Tv|v\rangle$ equal zero for every $v \in \mathcal{H}$,then T null operator.

3 T is a self-adjoint iff $\langle Tv|v\rangle \in \mathbb{R}$ for all $v \in \mathcal{H}$.

Remark 1.1.6

1. Note that the assertion (1) of proposition 1.1.4 fails to hold in general on real Hilbert spaces.Indeed,by taking

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we see that

$$\langle Tv|v\rangle = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\rangle = \langle \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} | \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\rangle = x_1x_2 - x_1x_2 = 0$$

but $T \neq 0$.

In spite of that,if T *is a self-adjoint acting on a Hilbert space such that* $\langle Tv|v\rangle = 0$ *for every* $v \in \mathcal{H}$,*then* $T = 0$.

2 If $\mathcal{H}$ *is a real Hilbert space,then the condition* $\langle Tv|v\rangle \geq 0$ *for all* $v \in \mathcal{H}$ *does not imply that* $T^\star = T$.*Indeed,consider the matrix* $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *on* $\mathbb{R}^2$,*clearly* T *is positive.*
however,

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq T^\star = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Remark 1.1.7 *For a diagonal matrix with real,non-negative entries,finding its square root is simple: just take the square root of each individual diagonal element.For example,*

$$\text{If } M = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},\text{then its square root is } M^{\frac{1}{2}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

In operator theory,orthogonal projections forms an important class of operators.We recall first the definition of a projection.

Definition 1.1.10 *An operator* P *(a bounded linear map on a Hilbert space* $\mathcal{H}$*) is called a projection if it is idempotent,meaning applying it twice is the same as applying it once:*

$$P^2 = P.$$

Proposition 1.1.5 *If* P *is a projection.Then,*
(1) The complementary operator $Q = I - P$ *is also a projection.*
(2) The ranges of P *and* Q *are closed subspaces.Furthermore,the entire space* $\mathcal{H}$ *can be expressed as the direct sum of these two ranges:* $\mathcal{H} = \mathcal{R}(P) \oplus \mathcal{R}(Q)$.

Definition 1.1.11 *An operator* $P \in \mathcal{B}(\mathcal{H})$ *is an orthogonal projection if* P *is a projection and* $P^\star = P$.

As the above definition mentioned,a projection which is self-adjoint is called an orthogonal projection for the following reason.

Theorem 1.1.5 *Let* $P \in \mathcal{B}(\mathcal{H})$ *be a nonzero projection.Then the following are equivalent:*
(1) $P^\star = P$.
(2) $\mathcal{R}(P)$ *is orthogonal to* $\mathcal{R}(I-P)$,*i.e:*$\mathcal{R}(P)^\perp = \mathcal{R}(I-P)$.

1.1.6 Tensor Products Hilbert Space

The tensor product of operators on Hilbert spaces is a powerful construction that allows combining operators acting on different Hilbert spaces.Let $(\mathcal{H}_1, \langle .|.\rangle_1)$ and $(\mathcal{H}_2, \langle .|.\rangle_2)$ are non-trivial complex Hilbert spaces.This subsection is intended to remind you of the basic material about tensor products of Hilbert space operators that will be required in the work.The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by:

$$\langle x_1 \otimes y_1 | x_2 \otimes y_2\rangle = \langle x_1|x_2\rangle_1 \langle y_1|y_2\rangle_2.$$

Definition 1.1.12 *(see[9,13]) If T_1 is an operator on $\mathcal{H}_1$ and T_2 is an operator on $\mathcal{H}_2$,the tensor product $T_1 \otimes T_2$ operator acts on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ as:*

$$\begin{aligned} T_1 \otimes T_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathbb{C} \\ (x_1 \otimes x_2) \mapsto (T_1 \otimes T_2)(x_1 \otimes x_2) = \langle T_1 | x_1 \rangle_1 \langle T_2 | x_2 \rangle_2 \end{aligned}$$

Following Kubrusly' s work,we present some basic properties of the single tensor product of vectors.

Proposition 1.1.6

1. ***Linear Operator:*** *the tensor product of two bounded linear operators is itself a bounded linear operator.If T_1 and T_2 are bounded operators,then $T_1 \otimes T_2$ is bounded operator.*

2 ***Adjoint of tensor Product:***

$$(T_1 \otimes T_2)^\star = T_1^\star \otimes T_2^\star.$$

3 ***Commutativity:*** *The tensor product of two operators is not generally commutative.That is:*

$$(T_1 \otimes T_2)(T_3 \otimes T_4) \neq (T_3 \otimes T_4)(T_1 \otimes T_2).$$

4 ***Associativity:*** *The tensor product is associative.That is:*

$$(T_1 \otimes (T_1 \otimes T_3)) = ((T_1 \otimes T_2) \otimes T_3).$$

5 ***Identity Operator:*** *The identity operator on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as: $I_1 \otimes I_2$.*

6 ***Distributivity:*** *The tensor product is Distributive of addition,that is*

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_1 \quad \text{and} \quad (T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3.$$

Example 1.1.5 *Consider $T_1 = 6I_1$ and $T_2 = 3I_2$ two operators.The tensor product operator $T_1 \otimes T_2$ acts on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ as:*

$$(T_1 \otimes T_2)(v \otimes y) = 6I_1 v \otimes 3I_2 y = 6x \otimes 3y = 18(v \otimes y) = 18(I_1 \otimes I_2).$$

Example 1.1.6 *Let*

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

The tensor product operator

$$M_1 \otimes M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \otimes \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

In the following we recall the definition of the algebraic tensor product on Hilbert spaces.

Definition 1.1.13 *Let $\mathcal{H}$ be complex Hilbert spaces.The algebraic tensor product of $\mathcal{H}$ and itself is given by:*

$$\mathcal{H} \otimes \mathcal{H} = \Big\{ \sum_{i=1}^{n} x_i \otimes y_i, n \in \mathbb{N}^{\star} \text{ and } x_i, y_i \in \mathcal{H} \Big\}.$$

In $\mathcal{H} \otimes \mathcal{H}$ we define the following sesquilinear form:

$$\langle u|\eta \rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} \langle x_i|z_j\rangle \langle y_i|w_j\rangle$$

For $u = \sum_{i=1}^{n} x_i \otimes y_i \in \mathcal{H} \otimes \mathcal{H}$ and $\eta = \sum_{j=1}^{m} z_i \otimes w_i \in \mathcal{H} \otimes \mathcal{H}$.It can be seen that the sesquilinear form defined in 1.1.12 is an inner product in $\mathcal{H} \otimes \mathcal{H}$.

$\mathcal{H} \otimes \mathcal{H}$ can be competed with respect to this inner product and thus a new Hilbert space is obtained.More precisely,we have the following definition.

Definition 1.1.14 *If $\mathcal{H}$ is Hilbert spaces,then the Hilbert tensor product on $\mathcal{H}$ denoted $\mathcal{H}\overline{\otimes}\mathcal{H}$,is the Hilbert space obtained by completing $\mathcal{H} \otimes \mathcal{H}$ with respect to the inner product defined in 1.1.12*

The tensor product on $\mathcal{H}\overline{\otimes}\mathcal{H}$ of two operators $T, S \in \mathcal{B}(\mathcal{H})$ is defined as follows.

Definition 1.1.15 *Given non-zero operators $T, S \in \mathcal{B}(\mathcal{H})$.Let $T \otimes S \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})$ denoted tensor product defined on the Hilbert space $\mathcal{H}\overline{\otimes}\mathcal{H}$ as follows*

$$\langle T \otimes S(\xi_1 \otimes \eta_1)|(\xi_2 \otimes \eta_2)\rangle = \langle T\xi_1|\xi_2\rangle \langle S\eta_1|\eta_2\rangle$$

The proposition below summarizes some basic well-known properties of tensor products of Hilbert Space operators that will be required in the sequel.

Proposition 1.1.7 *For operators $T_1, T_2, S_1, S_2 \in \mathcal{B}(\mathcal{H})$ and scalars $\rho, \rho' \in \mathbb{C}$.The following hold:*

1. $(T_1 \otimes S_1)(T_2 \otimes S_2) = (T_1T_2 \otimes S_1S_2)$.

2 $(T_1 \otimes T_2)^{\star} = T_1^{\star} \otimes T_2^{\star}$.

3 $(T_1 + T_2) \otimes (S_1 + S_2) = T_1 \otimes S_1 + T_1 \otimes S_2 + T_2 \otimes S_1 + T_2 \otimes S_2$.

4 $\rho\rho'(T_1 \otimes T_2) = \rho T_1 \otimes \rho' T_2$.

1.2 Operator Theory in Semi-Hilbertian Space

A semi-Hilbert space differs from a Hilbert space in that it need not be complete.It is simply equipped with a semi-inner product.Of certain conditions,making it a useful concept.Semetime the full structure of a Hilbert space might not be necessary or available.Our aim in this section is to survey the theory of operators on a complex Hilbert space when a semi-inner product defined by a positive operator is introduced.

1.2.1 Semi-Hilbertian Space

Definition 1.2.1 *Let $A \in \mathcal{B}(\mathcal{H})^+$ be any positive operator,then A defines the following semi-inner product:*

$$\langle .|. \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$
$$(u; v) \mapsto \langle u|v \rangle_A = \langle Au|v \rangle$$

unlike a true inner product,this semi-inner product may not be definite.This means $\langle u|u \rangle_A = 0$ can happen even if $u \neq 0$ (i.e:$\langle u|u \rangle_A = 0$ doesn't necessarily imply $u = 0$).

Remark 1.2.1 *we have*

$$\langle u|u \rangle_A^{\frac{1}{2}} = \langle Au|u \rangle^{\frac{1}{2}} = \|A^{\frac{1}{2}}u\| = \|u\|_A \text{ for all } u \in \mathcal{H}.$$

we denote $\|.\|_A$ is semi-norm induced by $\langle .|. \rangle_A$. The semi-Hilbertian space $(\mathcal{H}, \langle .|. \rangle_A)$,it is a semi norm Space $(\mathcal{H}, \|.\|_A)$.

Notice that we have

$$|\langle u|v \rangle_A| \leq \|u\|_A.\|v\|_A \text{ for all } u, v \in \mathcal{H}$$

Lemma 1.2.1 *Given $A \in \mathcal{B}(\mathcal{H})^+$.Then,*

1. *$\mathcal{N}(A^{\frac{1}{2}}) - \mathcal{N}(A)$ is equal zero.*

2 *$\mathcal{R}(A) \subset \mathcal{R}(A^{\frac{1}{2}}) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{R}(A)$ is closed precisely when $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$.*

Proof 1.2.1 (1) *If $x \in \mathcal{N}(A^{\frac{1}{2}})$,then $A^{\frac{1}{2}}x = 0$.This implies that $Ax = A^{\frac{1}{2}}A^{\frac{1}{2}}x = 0$ thus $x \in \mathcal{N}(A)$.Conversely,if $Ax = 0$,then $\| A^{\frac{1}{2}}x \|_A^2 = \langle Ax|x \rangle = 0$,so $A^{\frac{1}{2}}x = 0$*
(2) *It is obvious that $\mathcal{R}(A) \subset \mathcal{R}(A^{\frac{1}{2}})$.Moreover,since $\mathcal{N}(A^{\frac{1}{2}})$ is a closed subspace of $\mathcal{H}$,then theorem 1.1.1 implies that $\mathcal{H}$ can be decomposed into*

$$\mathcal{H} = \mathcal{N}(A^{\frac{1}{2}}) \oplus \mathcal{N}(A^{\frac{1}{2}})^\perp = \mathcal{N}(A^{\frac{1}{2}}) \oplus \overline{\mathcal{R}(A^{\frac{1}{2}})}$$

Let $y \in \mathcal{R}(A^{\frac{1}{2}})$,then there exists $x \in \mathcal{H}$ such that $y = A^{\frac{1}{2}}x$.Moreover x can be decomposed into $x = z+t$ with $(z,t) \in \mathcal{N}(A^{\frac{1}{2}}) \times \overline{\mathcal{R}(A^{\frac{1}{2}})}$.Since $t \in \overline{\mathcal{R}(A^{\frac{1}{2}})}$,then $t = lim_{n\to\infty}A^{\frac{1}{2}}\psi_n$ where $(\psi_n)_n \subset \mathcal{H}$.So $y = A^{\frac{1}{2}}t = lim_{n\to\infty}A\psi_n \in \overline{\mathcal{R}(A)}$.
(3) *If $\mathcal{R}(A)$ is closed,then it follows from the above assertion that $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$. conversely,assume that $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$ and we shall prove that $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$.It suffices to prove that $\mathcal{N}(A)^\perp \subset \mathcal{R}(A)$ because we have $\mathcal{R}(A) \subset \mathcal{N}(A)^\perp = \overline{\mathcal{R}(A)}$.*
Let $y \in \mathcal{N}(A)^\perp$,then $A^{\frac{1}{2}}y \in \mathcal{R}(A^{\frac{1}{2}})$.Since $\mathcal{R}(A) = \mathcal{R}(A^{\frac{1}{2}})$,then there exists $x \in \mathcal{H}$,such that $A^{\frac{1}{2}}y = Ax$.This implies that$A^{\frac{1}{2}}(A^{\frac{1}{2}}x - y) = 0$ i.e:$A^{\frac{1}{2}}x - y \in \mathcal{N}(A^{\frac{1}{2}})$.Moreover,since $A^{\frac{1}{2}}x \in \mathcal{R}(A^{\frac{1}{2}}) \subset \mathcal{N}(A^{\frac{1}{2}})^\perp$ and $y \in \mathcal{N}(A)^\perp = \mathcal{N}(A^{\frac{1}{2}})^\perp$,then we deduce that $A^{\frac{1}{2}}x - y \in \mathcal{N}(A^{\frac{1}{2}})^\perp$.Thus,we obtain $y = A^{\frac{1}{2}}x \in \mathcal{R}(A^{\frac{1}{2}}) = \mathcal{R}(A)$.Therefore,$\mathcal{N}(A)^\perp \subset \mathcal{R}(A)$ and so $\mathcal{R}(A)$ is closed.

Proposition 1.2.1 *The following assertions hold:*

1. A necessary and sufficient condition for $(\mathcal{H}; \|.\|_A)$ to be Norm space if A is injective operator.

2 $(\mathcal{H}; \|.\|_A)$ is complete iff $\mathcal{R}(A)$ is closed.

Proof 1.2.2 *We suffice to prove the assertions* (1) *note that for every* $x \in \mathcal{H}$.*we have,*

$$\begin{aligned} \| x \|_A = 0 &\Leftrightarrow \| Ax \| = 0 \\ &\Leftrightarrow x \in \mathcal{N}(A^{\frac{1}{2}}) \\ &\Leftrightarrow x \in \mathcal{N}(A) \end{aligned}$$

So,$\| \, . \, \|_A$ *is a norm on* $\mathcal{H}$ *if and only if* A *is an injective operator.*

The following corollary is an immediate consequence of Proposition 1.2.1.

Corollary 1.2.1 *Combining these results,*
$(\mathcal{H}, \langle .|. \rangle_A)$ *is a Hilbert space if and only if* A *is an injective positive operator with closed range.*

Not all bounded operators on $\mathcal{H}$ are well-behaved with respect to the semi-norm $\| \cdot \|_A$.We define a special class of operators that are bounded in this new sense:

$$\mathcal{B}^A(\mathcal{H}) = \{ T \ : \ \exists \, \rho > 0, \|Tv\|_A \leq \rho \|v\|_A \} \, .$$

Now,we introduce the following proposition.

Proposition 1.2.2 *Let* $T \in \mathcal{B}^A(\mathcal{H})$.

$$T(\overline{\mathcal{R}(A)}) \subset \mathcal{N}(A)$$

equivalent

$$ATA = 0.$$

Remark 1.2.2 *It's important to note that the set* $\mathcal{B}^A(\mathcal{H})$,*while consisting of operators bounded in the A-norm,doesn't form a subalgebra.This means that if you take two operators from this set,their product isn't guaranteed to also be in* $\mathcal{B}^A(\mathcal{H})$. $\mathcal{B}^A(\mathcal{H})$ *isn't a subalgebra of* $\mathcal{B}(\mathcal{H})$.

1.2.2 The pseudoinverse of Hilbert Space

The Moore-Penrose inverse (also called the pseudoinverse) of a Hilbert space operator is a generalization of the concept of the inverse of a linear operator,particularly useful when the operator is not invertible in the traditional sense.

For given operators $T, S \in \mathcal{B}(\mathcal{H})$,the reduced solution of the equation $TX = S$ can be explicitly obtained by means of the Moore-Penrose inverse of T.More details will be given in the next subsection.For this and other results concerning different generalized inverses of T and solutions of the equations $TX = S$ the reader may see ([26]).
In the following we recall the definition of the Moore-Penrose inverse of an operator $T \in \mathcal{B}(\mathcal{H})$.

Definition 1.2.2 *Let* $T \in \mathcal{B}(\mathcal{H})$.*The pseudoinverse of* T *denoted by* $T^\dagger$ *is the unique operator that satisfies these four* ***Moore-Penrose*** *equations:*

1. $T\mathcal{X}T = T$.

2 $\mathcal{X}T\mathcal{X} = \mathcal{X}$.

3 $(T\mathcal{X})^\star = T\mathcal{X}$.

4 $(\mathcal{X}T)^\star = \mathcal{X}T$.

Remark 1.2.3

1. *If $T \in \mathcal{B}(\mathcal{H})$ is an invertible operator,then its* ***Moore-Penrose*** *inverse coincides with its regular inverse: $T^{\dagger} = T^{-1}$.*

2 *When the range of $\mathcal{R}(T)$,the space $\mathcal{H}$ decomposes as direct sum between $\mathcal{R}(T)$ and $\mathcal{R}(T)^{\perp}$.However, if $\mathcal{R}(T)$ is not closed,such a direct sum decomposition fails and $T^{\dagger}$ becomes an unbounded operator. The example below illustrates.*

Example 1.2.1 (1) *Let T_r the forward shift operator on $\mathcal{H} = l^2_{\mathbb{N}^\star}(\mathbb{C})$,then one can check directly from the Moore-Penrose equations that $T_r^{\dagger} = T_r^{\star} = T_l$.Remark here that $T^{\dagger} \in \mathcal{B}(l^2_{\mathbb{N}^\star}(\mathbb{C}))$.*
(2) *Let T be the diagonal operator on $l^2_{\mathbb{N}^\star}(\mathbb{C})$ given by $Te_n = \frac{1}{n}e_n$ for all $n \in \mathbb{N}^\star$,it can be seen that*

$$T^{\dagger}e_n = ne_{n+1}$$

Observe that $\mathcal{R}(T)$ is not closed.Indeed,we have $Te_n = \frac{1}{n}e_n$ so $e_n = nTe_n \in \mathcal{R}(T)$ for all n.Thus,the range is dense in $l^2_{\mathbb{N}^\star}(\mathbb{C})$.However,$T$ is not surjective because $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ isn't belongs to $\mathcal{R}(T)$. hence,$\mathcal{R}(T)$ isn't closed.Moreover,$T^{\dagger}$ is an unbounded operator.To see this,assume otherwise.Then,there exists $\rho > 0$,

$$\|T^{\dagger}u\|_A \leq \rho\|u\|_A$$

In particular,for $u = e_n$ we get $n \leq \rho$ for all $n \in \mathbb{N}^\star$.This leads to a contradiction.
hence,$T^{\dagger}$ is unbounded.

Proposition 1.2.3 *If $T \in \mathcal{B}(\mathcal{H})$,then*

1. *If $T = T^\star$,then $(T^{\dagger})^* = T^{\dagger}$.*

2 *If T is a positive,then its Moore-Penrose inverse can be related to the square root of T:*

$$T^{\dagger} = (T^{1/2})^{\dagger}(T^{1/2})^{\dagger}.$$

It should be noted that the calculus of $T^{\dagger}$ is not trivial in general even if T is a matrix.As we will see later,we will only compute $T^{\dagger}$ when T is a matrix.So,before we move on let us state the following remark which will be very useful in the calculation of $T^{\dagger}$.

Remark 1.2.4 *this remark provides practical formulas for calculating the Moore-Penrose inverse: Let T be an $m \times n$ matrix whose entries come from the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.Then,*

$$T^{\dagger} = \lim_{t \to 0^+} T^\star(TT^\star + tI_m)^{-1}$$

and

$$T^{\dagger} = \lim_{t \to 0^+} (TT^\star + tI_n)^{-1}T^\star.$$

here,I_n and I_m are identity matrices of size n and m,respectively.These formulas are very useful for theoretical analysis.
(3)*If a matrix T is self-adjoint ($T = T^\star$),the formulas simplify further:*

$$T^{\dagger} = \lim_{t \to 0^+} (T^2 + tI)^{-1}T = \lim_{t \to 0^+} T(T^2 + tI)^{-1}.$$

where I denotes the identity matrix.

1.2.3 A-adjoint Operators in Semi-Hilbertian Spaces

The concept of an adjoint operator arises naturally in the context of functional analysis and operator theory.In a semi-Hilbert space (or more generally),which is a vector space endowed with an inner product but possibly not complete,the adjoint of an operator in a semi-Hilbert space is defined similarly to how it is defined in a Hilbert space.This subsection presents key results regarding A-adjoint operators for use in later sections.First,we state the next theorem of R.G.Douglas which plays a key role in the sequel([26]).For its proof one can see ([26],[16],[34]).

Definition 1.2.3 *For a bounded operator $T : \mathcal{H} \to \mathcal{H}$.The operator $S \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if*

$$\langle Tu|v\rangle_A = \langle u|Sv\rangle_A \quad \forall u, v \in \mathcal{H}.$$

equivalently,

$$\langle ATu|v\rangle = \langle Au|Sv\rangle \quad \forall u, v \in \mathcal{H}.$$

thus,

$$T^{\star}A = AS.$$

1.2.4 Existence of A-adjoint in Semi-Hilbertian Space

we need the next theorem of R.G.Douglas which plays a key role in the sequel.

Theorem 1.2.1 *Let T_1 and T_2 be bounded linear operators on a Hilbert space $\mathcal{H}$.The following conditions are equivalent:*

1. $\mathcal{R}(T_2) \subset \mathcal{R}(T_1)$.

2 *There exists a bounded linear operator $X : \mathcal{H} \to \mathcal{H}$ such that:*

$$T_1 = T_2X.$$

3 *There exists a positive number $\rho > 0$ such that:*

$$T_2T_2^{\star} \leq \rho T_1T_1^{\star}$$

If one of these conditions holds,then there exists a unique operator $S \in \mathcal{B}(\mathcal{H})$ such that
(i) $T_1S = T_2$.
(ii) $\mathcal{R}(S) \subset \overline{\mathcal{R}(T_1^{\star})} = \mathcal{N}(T_1)^{\perp}$.
Furthermore,$\mathcal{N}(S) = \mathcal{N}(T_2)$
S is called the reduced solution or Douglas's solution of the equation $TX = S$

Proposition 1.2.4 *Let T_1 and T_2 be bounded linear operators.If $T_1X = T_2$ has solution.Then,the reduced solution is given by $T_1^{\dagger}T_2$.*

Proof 1.2.3 *Since $T_1X = T_2$ has solution,then Douglas theorem implies that $\mathcal{R}(T_2) \subset \mathcal{R}(T_1)$ which in turn yields $\mathcal{R}(T_2)$ lies inside domain of$T_1^{\dagger}$*
thus,gives $T^{\dagger}T' \in \mathcal{B}(\mathcal{H})$
Further,

$$T_1(T_1^{\dagger}T_2) = T_2 \quad \wedge \quad \mathcal{R}(T_1^{\dagger}T_2) \subset \overline{\mathcal{R}(T_1^{\star})}$$

therefore,the reduced solution of $T_1X = T_2$ is given by $T_1^{\dagger}T_2 = \mathcal{X}$.

According to Douglas's theorem,$T \in \mathcal{B}(\mathcal{H})$ has an $A-$adjoint iff $\mathcal{R}(T^{\star}A) \subset \mathcal{R}(A))$.Some operators have no $A-$adjoint,and when one exists,it is generally not unique.Of an A-adjoint operator is guaranteed.In fact,an operator $T \in \mathcal{B}(\mathcal{H})$ may admit none,one or many A-adjoints.
From now,we denote the set of all bounded operators that admit at least one $A-$adjoint by:

$$\mathcal{B}_A(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \ : \ \mathcal{R}(T^{*}A) \subset \mathcal{R}(A)\}$$

We would like to emphasize that $T \in \mathcal{B}_A(\mathcal{H})$, then $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$.

Definition 1.2.4 *If $T \in \mathcal{B}_A(\mathcal{H})$,then the distinguished A-adjoint of T,is denoted by $T^{\sharp}$ and given by $T^{\sharp} = A^{\dagger}T^{\star}A$.Further,properties of $T^{\sharp}$ are:*

1. *$AT^{\sharp} = T^{\star}A$.*

2 *its range is contained in $\overline{\mathcal{R}(A)}$.*

3 *its null space coincides with $\mathcal{N}(T^{\star}A)$.*

Remark 1.2.5 **1.** *we infer that $T^{\sharp} \in \mathcal{B}(\mathcal{H})$.*

2 *Let $\mathcal{K}$ is an A-adjoint of T.Then,one may observe that $\mathcal{K} = T^{\sharp} + \mathcal{R}$,with $\mathcal{AR} = 0$.*

3 *if A is an injective operator,then T has a unique A-adjoint.*

Proposition 1.2.5 *Let $T \in \mathcal{B}_A(\mathcal{H})$.Then,*

1. *$I^{\sharp} = P$ and $(T^n)^{\sharp} = (T^{\sharp})^n$ for any natural number n.*

2 *$T^{\sharp} \in \mathcal{B}_A(\mathcal{H})$,$(T^{\sharp})^{\sharp} = PTP$ and $((T^{\sharp})^{\sharp})^{\sharp} = T^{\sharp}$.*

3 *If Tcommutes with A,then $T^{\sharp} = P_{\overline{\mathcal{R}(A)}}T^{\star}$.*

4 *If $\mathcal{R} \in \mathcal{B}_A(H)$ then $T\mathcal{R} \in \mathcal{B}_A(\mathcal{H})$ and $(T\mathcal{R})^{\sharp} = \mathcal{R}^{\sharp}T^{\sharp}$.*

1.2.5 $A-$bounded operators

The study of semi-Hilbertian spaces,concept of $A-$bounded operators extends traditional operator theory to accommodate a modified inner product structure.
In this subsection we investigate the set of all A-bounded operators in $\mathcal{B}(\mathcal{H})$.

Definition 1.2.5 *An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be A-bounded operators when one can find a constant $\mu \geq 0$.Such that,*

$$\| Tu \|_A \leq \mu \| u \|_A$$

the set of all A-bounded operators denoted by $\mathcal{B}^A(\mathcal{H})$,i.e:

$$\mathcal{B}^A(\mathcal{H}) = \Big\{T \in \mathcal{B}(\mathcal{H}); \exists \mu > 0 | \, \| Tu \|_A \leq \mu \| u \|_A, \forall u \in \mathcal{H}\Big\}$$

It can be seen that $\mathcal{B}^A(\mathcal{H})$ isn't a subalgabra of $\mathcal{B}(\mathcal{H})$.

Proposition 1.2.6 *Let A bounded operators and T bounded operators.Then the following conditions are equivalent:*

1. $T \in \mathcal{B}^A(\mathcal{H})$.

2 $A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}} \in \mathcal{B}(\mathcal{H})$.

3 $R(A^{\frac{1}{2}}TA^{\frac{1}{2}}) \subset R(A)$.

Proof 1.2.4 $1 \Rightarrow 2$ *If* $T \in \mathcal{B}^A(\mathcal{H})$ *then there exists* $c > 0$ *such that* $\|Tv\|_A \leq c\|v\|_A$ *for every* $v \in \overline{R(A)}$. *Then,*

$$\left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}u\right\| = \left\|TA^{\dagger\frac{1}{2}}u\right\|_A \leq \|T\|_A \left\|A^{\dagger\frac{1}{2}}u\right\|_A \leq \|T\|_A\|u\|$$

therefore,$A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}$ *is bounded and* $\left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}\right\| \leq \|T\|_A$.

$2 \Rightarrow 1$ *Let* $A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}$ *be a bounded linear operator. Then,for every* $u \in \overline{R(A)}$ *we have that*

$$\begin{aligned}\|Tu\|_A = \left\|TP_{\overline{R(A)}}u\right\|_A &= \left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}A^{\frac{1}{2}}u\right\| \\ &\leq \left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}\right\| \left\|A^{\frac{1}{2}}u\right\| \\ &= \left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}\right\| \|u\|_A\end{aligned}$$

Moreover,$\|T\|_A \leq \left\|A^{\frac{1}{2}}TA^{\frac{1}{2}}\right\|$.

$2 \Rightarrow 3$ *Let* $A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}$ *be a bounded linear operator. Then,for every* $u \in \overline{R(A)}$ *we have that*

$$\left\|A^{\frac{1}{2}}TA^{\frac{1}{2}}u\right\| = \left\|TP_{R(A)}A^{\frac{1}{2}}u\right\|_A = \left\|TA^{\dagger\frac{1}{2}}Au\right\|_A = \left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}Au\right\| \leq \mu\left\|Au\right\|$$

by Douglas theorem this is equivalent to $R\left(A^{\frac{1}{2}}T^*A^{\frac{1}{2}}\right) \subseteq R(A)$.

$3 \Rightarrow 1$ *we prove that exist* μ *such that* $\|Tu\|_A \leq \mu\|u\|_A$ *for all* $u \in R(T)$

$$\begin{aligned}\|Tu\|_A &= \left\|TA^{\dagger\frac{1}{2}}A^{\frac{1}{2}}u\right\|_A \\ &= \left\|A^{\frac{1}{2}}TA^{\dagger\frac{1}{2}}A^{\frac{1}{2}}u\right\| \\ &\leq \mu\left\|A^{\frac{1}{2}}u\right\| = \mu\left\|u\right\|_A\end{aligned}$$

then $T \in \mathcal{B}^A(\mathcal{H})$.

By proposition if $A \in \mathcal{B}(\mathcal{H})^+$ *has closed range then* $\mathcal{B}^A(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ *because* $A^{\dagger\frac{1}{2}}$ *is bounded. But,as the next example shows,if* A *has not closed range then* $\mathcal{B}^A(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H})$.

More precisely,a straightforward application of Douglas theorem gives

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) = \left\{T; T^\star\mathcal{R}(A^{\frac{1}{2}}) \subset \mathcal{R}(A^{\frac{1}{2}})\right\}$$

If $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$,we say that T is A-bounded.Note that like $\mathcal{B}_A(\mathcal{H})$,the subspace $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is neither closed nor dense in $\mathcal{B}(\mathcal{H})$. before proceeding any further,it is necessary to point out the fact that,if $T \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$,then $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$.However,the inclusion $T^\star(\mathcal{N}(A)) \subset \mathcal{N}(A)$ need not hold in general for A-bounded operator.Moreover,it is easy to see that $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H})$.In general,this inclusion is strict,as illustrated below.

Example 1.2.2 *Let* $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$ *and* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$*.For every* $\mathcal{Z} = (z_1, z_2) \in \mathbb{C}^2$ *we have,*

$$\langle A\mathcal{Z}|\mathcal{Z}\rangle = |z_1 + z_2|^2 \geq 0$$

then A is positive operator. notice that subspaces with finite dimensions are always closed.Further,

$$\mathcal{R}(A) = \{(z, z), z \in \mathbb{C}\}$$

To check if T *is* $A-$*bounded,we see how it acts on a general vector* $\widetilde{\mathcal{Z}} = (z, z)$ *in* $\mathcal{R}(A)$*:*

$$\|T\widetilde{\mathcal{Z}}\|_A = \|(0, z)\|_A = |z| \quad and \quad \|\widetilde{\mathcal{Z}}\|_A = \sqrt{2}|z|$$

So,

$$\|T\widetilde{\mathcal{Z}}\|_A \leq \frac{1}{\sqrt{2}}\|\widetilde{\mathcal{Z}}\|_A$$

this confirms T *is* $A-$*bounded.*

in addition,$(1, -1) \in \mathcal{N}(A)$ *then,*

$$T(1, -1) = (0; -1) \not\subseteq \mathcal{N}(A)$$

this yields that T *is not in* $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$*.*
Consequently,we infer that $T \in \mathcal{B}^A(\mathcal{H}) \mid \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$*.*

Remark 1.2.6 (1) *If A is an injective operator,then obviously we have* $\mathcal{B}^A(\mathcal{H}) = \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$*.*
(2) *Since* $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H})$*,then the inclusions* $\mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H})$ *hold.Observe that these inclusions assure that* $\| \,.\, \|_A$ *is finite for every* T *which admits an A-adjoint.*

Proof 1.2.5 *Clearly we have* $\mathcal{B}_A(\mathcal{H}) \subset \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subset \mathcal{B}^A(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$*.Moreover since A has closed range,then* $\mathcal{B}_A(\mathcal{H}) = \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ *and* $\mathcal{B}^A(\mathcal{H}) = \mathcal{B}(\mathcal{H})$*.On other hand,A is injective implies that* $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) = \mathcal{B}^A(\mathcal{H})$*.*
The proof is complete.

1.3 Some classes of operators in semi-Hilbertian space

Recently,the study of operators in semi-Hilbertian space received important attention.In particular,several results concerning various classes of operators on a complex Hilbert space were extended to the general frame of semi-Hilbertian spaces.One may see for example and their references.In this section we describe some classes of operators which are bounded,normal,self-adjoint,unitary,Hyponormal and isometry with respect to $\langle .|.\rangle_A$.These classes share many properties with their classical analogues,notably in their behavior.It should be noticed that the extension is not trivial because the existence of an adjoint for $\langle .|.\rangle_A$ is not always guaranteed(see [16],[21],[23],[24],[26],[34],[39]).

1.3.1 $A-$normal operators

This section introduces a generalization of normal operators for semi-Hilbertian space.The concept of A-normal operators was introduced by Arias et al as follows.

Definition 1.3.1 *We say that an operator $T \in \mathcal{B}_A(\mathcal{H})$ is A-normal when it commutes with its $A-$adjoint i,e:*

$$(TT^\sharp - T^\sharp T)v = 0, \quad \forall v \in \mathcal{H}.$$

A-normal operators may be regarded a generalization of normal operators.

Example 1.3.1 *Let* $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$,*and* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)^+$
then $AT = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A$ *and* $A^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$
this implies that $T^\sharp = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}A$
therefore

$$\begin{aligned} \|Tu\|_A &= \langle ATu|Tu\rangle \\ &= \langle Au|Tu\rangle \\ &= \langle u|ATu\rangle \\ &= \langle u|Au\rangle \\ &= \|u\|_A. \end{aligned}$$

and

$$\begin{aligned} \|T^\sharp u\|_A &= \left\langle AT^\sharp u|T^\sharp u\right\rangle \\ &= \left\langle \frac{1}{4}A^3 u|u\right\rangle \\ &= \langle Au|u\rangle \\ &= \|u\|_A. \end{aligned}$$

then

$$\|Tu\|_A = \|T^\sharp u\|_A$$

alternatively,

$$T^\sharp T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and

$$TT^\sharp = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

hence

$$TT^\sharp \neq T^\sharp T$$

this example is crucial.It demonstrates that the property $\|Tu\|_A = \|T^\sharp u\|_A$ is necessary but not sufficient for an operator to be $A-$normal.

An important characterization of $A-$normal operators that will be used in the sequel reads as follows.

Theorem 1.3.1 *For an operator $T \in \mathcal{B}_A(\mathcal{H})$. the following are equivalent:*

1. *T is A-normal.*

2 $\|Tu\|_A = \|T^\sharp u\|_A$ *and* $\mathcal{R}\left(TT^\sharp\right) \subset \overline{\mathcal{R}(A)}$.

Proof 1.3.1 *Assume that T is a A-normal.This implies that $\mathcal{R}\left(TT^\sharp\right) \subset \mathcal{R}\left(T^\sharp T\right) \subset \overline{\mathcal{R}(A)}$ since $\mathcal{R}\left(T^\sharp\right) \subset \overline{\mathcal{R}(A)}$.*
Furthermore,

$$\begin{aligned}
T^\sharp Tu = TT^\sharp u &\Rightarrow \left\langle AT^\sharp Tu|u\right\rangle = \left\langle ATT^\sharp u|u\right\rangle \\
&\Rightarrow \langle T^* ATu|u\rangle = \left\langle TT^\sharp u|Au\right\rangle \\
&\Rightarrow \langle T^* ATu|u\rangle = \left\langle T^\sharp u|T^\star Au\right\rangle \\
&\Rightarrow \langle ATu|Tu\rangle = \left\langle T^\sharp u|AT^\sharp u\right\rangle \\
&\Rightarrow \langle Tu|Tu\rangle = \left\langle T^\sharp u|T^\sharp u\right\rangle_A \\
&\Rightarrow \|Tu\|_A = \left\|T^\sharp u\right\|_A
\end{aligned}$$

Assume that $\|Tu\|_A = \left\|T^\sharp u\right\|_A$ for every $u \in \mathcal{H}$ and $\mathcal{R}\left(TT^\sharp\right) \subset \overline{\mathcal{R}(A)}$ then

$$\begin{aligned}
\|Tu\|_A = \left\|T^\sharp u\right\|_A &\Rightarrow \left\langle AT^\sharp Tu|u\right\rangle = \left\langle AT^\sharp Tu|u\right\rangle \\
&\Rightarrow \left\langle A\left(T^\sharp T - TT^\sharp\right)u|u\right\rangle = 0 \\
&\Rightarrow A\left(T^\sharp T - TT^\sharp\right)u = 0
\end{aligned}$$

therefore

$$\mathcal{R}\left(T^\sharp T - TT^\sharp\right) \subset \mathcal{N}(A)$$

and other hand,since $\mathcal{R}\left(TT^\sharp\right) \subset \overline{\mathcal{R}(A)}$ and $\mathcal{R}\left(T^\sharp T\right) \subset \overline{\mathcal{R}(A)}$,hence

$$\mathcal{R}\left(T^\sharp T - TT^\sharp\right) \subset \overline{\mathcal{R}(A)} = \mathcal{N}(A)^\perp$$

hence $T^\sharp T = TT^\sharp$.

Remark 1.3.1 *The condition $T^\sharp = T^\star$reduces the notion of an A-normal operator to that of a classical normal operator.This occurs specifically if A is the identity operator or if T and A commute and A has dense range.*

Proposition 1.3.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$ is an A-normal operator,the following conditions holds:*

1. *$T^\sharp$ and $T - \lambda$ are Class A-normal operator.*

2 *T^k is Class A-normal operator.*

Proposition 1.3.2 *if $T_{1,2}$ are two A-normal for which $T_2T_1^\sharp - T_1^\sharp T_2 = 0$.Then,* *1. $T_1 + T_2$ is Class A-normal operator.*

2 T_1T_2 is Class A-normal operator.

Proof 1.3.2 *Let T_1 and T_2 are A-normal operator such that $T_2T_1^\sharp = T^\sharp T_2$*

$$\begin{aligned}(T_1+T_2)^\sharp(T_1+T_2)u =&(T_1^\sharp+T_2^\sharp)(T_1+T_2)u\\ =&T_1^\sharp T_1u+T_1^\sharp T_2u+T_2^\sharp T_1u+T_2^\sharp T_2u\\ =&T_1T_1^\sharp u+T_1^\sharp T_2u+T_2^\sharp T_1u+T_2T_2^\sharp u \quad (T_1 \text{ and } T_2 \text{ are } A\text{-normal})\\ =&T_1T_1^\sharp u+T_2T_1^\sharp u+T_1T_2^\sharp u+T_2T_2^\sharp u\\ =&(T_1+T_2)(T_1^\sharp+T_2^\sharp)u\\ =&(T_1+T_2)(T_1+T_2)^\sharp u\end{aligned}$$

then $T_1 + T_2$ is A-normal operator.

$$\begin{aligned}\langle (T_1T_2)^\sharp(T_1T_2)u|u\rangle_A =&\langle (T_2^\sharp T_1^\sharp T_1T_2u|u\rangle_A\\ =&\langle T_1^\sharp T_1T_2u|T_2u\rangle_A\\ =&\langle T_1T_1^\sharp T_2u|T_2u\rangle_A \quad (T_1 \text{ is } A\text{-normal})\\ =&\langle T_1T_2^\sharp T_2T_1^\sharp u|u\rangle_A\\ =&\langle T_2^\sharp T_2T_1^\sharp u|T_1^\sharp u\rangle_A\\ =&\langle T_2T_2^\sharp T_1^\sharp u|T_1^\sharp u\rangle_A \quad (T_2 \text{ is } A\text{-normal})\\ =&\langle (T_1T_2)(T_1T_2)^\sharp u|u\rangle_A\end{aligned}$$

then T_1T_2 is A-normal operator.

1.3.2 $A-$selfadjoint operators

In semi-Hilbertian spaces,the concept of A-selfadjoint operators extends the classical notion of selfadjoint (or Hermitian) operators to accommodate a modified inner product structure.

Definition 1.3.2 *An operator $T \in \mathcal{B}(\mathcal{H})$ is called A-selfadjoint if*

$$\langle Tv|v\rangle_A = \langle v|Tv\rangle_A$$

equivalent

$$AT = (AT)^\star$$

the operator AT is selfadjoint.

Remark 1.3.2

1. AT is a selfadjoint.

2 In general,if T is an A-self adjoint doesn't $T = T^\sharp$ and it isn't necessarily A-normal.

In the following example,we give an operator T which is A-self adjoint,but it is neither equal to $T^\sharp$ nor A-normal.

Example 1.3.2 *Consider the operators*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}^{+}(\mathbb{C}^2) \quad \wedge \quad T = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}_A(\mathbb{C}^2)$$

a direct computation gives

$$AT = T^{\star}A$$

which shows that T is A-selfadjoint.
and

$$T^{\sharp} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

accordingly $T^{\sharp} \neq T$.
also,a short calculation shows that

$$TT^{\sharp} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 0 & 0 \end{pmatrix} \quad \wedge \quad T^{\sharp}T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

then $T^{\sharp}T \neq TT^{\sharp}$.

The next theorem gives the precise condition under which the stronger identity $T = T^{\sharp}$ holds for an $A-$selfadjoint operator.

Theorem 1.3.2 *Let $T \in \mathcal{B}_A(\mathcal{H})$.Then,*

T is A-selfadjoint and $\mathcal{R}(T) \subset \overline{\mathcal{R}(A)}$ if only if $T^{\sharp} = T$.

Proof 1.3.3 $(2) \Rightarrow (1)$ *: Assume that T is A-selfadjoint such that $\mathcal{R}(T) \subset \overline{\mathcal{R}(A)}$ then*

$$\langle A(T - T^{\sharp})u|v\rangle = \langle (T^{\star}A - T^{\star}A)u|v\rangle = 0 \quad \forall u, v \in \mathcal{H}.$$

thus,$A(T - T^{\sharp}) = 0$ impel that

$$\mathcal{R}(T - T^{\sharp}) \subset \mathcal{N}(A) \tag{1.2}$$

given that $\mathcal{R}(T^{\sharp})$ and $\mathcal{R}(T)$ are both contained $\overline{\mathcal{R}(A)}$ it follows that

$$\mathcal{R}(T - T^{\sharp}) \subset \overline{\mathcal{R}(A)} = \mathcal{N}(A)^{\perp}$$

accordingly,

$$T^{\sharp} = T$$

$(1) \Rightarrow (2)$*If $T^{\sharp} = T$,we get $\mathcal{R}(T) = \mathcal{R}(T^{\sharp}) \subset \overline{\mathcal{R}(A)}$.*
Furthermore,the defining property of $T^{\sharp}$ is $AT^{\sharp} = T^{*}A$.Substituting $T^{\sharp} = T$ into this gives $AT = T^{*}A$,which is the definition of an $A-$selfadjoint operator.

1.3.3 $A-$isometry operators

Definition 1.3.3 *Let $A \in \mathcal{B}(\mathcal{H})^+$,an operator $T \in \mathcal{B}(\mathcal{H})$is called an A-isometry if*

$$\|Tu\|_A = \|u\|_A, \quad \forall u \in \mathcal{H}$$

or equivalently

$$T^*AT = A.$$

Proposition 1.3.3 *Let T_1 and T_2 be two an A-isometry operators*

1. $T_2^\sharp T_2 = T_1^\sharp T_1 = P$.

2 $N_A(T_1) = N_A(T_2)$.

3 Both $T_1 + T_2$ and T_1T_2 remain A-isometry.

Proof 1.3.4 *Let $T \in \mathcal{B}_A(\mathcal{H})$ then $T^\sharp \in \mathcal{B}_A(\mathcal{H})$ and T is A-unitary.We have*

$$T^\sharp T = (T^\sharp)^\sharp T^\sharp = P_{\overline{R(A)}}$$

then

$$\begin{aligned}
T^\sharp T = (T^\sharp)^\sharp T^\sharp = P_{\overline{R(A)}} &\Leftrightarrow \langle T^\sharp Tu|u\rangle_A = \langle (T^\sharp)^\sharp T^\sharp u|u\rangle_A = \langle P_{\overline{R(A)}}u|u\rangle_A \\
&\Leftrightarrow \langle AT^\sharp Tu|u\rangle = \langle A(T^\sharp)^\sharp T^\sharp u|u\rangle = \langle AP_{\overline{R(A)}}u|u\rangle \\
&\Leftrightarrow \langle T^*ATu|u\rangle = \langle (T^\sharp)^* AT^\sharp u|u\rangle = \langle Au|u\rangle \\
&\Leftrightarrow \langle ATu|Tu\rangle = \langle AT^\sharp u|T^\sharp u\rangle = \langle Au|u\rangle \\
&\Leftrightarrow \|Tu\|_A = \left\|T^\sharp u\right\|_A = \|u\|_A
\end{aligned}$$

1.3.4 $A-$unitary operators

Definition 1.3.4 *An operator T is called an A-unitary if it satisfies:*

$$T^\sharp T = \left(T^\sharp\right)^\sharp T^\sharp = P.$$

A-unitary operators may be regarded as a generalization of unitary operators in which $T^\sharp = T^*$.

Theorem 1.3.3 *Let $T \in \mathcal{B}_A(\mathcal{H})$.Then*

T is an A-unitary operator if only if $\|Tu\|_A = \|T^\sharp u\|_A = \|u\|_A$ for all $u \in \mathcal{H}$.

Proof 1.3.5 *Let $T \in \mathcal{B}_A(\mathcal{H})$. then,*

$$\begin{aligned}
T^\sharp T = (T^\sharp)^\sharp T^\sharp = P &\Leftrightarrow \langle T^\sharp Tu|u\rangle_A = \langle (T^\sharp)^\sharp T^\sharp u|u\rangle_A = \langle Pu|u\rangle_A \\
&\Leftrightarrow \langle ATu|Tu\rangle = \langle AT^\sharp u|T^\sharp u\rangle = \langle Au|u\rangle \\
&\Leftrightarrow \|Tu\|_A = \|T^\sharp u\|_A = \|u\|_A
\end{aligned}$$

Example 1.3.3 *Let* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *and* $T = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ *be operators acting on* $\mathbb{C}^2$ *and let* $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{C}^2$*.One can verify that*

$$T^\sharp = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and other hand

$$\begin{aligned}
\| Tu \|_A^2 &= \langle ATu | Tu \rangle \\
&= \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \end{pmatrix} | \begin{pmatrix} u_1 - u_2 \\ 2u_2 \end{pmatrix} \rangle \\
&= |u_1 + u_2|^2
\end{aligned}$$

and

$$\begin{aligned}
\| T^\sharp u \|_A^2 &= \langle AT^\sharp u | T^\sharp u \rangle \\
&= \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \end{pmatrix} | \begin{pmatrix} \frac{u_1+u_2}{2} \\ \frac{u_1+u_2}{2} \end{pmatrix} \rangle \\
&= |u_1 + u_2|^2
\end{aligned}$$

and

$$\begin{aligned}
\| u \|_A^2 &= \langle Au | u \rangle \\
&= \langle \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} | \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \end{pmatrix} | \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rangle \\
&= |u_1 + u_2|^2
\end{aligned}$$

hence,$\| Tu \|_A = \| T^\sharp u \|_A = \| u \|_A$.
therefore T *is A-unitary.*

In the following proposition,we sum up some basic properties of A-unitary operators.

Proposition 1.3.4 *if* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are an A-unitary operators.Then,*

1. $\mathcal{U}_1^\sharp$ *and* $\mathcal{U}_2^\sharp$ *are A-unitary.*

2 $\mathcal{U}_1\mathcal{U}_2$ *is A-unitary.*

Proof 1.3.6 *Let* $U_1, U_2 \in \mathcal{B}_A(\mathcal{H})$ *be A-unitary operators.Then,*

$$U_1^\sharp U_1 = \left(U_1^\sharp\right)^\sharp U_1^\sharp = P_{\overline{R(A)}} \quad \text{and} \quad U_2^\sharp U_2 = \left(U_2^\sharp\right)^\sharp U_2^\sharp = P_{\overline{R(A)}}$$

1. *By using the fact that U_1 is A-unitary. We see that,*

$$\left(U_1^\sharp\right)^\sharp U_1^\sharp = P_{\overline{R(A)}}$$

and

$$\left[\left(U_1^\sharp\right)^\sharp\right]^\sharp (U_1^\sharp)^\sharp = \left[\left(U_1^\sharp U_1\right)^\sharp\right]^\sharp = \left(P^\sharp_{\overline{R(A)}}\right)^\sharp = P_{\overline{R(A)}}$$

therefore $U_1^\sharp$ is A-unitary.

2 *Since U_1 and U_2 are A-unitary,then for all $v \in \mathcal{H}$ we have*

$$\|U_1U_2v\|_A = \|U_2v\|_A = \|v\|_A$$

Furthermore,since U_1 and U_2 are A-unitary,then $U_1^\sharp$ and $U_2^\sharp$ are also A-unitary.So,for all $v \in \mathcal{H}$ we see that

$$\left\|(U_1U_2)^\sharp v\right\|_A = \left\|U_2^\sharp U_1^\sharp v\right\|_A = \left\|U_1^\sharp v\right\|_A = \|v\|_A.$$

Consequently,it follows directly from the definition of A-unitary operators that U_1U_2 is A-unitary as desired.

1.3.5 $A-$Hyponormal operators

In the following we introduce the notion of A-Hyponormal operators.

Definition 1.3.5 *An operator $T \in \mathcal{B}_A(\mathcal{H})$ is called A-Hyponormal if it satisfies*

$$\langle (T^\sharp T - T^\sharp T)v|v\rangle_A \geq 0 \quad \forall v \in \mathcal{H}.$$

equivalently

$$T^\sharp T \geq_A TT^\sharp.$$

A-Hyponormal operators may be regarded as a generalization of Hyponormal operators in which $T^\sharp = T^*$.This last property is realized in particular if $A = I$ or if T and A commute and A has a dense range.

Theorem 1.3.4 *Let $T \in \mathcal{B}_A(\mathcal{H})$.The following conditions are equivalent :*

1. *T is A-Hyponormal operator.*

2 $\|Tv\|_A \geq \|T^\sharp v\|_A$.

Proof 1.3.7 *1⇒ 2 : Assume that T is an A-Hyponormal.Then,*

$$\begin{aligned} T^\sharp T \geq_A TT^\sharp &\Rightarrow \left\langle AT^\sharp Tu|u\right\rangle \geq \left\langle TT^\sharp u|Au\right\rangle \\ &\Rightarrow \langle T^*ATu|u\rangle \geq \left\langle T^\sharp u|T^*Au\right\rangle \\ &\Rightarrow \langle ATu|Tu\rangle \geq \left\langle T^\sharp u|AT^\sharp u\right\rangle \\ &\Rightarrow \|Tu\|_A \geq \left\|T^\sharp u\right\|_A. \end{aligned}$$

(2) $\Rightarrow$ *(1) suposse that* $\|Tu\|_A = \left\|T^\sharp u\right\|_A$ *for every* $u \in \mathcal{H}$,*we have*

$$\begin{aligned}\|Tu\|_A \geq \left\|T^\sharp u\right\|_A &\Rightarrow \langle ATu|Tu\rangle \geq \left\langle T^\sharp u|AT^\sharp u\right\rangle \\ &\Rightarrow \langle T^*ATu|u\rangle \geq \left\langle T^\sharp u|T^*Au\right\rangle \\ &\Rightarrow \left\langle AT^\sharp Tu|u\right\rangle \geq \left\langle ATT^\sharp u|u\right\rangle \\ &\Rightarrow T^\sharp T \geq_A TT^\sharp.\end{aligned}$$

Proof is complete.

Proposition 1.3.5 *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$ *are A-Hyponormal operator such that* $T_2T_1^\sharp = T_1^\sharp T_2$ *and* $T_1T_2^\sharp = T_2^\sharp T_1$,*then* $T_1 + T_2$ *is an A-Hyponormal operator.*

Proof 1.3.8 *Let* T_1 *and* T_2 *are A-Hyponormal.Then,*

$$\begin{aligned}(T_1+T_2)^\sharp(T_1+T_2) =&(T_1^\sharp + T_2^\sharp)(T_1+T_2) \\ =&T_1^\sharp T_1 + T_1^\sharp T_2 + T_2^\sharp T_1 + T_2^\sharp T_2 \\ \geq&T_1T_1^\sharp + T_1^\sharp T_2 + T_2^\sharp T_1 + T_2T_2^\sharp \quad (T_1 \text{ and } T_2 \text{ are A-Hyponormal}) \\ =&T_1T_1^\sharp + T_2T_1^\sharp + T_1T_2^\sharp + T_2T_2^\sharp \\ =&(T_1+T_2)(T_1^\sharp + T_2^\sharp)\end{aligned}$$

then $T_1 + T_2$ *is A-Hyponormal operator.*

Proposition 1.3.6 *Let* $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$ *are an A-Hyponormal operator such that* $T_2T_1^\sharp = T_1^\sharp T_2$ *and* $T_1T_2^\sharp = T_2^\sharp T_1$,*then* T_1T_2 *is Class A-Hyponormal operator.*

Proof 1.3.9 *Let* T_1 *and* T_2 *are an A-Hyponormal*

$$\begin{aligned}(T_1T_2)^\sharp(T_1T_2) =&T_2^\sharp T_1^\sharp T_1T_2 \\ \geq&T_2^\sharp T_1T_1^\sharp T_2 \quad (T_1 \text{ is A-Hyponormal}) \\ =&T_1T_2^\sharp T_2T_1^\sharp \\ \geq&T_1T_2T_2^\sharp T_1^\sharp \quad (T_2 \text{ is A-Hyponormal}) \\ =&(T_1T_2)(T_1T_2)^\sharp\end{aligned}$$

then T_1T_2 *is A-Hyponormal operator.*

1.3.6 Class $A^\sharp$ operators

Definition 1.3.6 *We call an operator* $T \in \mathcal{B}_A(\mathcal{H})$ *Class* $A^\sharp$ *if*

$$|T^2|_A \geq_A |T^\sharp|_A^2$$

i,e:

$$T^{2\sharp}T^2 \geq_A (TT^\sharp)^2$$

Theorem 1.3.5 *Let $T \in \mathcal{B}_A(\mathcal{H})$. Then,*

$T \in \mathcal{B}(\mathcal{H})$ *is Class* $A^\sharp$ *operator if only if* $\|T^2 v\|_A \geq \|TT^\sharp v\|_A$ *for all* $u \in \mathcal{H}$.

Proof 1.3.10 *Let $T \in \mathcal{B}_A(\mathcal{H})$ then $T^\sharp \in \mathcal{B}_A(\mathcal{H})$. Assume that T is a Class $A^\sharp$. Then,*

$$\begin{aligned}
T^{2\sharp}T^2 \geq_A (TT^\sharp)^2 &\Leftrightarrow \left\langle AT^{2\sharp}T^2u|u\right\rangle \geq \left\langle (TT^\sharp)^2u|Au\right\rangle \\
&\Leftrightarrow \langle T^{*2}AT^2u|u\rangle \geq \left\langle TT^\sharp u|\left(T^\sharp\right)^* T^*Au\right\rangle \\
&\Leftrightarrow \langle AT^2u|T^2u\rangle \geq \left\langle TT^\sharp u|\left(T^\sharp\right)^* AT^\sharp u\right\rangle \\
&\Leftrightarrow \langle AT^2u|T^2u\rangle \geq \left\langle TT^\sharp u|A\left(T^\sharp\right)^\sharp T^\sharp u\right\rangle \\
&\Leftrightarrow \langle AT^2u|T^2u\rangle \geq \left\langle TT^\sharp u|AP_{\overline{R(A)}}TP_{\overline{R(A)}}T^\sharp u\right\rangle \\
&\Leftrightarrow \langle AT^2u|T^2u\rangle \geq \left\langle TT^\sharp u|AP_{\overline{R(A)}}TT^\sharp u\right\rangle \\
&\Leftrightarrow \langle AT^2u|T^2u\rangle \geq \left\langle TT^\sharp u|ATT^\sharp u\right\rangle \\
&\Leftrightarrow \|T^2u\|_A \geq \left\|TT^\sharp u\right\|_A
\end{aligned}$$

Proposition 1.3.7 *Let $T \in \mathcal{B}_A(\mathcal{H})$. Here are some of the following results:*

1. In the case where A is the identity operator, Class $A^\sharp$ reduces to $\star$-Class A.

2 A-normal or A-Hyponormal both imply in Class $A^\sharp$.

3 For T a Class $A^\sharp$ satisfying $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. The A-adjoint $T^\sharp$ also belongs to Class $A^\sharp$.

Proof 1.3.11

1. When $A = I$ then $T^\sharp = T^*$, hence $T^{2*}T^2 \geq_A (TT^*)^2$ therefore T is $*$-Class A.

2 If T is A-Hyponormal. Then,

$$\begin{aligned}
T^{2\sharp}T^2 &= T^\sharp T^\sharp TT \\
&\geq_A T^\sharp TT^\sharp T \\
&\geq_A (T^\sharp T)^2 \\
&= (TT^\sharp)^2
\end{aligned}$$

then T is Class $A^\sharp$.

3 If T is Class $A^\sharp$ and $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Then,

$$\begin{aligned}
\left\langle A(T^{\sharp})^{2\sharp}T^{\sharp 2} - (T^{\sharp}(T^{\sharp})^{\sharp})^2 u|u \right\rangle &= \left\langle A(T^2 P_{\overline{R(A)}} T^{\sharp 2} - (T^{\sharp} P_{\overline{R(A)}} T P_{\overline{R(A)}})^2) u|u \right\rangle \\
&= \left\langle A(P_{\overline{R(A)}} T^2 T^{\sharp 2} - P_{\overline{R(A)}} (T^{\sharp} T)^2) u|u \right\rangle \\
&= \left\langle A(T^2 T^{\sharp 2} - (T^{\sharp} T)^2) u|u \right\rangle \\
&= \left\langle T^2 T^{\sharp 2} - (T^{\sharp} T)^2 u|u \right\rangle_A \\
&\geq 0.
\end{aligned}$$

then $T^{\sharp}$ is Class $A^{\sharp}$.

Chapter 2

The Class $A_k^\sharp$ Operators in Semi-Hilbertian Space

2.1 introduction

We present a new class of operators,termed Class $A_k^\sharp$ operators,within the framework of semi-Hilbertian spaces.It is a generalization of some previous studies in the field of classes of operators,especially for a A-Hyponormal operator.
Our aim in this section is to introduce the notion of Class $A_k^\sharp$ operators in semi-Hilbertian spaces and to present some of their essential properties.

2.2 The Class $A_k^\sharp$ Operators

Definition 2.2.1 *An operator $T \in \mathcal{B}_A(\mathcal{H})$ belongs to Class $A_k^\sharp$ for some positive integer k if*

$$(T^{\sharp k}T^k) \geq_A (TT^\sharp)^k$$

equivalent

$$A[T^{\sharp k}T^k - (TT^\sharp)^k] \geq 0$$

Remark 2.2.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$.Here are some of the following results:*

(i) When $A = I$,Class $I_k^\sharp$ coincide with Class A_k^* operator(see chapter 4).

(ii) If T belongs to Class $A_1^\sharp$,then it is A-Hyponormal.

(iii) If $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and T is Class $A_1^\sharp$,imply $T^\sharp$ is A-Hyponormal.

(iv) Every A-normal operator is of class $A_k^\sharp$

(v) Under the assumption that A is injective,Class $A_1^\sharp$ coincides with A-normal operators.

(vi) If $\mathcal{R}\left(TT^\sharp\right) \subset \overline{\mathcal{R}(A)}$,then Class $A_1^\sharp$ is the A-normal operator.

Theorem 2.2.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$.The following are equivalent:*

1. $T$to be a Class $A_k^\sharp$

2. $\begin{cases} \|T^k v\|_A \geq \|(TT^\sharp)^{\frac{k}{2}}\|_A, & \text{if } k \text{ even.} \\ \|T^k v\|_A \geq \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}}\|_A, & \text{if } k \text{ odd.} \end{cases}$

Proof 2.2.1 ***If*** k ***even number,***

$$\begin{aligned}
\left\langle (TT^\sharp)^k u|u \right\rangle_A &= \left\langle \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k\ time} u|u \right\rangle_A \\
&= \left\langle \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k\ time} u|Au \right\rangle \\
&= \left\langle T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^* Au \right\rangle \\
&= \left\langle T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|AT^\sharp u \right\rangle \\
&= \left\langle AT^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^\sharp u \right\rangle \\
&= \left\langle T^* A \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^\sharp u \right\rangle \\
&= \left\langle A \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|TT^\sharp u \right\rangle \\
&\ \vdots \\
&= \left\langle \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{\frac{k}{2}\ time} u| \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{\frac{k}{2}\ time} u \right\rangle_A \\
&= \left\langle (TT^\sharp)^{\frac{k}{2}} u|(TT^\sharp)^{\frac{k}{2}} u \right\rangle_A \\
&= \|(TT^\sharp)^{\frac{k}{2}} u\|_A^2
\end{aligned}$$

on other hand

$$\begin{aligned}
\langle (T^\sharp)^k T^k u|u\rangle_A &= \langle A \left(T^\sharp\right)^k T^k u|u\rangle \\
&= \langle T^{*k} A T^k u|u\rangle \\
&= \langle AT^k u|T^k u\rangle \\
&= \|T^k u\|_A^2
\end{aligned}$$

hance

$$\begin{aligned}
\langle \left(T^\sharp\right)^k T^k - (TT^\sharp)^k u|u\rangle_A \geq 0 &\Leftrightarrow \langle \left(T^\sharp\right)^k T^k u|u\rangle_A \geq \langle (TT^\sharp)^k u|u\rangle_A \\
&\Leftrightarrow \|T^k u\|_A \geq \|(TT^\sharp)^{\frac{k}{2}} u\|_A.
\end{aligned}$$

If k odd number,

$$\begin{aligned}
\left\langle (TT^\sharp)^k u|u\right\rangle_A &= \left\langle \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k\ time} u|u \right\rangle_A \\
&= \left\langle \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k\ time} u|Au \right\rangle \\
&= \left\langle T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^* Au \right\rangle \\
&= \left\langle T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|AT^\sharp u \right\rangle \\
&= \left\langle AT^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^\sharp u \right\rangle \\
&= \left\langle T^* A \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|T^\sharp u \right\rangle \\
&= \left\langle A \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{k-1\ time} u|TT^\sharp u \right\rangle \\
&\vdots \\
&= \left\langle T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{\frac{k-1}{2}\ time} u|T^\sharp \overbrace{(TT^\sharp)\cdots(TT^\sharp)}^{\frac{k-1}{2}\ time} u \right\rangle_A \\
&= \left\langle T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u \right\rangle_A \\
&= \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u\|_A^2
\end{aligned}$$

hance

$$\begin{aligned}
\langle \left(T^\sharp\right)^k T^k - (TT^\sharp)^k u|u\rangle_A \geq 0 &\Leftrightarrow \langle \left(T^\sharp\right)^k T^k u|u\rangle_A \geq \langle (TT^\sharp)^k u|u\rangle_A \\
&\Leftrightarrow \|T^k u\|_A \geq \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u\|_A.
\end{aligned}$$

Proposition 2.2.1 *Consider an operator $T \in \mathcal{B}_A(\mathcal{H})$ verified $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$ and $(TT^\sharp)^k - (T^\sharp T)^k = T^k T^{\sharp k} - T^{\sharp k} T^k = 0$. Then, T is a Class $A_k^\sharp$ precisely when $T^\sharp$ is a Class $A_k^\sharp$.*

Proof 2.2.2 *Under the assumption that $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$.*
It follows that

$$P_{\overline{R(A)}} T = T P_{\overline{R(A)}} \quad and \quad P_{\overline{R(A)}} T^\sharp = T^\sharp P_{\overline{R(A)}}$$

Assume that T is a Class $A_k^\sharp$ operator and prove that $T^\sharp$ is a Class $A_k^\sharp$
we have

$$T^{\sharp k} T^k \geq_A (TT^\sharp)^k$$

then

$$\begin{aligned}(T^\sharp)^{\sharp k}T^{\sharp k} =&(P_{\overline{R(A)}}TP_{\overline{R(A)}})^kT^{\sharp k}\\ =&P_{\overline{R(A)}}T^kP_{\overline{R(A)}}T^{\sharp k}\\ =&P_{\overline{R(A)}}T^kT^{\sharp k}P_{\overline{R(A)}}\\ =&P_{\overline{R(A)}}T^{\sharp k}T^kP_{\overline{R(A)}}\\ \geq_A&P_{\overline{R(A)}}(TT^\sharp)^kP_{\overline{R(A)}}\\ =&P_{\overline{R(A)}}(T^\sharp T)^kP_{\overline{R(A)}}\\ =&(T^\sharp P_{\overline{R(A)}}TP_{\overline{R(A)}})^k\\ =&(T^\sharp(T^\sharp)^\sharp)^k\end{aligned}$$

hence $T^\sharp$ is a Class $A_k^\sharp$ operator.
Assume that $T^\sharp$ is a Class $A_k^\sharp$ operator and prove that T is a Class $A_k^\sharp$
we have

$$(T^\sharp)^{\sharp k}T^{\sharp k} \geq_A (T^\sharp(T^\sharp)^\sharp)^k$$

and then

$$\begin{aligned}(T^\sharp)^{\sharp k}T^{\sharp k} \geq_A (T^\sharp(T^\sharp)^\sharp)^k \Longrightarrow&P_{\overline{R(A)}}T^kP_{\overline{R(A)}}T^{\sharp k} \geq_A (T^\sharp P_{\overline{R(A)}}TP_{\overline{R(A)}})^k\\ \Longrightarrow&P_{\overline{R(A)}}T^kT^{\sharp k}P_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(T^\sharp T)^kP_{\overline{R(A)}}\\ \Longrightarrow&P_{\overline{R(A)}}T^{\sharp k}T^kP_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(TT^\sharp)^kP_{\overline{R(A)}}\\ \Longrightarrow&T^{\sharp k}T^k \geq_A (TT^\sharp)^k\end{aligned}$$

hence T is a Class $A_k^\sharp$ operator.

Proposition 2.2.2 *Let $T \in \mathcal{B}_A(\mathcal{H})$ is a Class $A_k^\sharp$.*
$\left[T^{\sharp k}T^k - (TT^\sharp)^k\right] u \in \mathcal{N}(A)$ *if and only if* $\|T^ku\|_A = \begin{cases}\|(TT^\sharp)^{\frac{k}{2}}u\|_A, & \text{if } k \text{ even.}\\ \|T^\sharp(TT^\sharp)^{\frac{k-1}{2}}u\|_A, & \text{if } k \text{ odd.}\end{cases}$

Proof 2.2.3

(1) Assume that k is even number.
Suppose that

$$\|T^ku\|_A = \|(TT^\sharp)^{\frac{k}{2}}u\|_A \quad \forall u \in \mathcal{H}$$

then

$$\begin{aligned}\|T^ku\|_A = \|(TT^\sharp)^{\frac{k}{2}}u\|_A \Longleftrightarrow&\langle T^ku|T^ku\rangle_A = \langle(TT^\sharp)^{\frac{k}{2}}u|(TT^\sharp)^{\frac{k}{2}}u\rangle_A\\ \Longleftrightarrow&\langle T^{\sharp k}T^ku|u\rangle_A = \langle(TT^\sharp)^ku|u\rangle_A\\ \Longleftrightarrow&\langle(T^{\sharp k}T^k - (TT^\sharp)^k)u|u\rangle_A = 0\end{aligned}$$

Since $(T^{\sharp k}T^k - (TT^\sharp)^k)$ is A-positive by the A-Cauchy Schwarz inequality,

$$\langle(T^{\sharp k}T^k - (TT^\sharp)^k)u|v\rangle_A = 0 \quad \forall v \in \mathcal{H}; \forall u \in \mathcal{H}$$

this implies that

$$(T^{\sharp k}T^k - (TT^\sharp)^k)u \in \mathcal{N}(A).$$

Conversely if

$$A(T^{\sharp k}T^k - (TT^\sharp)^k)u = 0$$

this implies that

$$\langle (T^{\sharp k}T^k - (TT^\sharp)^k)u|u\rangle_A = 0$$

hence

$$\|T^k u\|_A = \|(TT^\sharp)^{\frac{k}{2}}u\|_A.$$

Example 2.2.1 *Let* $T = \begin{pmatrix} 1 & 1-\rho \\ 0 & \rho \end{pmatrix}$*;* $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
then $A^\dagger = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$
and $T^\sharp = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
this yields

$$\begin{aligned} T^{\sharp k}T^k &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}^k \begin{pmatrix} 1 & 1-\rho \\ 0 & \rho \end{pmatrix}^k \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1-\rho^k \\ 0 & \rho^k \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} (TT^\sharp)^k &= \left(\begin{pmatrix} 1 & 1-\rho \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \right)^k \\ &= \begin{pmatrix} 1-\frac{\rho}{2} & 1-\frac{\rho}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} \end{pmatrix} \end{aligned}$$

hence

$$A(T^{\sharp k}T^k - (TT^\sharp)^k) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1-\frac{\rho}{2} & 1-\frac{\rho}{2} \\ \frac{\rho}{2} & \frac{\rho}{2} \end{pmatrix} \right] \geq 0$$

so T *is Class* $A_k^\sharp$.

Proposition 2.2.3 *Consider an operator* $T \in \mathcal{B}_A(\mathcal{H})$ *with* $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$.

T *is a Class* $A_k^\sharp$ *and it's commutes with* $T^{\sharp k+1}$ *,then* T *is also a Class* $A_{k+1}^\sharp$.

Proof 2.2.4 *suppose that* T *is Class* $A_k^\sharp$ *then,*

$$T^{\sharp k}T^k - (TT^\sharp)^k \geq_A 0 \quad \text{and} \quad TT^\sharp \geq_A 0$$

and we have

$$\begin{aligned} TT^\sharp(T^{\sharp k}T^k - (TT^\sharp)^k) \geq_A 0 &\Longrightarrow TT^\sharp T^{\sharp k}T^k - TT^\sharp(TT^\sharp)^k \geq_A 0 \\ &\Longrightarrow TT^{\sharp k+1}T^k - (TT^\sharp)^{k+1} \geq_A 0 \\ &\Longrightarrow T^{\sharp k+1}T^{k+1} - (TT^\sharp)^{k+1} \geq_A 0 \end{aligned}$$

hence T *is a Class* $A_{k+1}^\sharp$ *operator.*

Proposition 2.2.4 *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $TT^{\sharp k+n} = T^{\sharp k+n}T$ for all positive integers number n and $T(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Then,*
$Class A_k^\sharp \subseteq Class A_{k+1}^\sharp \subseteq ... \subseteq Class A_{k+n}^\sharp$.

Proof 2.2.5 *let T is Class $A_k^\sharp$. Then,*

$$\begin{aligned}
(|T^\sharp|_A^2)^n \left[|T^k|_A^2 - (|T^\sharp|_A^2)^k\right] \geq_A 0 &\Longrightarrow (|T^\sharp|_A^2)^n |T^k|_A^2 - (|T^\sharp|_A^2)^n (|T^\sharp|_A^2)^k \geq_A 0 \\
&\Longrightarrow (|T^\sharp|_A^2)^{n-1} |T^{k+1}|_A^2 - (|T^\sharp|_A^2)^{k+n} \geq_A 0 \\
&\Longrightarrow (|T^\sharp|_A^2)^{n-2} TT^{\sharp k+2} T^{k+1} - (|T^\sharp|_A^2)^{k+n} \geq_A 0 \\
&\Longrightarrow (|T^\sharp|_A^2)^{n-2} |T^{k+1}|_A^2 - (|T^\sharp|_A^2)^{k+n} \geq_A 0 \\
&\vdots \\
&\Longrightarrow |T^{k+1}|_A^2 - (|T^\sharp|_A^2)^{k+n} \geq_A 0
\end{aligned}$$

thus T is a Class $A_{k+n}^\sharp$ operator for all positive integers number n.

Theorem 2.2.2 *For $T \in \mathcal{B}_A(\mathcal{H})$ with $|T^\sharp|_A^{2k} = |T^{\sharp k}|_A^2$ and $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$,the following are equivalent:*

1. *Both T and $T^\sharp$ are Class $A_k^\sharp$ if and only if $\|T^k v\|_A$ and $\|T^{\sharp k} v\|_A$ are equal.*
2. *Provided A is injective,T and $T^\sharp$ are Class $A_k^\sharp$ if and only if T^k and $T^{\sharp k}$ are commute.*

Proof 2.2.6

1. *Assume T and $T^\sharp$ are Class $A_k^\sharp$. Then,*

$$\begin{aligned}
&\left\langle \left[\left(T^\sharp\right)^{\sharp k} T^{\sharp k} - (T^\sharp \left(T^\sharp\right)^\sharp)^k \right] v | v \right\rangle_A \geq 0 \\
&\Longrightarrow \left\langle \left[P_{\overline{\mathcal{R}(A)}} T^k P_{\overline{\mathcal{R}(A)}} T^{\sharp k} - (T^\sharp P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^k \right] v | v \right\rangle_A \geq 0 \\
&\Longrightarrow \left\langle \left[P_{\overline{\mathcal{R}(A)}} T^k P_{\overline{\mathcal{R}(A)}} T^{\sharp k} - P_{\overline{\mathcal{R}(A)}} (T^\sharp T)^k P_{\overline{\mathcal{R}(A)}} \right] v | v \right\rangle_A \geq 0 \\
&\Longrightarrow \left\langle P_{\overline{\mathcal{R}(A)}} \left[T^k T^{\sharp k} - (T^\sharp T)^k \right] v | v \right\rangle_A \geq 0 \\
&\Longrightarrow \left\langle A P_{\overline{\mathcal{R}(A)}} \left[T^k T^{\sharp k} - (T^\sharp T)^k \right] v | v \right\rangle \geq 0 \\
&\Longrightarrow A \left[T^k T^{\sharp k} - (T^\sharp T)^k \right] \geq 0 \\
&\Longrightarrow T^k T^{\sharp k} - T^{\sharp k} T^k \geq_A 0
\end{aligned}$$

on other hand

$$T^{\sharp k} T^k \geq_A (TT^\sharp)^k \Longrightarrow 0 \geq_A T^{\sharp k} T^k - T^k T^{\sharp k}$$

hence

$$T^k T^{\sharp k} - T^{\sharp k} T^k = 0$$

therefore

$$\left\| T^k v \right\|_A = \left\| T^{\sharp k} v \right\|_A .$$

For the converse,suppose that $\|T^k v\|_A = \|T^{\sharp k} v\|_A \, \forall v \in \mathcal{H}$ *holds for every* $v \in \mathcal{H}$ *this assumption directly yields*

$$\left\langle \left(T^{\sharp k}T^k - T^k T^{\sharp k}\right) v|v \right\rangle_A = 0$$

then

$$\left\langle \left(T^{\sharp k}T^k - (TT^\sharp)^k\right) v|v \right\rangle_A = 0$$

therefore T *is a Class* $A_k^\sharp$.
Furthermore,

$$\begin{aligned} 0 &= \left\langle \left(T^k T^{\sharp k} - T^{\sharp k}T^k\right) u \mid u \right\rangle_A \\ &= \left\langle A\left(T^k T^{\sharp k} - (T^\sharp T)^k\right) u \mid u \right\rangle \\ &= \left\langle AP_{\overline{\mathcal{R}(A)}}\left(T^k T^{\sharp k} - (T^\sharp T)^k\right) P_{\overline{\mathcal{R}(A)}} u \mid u \right\rangle \\ &= \left\langle \left(P_{\overline{\mathcal{R}(A)}} T^k P_{\overline{\mathcal{R}(A)}} T^{\sharp k} - P_{\overline{\mathcal{R}(A)}} (T^\sharp T)^k P_{\overline{\mathcal{R}(A)}}\right) u \mid u \right\rangle_A \\ &= \left\langle \left(P_{\overline{\mathcal{R}(A)}} T^k P_{\overline{\mathcal{R}(A)}} T^{\sharp k} - (T^\sharp P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^k\right) u \mid u \right\rangle_A \\ &= \left\langle \left(\left(T^\sharp\right)^{\sharp k} T^{\sharp k} - \left(T^\sharp (T^\sharp)^\sharp\right)^k\right) u \mid u \right\rangle_A \end{aligned}$$

thus,$T^\sharp$ *is a Class* $A_k^\sharp$.

2. *we assume that* T *and* $T^\sharp$ *are a Class* $A_k^\sharp$ *from statement* (1) *it follows that*

$$\begin{aligned} \left\|T^k u\right\|_A = \left\|T^{\sharp k} u\right\|_A &\Longrightarrow \langle \left[T^{\sharp k}T^k - T^k T^{\sharp k}\right] u|u\rangle_A = 0 \\ &\Longrightarrow A\left[T^{\sharp k}T^k - T^k T^{\sharp k}\right] = 0 \\ &\Longrightarrow T^{\sharp k}T^k - T^k T^{\sharp k} = 0 \quad (A \text{ is injective}). \end{aligned}$$

Proof is complete.

Theorem 2.2.3 *For* $T \in \mathcal{B}_A(\mathcal{H})$. *Then,*

$$T \text{ is Class } A_k^\sharp \text{ iff } \quad T^{\sharp k}T^k \geq_A \tfrac{2\rho}{\rho^2+1}(TT^\sharp)^k, \quad \forall \rho \in \mathbb{R}.$$

Proof 2.2.7 ***If*** k ***even number,***
we have

$$\begin{aligned} \|T^k u\|_A^4 \geq \|(TT^\sharp)^{\frac{k}{2}} u\|_A^4 &\Longleftrightarrow 0 \geq \|(TT^\sharp)^{\frac{k}{2}} u\|_A^4 - \|T^k u\|_A^4 \\ &\Longleftrightarrow \rho^2 \|T^k u\|_A^2 - 2\rho \|(TT^\sharp)^{\frac{k}{2}} u\|_A^2 + \|T^k u\|_A^2 \geq 0 \\ &\Longleftrightarrow (\rho^2+1) < T^k u|T^k u >_A - 2\rho < (TT^\sharp)^{\frac{k}{2}} u|(TT^\sharp)^{\frac{k}{2}} u >_A \geq 0 \\ &\Longleftrightarrow < ((\rho^2+1)T^{\sharp k}T^k - 2\rho (TT^\sharp)^k)u|u >_A \geq 0 \\ &\Longleftrightarrow T^{\sharp k}T^k \geq_A \frac{2\rho}{\rho^2+1}(TT^\sharp)^k. \end{aligned}$$

If k ***odd number,***
we have

$$\begin{aligned}
\|T^k u\|_A^4 \geq \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u\|_A^4 &\Longleftrightarrow 0 \geq \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u\|_A^4 - \|T^k u\|_A^4 \\
&\Longleftrightarrow \rho^2 \|T^k u\|_A^2 - 2\rho \|T^\sharp (TT^\sharp)^{\frac{k-1}{2}} u\|_A^2 + \|T^k u\|_A^2 \geq 0 \\
&\Longleftrightarrow (\rho^2+1) < T^k u | T^k u >_A - 2\rho < (TT^\sharp)^{\frac{k}{2}} u | (TT^\sharp)^{\frac{k}{2}} u >_A \geq 0 \\
&\Longleftrightarrow < ((\rho^2+1) T^{\sharp k} T^k - 2\rho (TT^\sharp)^k) u | u >_A \geq 0 \\
&\Longleftrightarrow T^{\sharp k} T^k \geq_A \frac{2\rho}{\rho^2+1} (TT^\sharp)^k.
\end{aligned}$$

Theorem 2.2.4 *For* $T_1; T_2 \in \mathcal{B}_A(\mathcal{H})$ *such that* $T_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$*,the commutations* $T_1 T_2^\sharp = T_2^\sharp T_1$ *and* $T_2 T_1^\sharp = T_1^\sharp T_2$ *are equivalent.*

Proof 2.2.8 *Let* $T_1; T_2 \in \mathcal{B}_A(\mathcal{H})$*,such that* $T_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$*. Then,*

$$P_{\overline{R(A)}} T_1 = T_1 P_{\overline{R(A)}} \quad \textit{and} \quad P_{\overline{R(A)}} T_1^\sharp = T_1^\sharp P_{\overline{R(A)}} \quad \textit{and} \quad A P_{\overline{R(A)}} = A$$

and

$$P_{\overline{R(A)}} T_2 = T_2 P_{\overline{R(A)}} \quad \textit{and} \quad P_{\overline{R(A)}} T_2^\sharp = T_2^\sharp P_{\overline{R(A)}}$$

therefore

$$\begin{aligned}
T_1 T_2^\sharp = T_2^\sharp T_1 &\Longleftrightarrow (T_1 T_2^\sharp)^\sharp = (T_2^\sharp T_1)^\sharp \\
&\Longleftrightarrow (T_2^\sharp)^\sharp T_1^\sharp = T_1^\sharp (T_2^\sharp)^\sharp \\
&\Longleftrightarrow \langle (T_2^\sharp)^\sharp T_1^\sharp u | v \rangle_A = \langle T_1^\sharp (T_2^\sharp)^\sharp u | v \rangle_A | \quad \forall u, v \in \mathcal{H} \\
&\Longleftrightarrow \langle P_{\overline{R(A)}} T_2 P_{\overline{R(A)}} T_1^\sharp u | v \rangle_A = \langle T_1^\sharp P_{\overline{R(A)}} T_2 P_{\overline{R(A)}} u | v \rangle_A | \quad \forall u, v \in \mathcal{H} \\
&\Longleftrightarrow \langle A P_{\overline{R(A)}} T_2 T_1^\sharp u | v \rangle_A = \langle A P_{\overline{R(A)}} T_1^\sharp T_2 u | v \rangle | \quad \forall u, v \in \mathcal{H} \\
&\Longleftrightarrow \langle A T_2 T_1^\sharp u | v \rangle = \langle A T_1^\sharp T_2 u | v \rangle | \quad \forall u, v \in \mathcal{H} \\
&\Longleftrightarrow \langle T_2 T_1^\sharp u | v \rangle_A = \langle T_1^\sharp T_2 u | v \rangle_A | \quad \forall u, v \in \mathcal{H} \\
&\Longleftrightarrow T_2 T_1^\sharp = T_1^\sharp T_2
\end{aligned}$$

Proposition 2.2.5 *Let* $T_{1,2}$ *be Class* $A_k^\sharp$ *satisfying* $T_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$*.If* $T_1^k T_2 = T_1^{\mathrm{k}} T_2^\sharp = T_2^{\mathrm{k}} T_1 = T_2^{\mathrm{k}} T_1^\sharp = 0$ *for some* $k \in \mathbb{N}^*$*,then* $T_1 + T_2$ *is also a Class* $A_k^\sharp$*.*

Proof 2.2.9 *Let $T_1, T_2 \in \mathcal{B}_A(\mathcal{H})$ are Class $A_k^\sharp$. Then,*

$$\begin{aligned}
&(T_1+T_2)^{\sharp k}(T_1+T_2)^k - ((T_1+T_2)(T_1+T_2)^\sharp)^k \\
&= \left[T_1^{\sharp k} + T_2^{\sharp k} + \sum_{i=1}^{k-1}\binom{i}{k-1} T_1^{\sharp i}(T_2^\sharp)^{k-1-i}\right]\left[T_1^k + T_2^k + \sum_{i=1}^{k-1}\binom{i}{k-1} T_1^{i}(T_2)^{k-1-i}\right] \\
&- (T_1T_1^\sharp + T_1T_2^\sharp + T_2T_1^\sharp + T_2T_2^\sharp)^k \\
&= (T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_1^k + T_1^{\sharp k}T_2^k + T_2^{\sharp k}T_2^k) \\
&- \left[(T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k + \sum_{i=1}^{k-1}\binom{i}{k-1}(T_1T_1^\sharp)^i(T_2T_2^\sharp)^{k-1-i}\right] \\
&= (T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k) - (T_1T_1^\sharp)^k - (T_2T_2^\sharp)^k \\
&= T_1^{\sharp k}T_1^k - (T_1T_1^\sharp)^k + T_2^{\sharp k}T_2^k - (T_2T_2^\sharp)^k \\
&\geq_A \cdot
\end{aligned}$$

then $T_1 + T_2$ is Class $A_k^\sharp$.

Proposition 2.2.6 *Let $T_2 \in \mathcal{B}_A(\mathcal{H})$ and T_1 is Class $A_k^\sharp$ obeying $T_1T_2 - T_2T_1 = T_1^\sharp T_2 - T_2T_1^\sharp = 0$. Then,*

1. *If T_2 being A-selfadjoint implies T_1T_2 is a Class $A_k^\sharp$.*

2. *If T_2 is A-normal and satisfies $T_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ then T_2T_1 belongs to Class $A_k^\sharp$.*

Proof 2.2.10 *Let T_1 is Class $A_k^\sharp$. Then,*

$$\begin{aligned}
T_1^{\sharp k}T_1^k - (T_1T_1^\sharp)^k \geq_A 0 &\Longrightarrow T_2^k\left[T_1^{\sharp k}T_1^kT_2^k - T_2^k(T_1T_1^\sharp)^k\right]T_2^k \geq_A 0 \\
&\Longrightarrow T_2^kT_1^{\sharp k}T_1^kT_2^k \geq_A T_2^k(T_1T_1^\sharp)^kT_2^k \\
&\Longrightarrow (T_2T_1^\sharp)^k(T_1T_2)^k \geq_A (T_2T_1T_1^\sharp T_2)^k \\
&\Longrightarrow (T_2^\sharp T_1^\sharp)^k(T_1T_2)^k \geq_A (T_2T_1T_1^\sharp T_2^\sharp)^k \\
&\Longrightarrow (T_2T_1)^{\sharp k}(T_2T_1)^k \geq_A (T_2T_1(T_2T_1)^\sharp)^k
\end{aligned}$$

hence T_2T_1 Class $A_k^\sharp$.
Suppose T_2 is A-normal ($T_2T_2^\sharp = T_2^\sharp T_2$) and T_1 belongs to Class $A_k^\sharp$. We then obtain the following

relations:

$$\begin{aligned}
&\langle\left[(T_1T_2)^{\sharp k}(T_1T_2)^k-\Big((T_1T_2)^\sharp(T_1T_2)\Big)^k\right]v|v\rangle_A\\
&=\langle\left[T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k-(T_2^\sharp T_1^\sharp T_1T_2)^k\right]v|v\rangle_A\\
&=\langle\left[T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k-T_2^{\sharp k}(T_1^\sharp T_1)^kT_2^k\right]v|v\rangle_A\\
&=\langle AT_2^{\sharp k}\left[T_1^{\sharp k}T_1^kT_2^k-(T_1^\sharp T_1)^kT_2^k\right]v|v\rangle\\
&=\langle T_2^{*k}A\left[T_1^{\sharp k}T_1^kT_2^k-(T_1^\sharp T_1)^kT_2^k\right]v|v\rangle\\
&=\langle A\left[T_1^{\sharp k}T_1^k-(T_1^\sharp T_1)^k\right]T_2^kv|T_2^kv\rangle\\
&=\langle\left[T_1^{\sharp k}T_1^k-(T_1^\sharp T_1)^k\right]T_2^kv|T_2^kv\rangle_A\geq 0
\end{aligned}$$

then T_2T_1 Class $A_k^\sharp$.

Proposition 2.2.7 *Let $T_1;T_2\in\mathcal{B}_A(\mathcal{H})$ such that $T_1^\sharp T_2-T_2T_1^\sharp=T_1T_2-T_2T_1=0$. when $T_1;T_2$ are Class $A_k^\sharp$ and $T_{1,2}(\mathcal{N}(A))\subseteq\mathcal{N}(A)$,then T_1T_2 is Class $A_k^\sharp$.*

Proof 2.2.11 *Let T_1 and T_2 are Class $A_k^\sharp$ operators.*
If k is even number.Then,

$$\begin{aligned}
\|(T_1T_2)^ku\|_A&=\|T_1^kT_2^ku\|_A\\
&\geq\|(T_1T_1^\sharp)^{\frac{k}{2}}T_2^ku\|_A\\
&=\|T_2^k(T_1T_1^\sharp)^{\frac{k}{2}}u\|_A\\
&\geq\|(T_2T_2^\sharp)^{\frac{k}{2}}(T_1T_1^\sharp)^{\frac{k}{2}}u\|_A\\
&=\|(T_2T_2^\sharp T_1T_1^\sharp)^{\frac{k}{2}}u\|_A\\
&=\|(T_1T_2T_2^\sharp T_1^\sharp)^{\frac{k}{2}}u\|_A\\
&=\|(T_1T_2(T_1T_2)^\sharp)^{\frac{k}{2}}u\|_A
\end{aligned}$$

so T_1T_2 is a Class $A_k^\sharp$.
If k is odd number.Then,

$$\begin{aligned}
\|(T_1T_2)^ku\|_A&=\|T_1^kT_2^ku\|_A\\
&\geq\|T_1^\sharp(T_1T_1^\sharp)^{\frac{k-1}{2}}T_2^ku\|_A\\
&=\|T_2^kT_1^\sharp(T_1T_1^\sharp)^{\frac{k-1}{2}}u\|_A\\
&\geq\|T_2^\sharp(T_2T_2^\sharp)^{\frac{k-1}{2}}T_1^\sharp(T_1T_1^\sharp)^{\frac{k-1}{2}}u\|_A\\
&=\|T_2^\sharp T_1^\sharp(T_2T_2^\sharp T_1T_1^\sharp)^{\frac{k-1}{2}}u\|_A\\
&=\|(T_2T_1)^\sharp(T_2T_1T_2^\sharp T_1^\sharp)^{\frac{k-1}{2}}u\|_A\\
&=\|(T_1T_2)^\sharp(T_1T_2(T_1T_2)^\sharp)^{\frac{k-1}{2}}u\|_A
\end{aligned}$$

so T_1T_2 is a Class $A_k^\sharp$.

2.3 Tensor Product of Class $A_k^\sharp$ in Semi-Hilbertian Spaces

The tensor product of operators is a bilinear operation that constructs a new operator on a tensor product space from two given operators.If you have two operators T_1, T_2 acting on spaces $\mathcal{H}_1, \mathcal{H}_2$. respectively,the tensor product operator $T_1 \otimes T_2$ is the induced operator on the tensor product space $\mathcal{H}_1 \bar{\otimes} \mathcal{H}_2$.It is defined as

$$(T_1 \otimes T_2)(u_1 \otimes u_2) = (T_1 u_1) \otimes (T_2 u_2).$$

We need in the following lemma.

Lemma 2.3.1 *Suppose $T_1, T_2, T_1', T_2' \in \mathcal{B}(\mathcal{H})$ and $A, A' \in \mathcal{B}(\mathcal{H})^+$,satisfy the inequalities* $\begin{cases} T_1 \geq_A T_2 \geq_A 0 \\ T_1' \geq_{A'} T_2' \geq_{A'} 0 \end{cases}$ *,then $(T_1 \otimes T_1') \geq_{A \otimes A'} (T_2 \otimes T_2') \geq_{A \otimes A'} 0$*

Proposition 2.3.1 *Let $A, A' \in \mathcal{B}(\mathcal{H})^+$ and $T_1, T_2, T_1', T_2' \in \mathcal{B}(\mathcal{H})$.Assume that T_i is A-positive and T_i' is A'-positive for $i = 1, 2$.Then the following are equivalent:*

1. *$T_2 \otimes T_2' \geq_{A \otimes A'} T_1 \otimes T_1'$.*
2. *There exists $\rho > 0$ such that $\rho T_2 \geq_A T_1$ and $\rho^{-1} T_2' \geq_{A'} T_1'$.*

We will require the following properties in the proof of the subsequent theorem.

1. $(T_1 \otimes T_2)(T_1 \otimes T_2)^* = (T_1 T_1^* \otimes T_2 T_2^*)$.
2. $\rho_1 \rho_2 T_1 \otimes T_2 = \rho_1 T_1 \otimes \rho_2 T_2$,for any real numbers ρ_1, ρ_2.
3. $((T_1 \otimes T_2)(T_1 \otimes T_2)^*)^\rho = (T_1 T_1^*)^\rho \otimes (T_2^* T_2)^\rho \quad \forall \rho \in \mathbb{R}^+$.

Theorem 2.3.1 *Let $A, A' \in \mathcal{B}(\mathcal{H})^+$ and $T_1 \in \mathcal{B}_A(\mathcal{H})$ and $T_2 \in \mathcal{B}_{A'}(\mathcal{H})$.The following are equivalent:*

i. The operators T_1, T_2 belong to Class $A_k^\sharp$ and Class $A_k'^\sharp$,respectively.

ii. $T_1 \otimes T_2$ is a Class $(A \otimes A')_k^\sharp$.

Proof 2.3.1 *Suppose T_1 are Class $A_k^\sharp$ and T_2 Class $A_k'^\sharp$.Then,*

$$\begin{aligned} \left((T_1 \otimes T_2)^\sharp\right)^k (T_1 \otimes T_2)^k &= \left(T_1^{\sharp k} \otimes T_2^{\sharp k}\right)\left(T_1^k \otimes T_2^k\right) \\ &= T_1^{\sharp k} T_1^k \otimes T_2^{\sharp k} T_2^k \\ &\geq_{A \otimes A'} (T_1 T_1^\sharp)^k \otimes (T_2 T_2^\sharp)^k \\ &= (T_1 T_1^\sharp \otimes T_2 T_2^\sharp)^k \\ &= ((T_1 \otimes T_2)(T_1^\sharp \otimes T_2^\sharp))^k \\ &= ((T_1 \otimes T_2)(T_1 \otimes T_2)^\sharp)^k. \end{aligned}$$

hence $T_1 \otimes T_2$ *is Class* $(A \otimes A')_k^\sharp$.
Assuming the converse,$T_1 \otimes T_2$ *belongs to class* $(A \otimes A')_k^\sharp$.*This yields*

$$\begin{aligned}
&\left((\mathcal{T}_1 \otimes \mathcal{T}_2)^\sharp\right)^k (\mathcal{T}_1 \otimes \mathcal{T}_2)^k \geq_{A\otimes A'} ((\mathcal{T}_1 \otimes \mathcal{T}_2)(\mathcal{T}_1 \otimes \mathcal{T}_2)^\sharp)^k \\
\Longleftrightarrow &\left(\mathcal{T}_1^{\sharp k} \otimes \mathcal{T}_2^{\sharp k}\right)(\mathcal{T}_1^k \otimes \mathcal{T}_2^k) \geq_{A\otimes A'} (\mathcal{T}_1\mathcal{T}_1^\sharp \otimes \mathcal{T}_2\mathcal{T}_2^\sharp)^k \\
\Longleftrightarrow &\mathcal{T}_1^{\sharp k}\mathcal{T}_1^k \otimes \mathcal{T}_2^{\sharp k}\mathcal{T}_2^k \geq_{A\otimes A'} (\mathcal{T}\mathcal{T}^\sharp)^k \otimes (\mathcal{T}_2\mathcal{T}_2^\sharp)^k \\
\Longleftrightarrow &\mathcal{T}_1^{\sharp k}\mathcal{T}_1^k \otimes \mathcal{T}_2^{\sharp k}\mathcal{T}_2^k \geq_{A\otimes A'} \rho(\mathcal{T}_1\mathcal{T}_1^\sharp)^k \otimes \rho^{-1}(\mathcal{T}_2\mathcal{T}_2^\sharp)^k
\end{aligned}$$

It follows from Proposition 2.3.1 that we can find $\rho > 0$ *satisfying*

$$\begin{cases} \rho T_1^{\sharp k} T_1^k \geq_A \rho(T_1T_1^\sharp)^k \\ \wedge \\ \rho^{-1}T_2^{\sharp k}T_2^k \geq_A \rho^{-1}(T_2T_2^\sharp)^k \end{cases}$$

a simple computation

$$T_1^{\sharp k}T_1^k \geq_A (T_1T_1^\sharp)^k \quad \textit{and} \quad T_2^{\sharp k}T_2^k \geq_{A'} (T_2T_2^\sharp)^k$$

we conclude T_1 *is Class* $A_k^\sharp$ *and* T_2 *is Class* $A_k'^\sharp$.

Proposition 2.3.2 *Consider two operators* T_1 *and* T_2 *in Class* $A_k^\sharp$ *and both satisfy* $T_{1,2}(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ *and the conditions* $T_1T_2 - T_2T_1 = T_1^\sharp T_2 - T_2T_1^\sharp = 0$.*Under these assumptions,*

$T_1T_2 \otimes T_1$ *and* $T_1T_2 \otimes T_2 \in \mathcal{B}_{A\otimes A}(\mathcal{H}\bar{\otimes}\mathcal{H})$ *are Class* $(A \otimes A)_k^\sharp$.

Proof 2.3.2

1 Suppose T_1 *and* T_2 *belong to Class* $A_k^\sharp$.*It follows that*

$$\begin{aligned}
T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k &\geq_A T_2^{\sharp k}(T_1T_1^\sharp)^kT_2^k \\
&= T_2^{\sharp k}T_2^k(T_1T_1^\sharp)^k \\
&\geq_A (T_2T_2^\sharp)^k(T_1T_1^\sharp)^k \\
&= (T_2T_2^\sharp T_1T_1^\sharp)^k \\
&= (T_2T_1(T_2T_1)^\sharp)^k
\end{aligned}$$

hence

$$\begin{cases} T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k \geq_A (T_1T_2(T_1T_2)^\sharp)^k \geq_A 0. \\ \wedge \\ T_1^{\sharp k}T_1^k \geq_A (T_1T_1^\sharp)^k \geq_A 0. \end{cases}$$

alternatively,

$$\begin{aligned}
(T_1T_2 \otimes T_1)^{\sharp k}(T_1T_2 \otimes T_1)^k &= \left((T_1T_2)^{\sharp k} \otimes T_1^{\sharp k}\right)\left((T_1T_2)^k \otimes T_1^k\right) \\
&= \left(T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k \otimes T_1^{\sharp k}T_1^k\right)
\end{aligned}$$

lemma 2.3.1 implies that

$$\left(T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k \otimes T_1^{\sharp k}T_1^k\right) \geq_A (T_1T_2(T_1T_2)^\sharp)^k \otimes (T_1T_1^\sharp)^k$$

then

$$(T_1T_2 \otimes T_1)^{\sharp k}(T_1T_2 \otimes T_1)^k \geq_A \left((T_1T_2 \otimes T_1)(T_1T_2 \otimes T_1)^\sharp\right)^k$$

therefore $(T_1T_2 \otimes T_1)$ *is Class* $(A \otimes A)_k^\sharp$.

2. *In the same way,we may deduce* $T_1T_2 \otimes T_2$ *belong to Class* $(A \otimes A)_k^\sharp$ *operators.*

Chapter 3

The Quasi Class $A_k^\sharp$ Operators in Semi-Hilbertian Space

3.1 Introduction

Our aim here is to introduce a novel Class of operators,called Quasi Class $A_k^\sharp$ operators,in the setting of semi-Hilbertian space $(\mathcal{H}, \langle . \mid . \rangle_A)$.We explore their fundamental properties and analyze their structural characteristics in detail.additionally,tensor product formulations specific to Quasi Class $A_k^\sharp$ operators are investigated.The discussion further establishes key structural properties intrinsic to this class,shedding light on their algebraic and functional behavior in semi-Hilbertian settings.

3.2 The Concept of Quasi Class $A_k^\sharp$ Operators

we now present definitions that generalize the concepts of Class $A_k^\sharp$ and Quasi A-Hyponormal operators([6],[39]).

Definition 3.2.1 *we say that an operator $T \in \mathcal{B}_A(\mathcal{H})$ is of Quasi Class $A_k^\sharp$ if for some positive integer k,*

$$A[T^{\sharp k+1}T^{k+1} - T^\sharp(TT^\sharp)^k T] \geq 0$$

equivalent

$$T^\sharp \left[|T^k|_A^2 - |T|_A^{2k} \right] T \geq_A 0$$

Remark 3.2.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$.The following observations hold:*

1. *For $A = I$,the class $I_k^\sharp$ is simply Quasi Class $A_k^\star$.*
2. *Under the assumption that $\mathcal{N}(A)$ is an invariant subspace for the Quasi Class $A_1^\sharp$ operator T,it follows that $T^\sharp$ is Quasi A-Hyponormal.*
3. *If $\mathcal{R}\left(TT^\sharp\right) \subset \overline{\mathcal{R}(A)}$,then Class $A_1^\sharp$ is A-normal.*
4. *Assuming that A is injective,Class $A_1^\sharp$ coincides with the class of A-normal.*

Theorem 3.2.1 *For an operator $T \in \mathcal{B}_A(\mathcal{H})$,the following are equivalent:*

i. T is Quasi Class $A_k^\sharp$.

ii. T satisfies the condition below: $\begin{cases} \|T^{k+1}u\|_A \geq \|(TT^\sharp)^{\frac{k}{2}}Tu\|_A, & \text{if } k \text{ even.} \\ \|T^{k+1}u\|_A \geq \|(T^\sharp T)^{\frac{k+1}{2}}u\|_A, & \text{if } k \text{ odd.} \end{cases}$

Proof 3.2.1 *Assume that T is a Class $A_k^\sharp$*

$$\begin{aligned}
\left\langle T^\sharp(TT^\sharp)^k Tu|u\right\rangle_A &= \begin{cases} \langle AT^\sharp(TT^\sharp)^k Tu|u\rangle; \text{ if } k \text{ even.} \\ \langle AT^\sharp(TT^\sharp)^k Tu|u\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \langle T^* A(TT^\sharp)^k Tu|u\rangle; \text{ if } k \text{ even.} \\ \langle T^* A(TT^\sharp)^k Tu|u\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \langle A(TT^\sharp)^k Tu|Tu\rangle; \text{ if } k \text{ even.} \\ \left\langle \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{ktime} Tu|ATu\right\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \left\langle T^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{k-1\ time} Tu|T^*ATu\right\rangle; \text{ if } k \text{ even.} \\ \left\langle T^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{(k-1)time} Tu|T^*ATu\right\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \left\langle T^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{k-1\ time} Tu|AT^\sharp Tu\right\rangle; \text{ if } k \text{ even.} \\ \left\langle T^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{(k-1)time} Tu|AT^\sharp Tu\right\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \left\langle AT^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{k-1\ time} Tu|T^\sharp Tu\right\rangle; \text{ if } k \text{ even.} \\ \left\langle AT^\sharp \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{(k-1)time} Tu|T^\sharp Tu\right\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \left\langle T^*A \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{k-1\ time} Tu|T^\sharp Tu\right\rangle; \text{ if } k \text{ even.} \\ \left\langle T^*A \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{(k-1)time} Tu|T^\sharp Tu\right\rangle; \text{ if } k \text{ odd.} \end{cases} \\
&= \begin{cases} \left\langle A \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{k-1\ time} Tu|TT^\sharp Tu\right\rangle; \text{ if } k \text{ even.} \\ \left\langle A \underbrace{(TT^\sharp)\cdots(TT^\sharp)}_{(k-1)time} Tu|TT^\sharp Tu\right\rangle; \text{ if } k \text{ odd.} \end{cases}
\end{aligned}$$

$$\vdots$$

$$= \begin{cases} \left\langle \underbrace{(TT^{\sharp})\cdots(TT^{\sharp})}_{\frac{k}{2}\,time} Tu | \underbrace{(TT^{\sharp})\cdots(TT^{\sharp})}_{\frac{k}{2}\,time} Tu \right\rangle_A & ;\ if\ k\ even. \\ \left\langle T^{\sharp} \underbrace{TT^{\sharp}\cdots TT^{\sharp}}_{\frac{k-1}{2}\,time} Tu | T^{\sharp} \underbrace{TT^{\sharp}\cdots TT^{\sharp}}_{\frac{k-1}{2}\,time} Tu \right\rangle_A & ;\ if\ k\ odd. \end{cases}$$

$$= \begin{cases} \left\langle (TT^{\sharp})^{\frac{k}{2}} Tu | (TT^{\sharp})^{\frac{k}{2}} Tu \right\rangle_A ;\ if\ k\ even. \\ \left\langle T^{\sharp}(TT^{\sharp})^{\frac{k-1}{2}} Tu | T^{\sharp}(TT^{\sharp})^{\frac{k-1}{2}} Tu \right\rangle_A ;\ if\ k\ odd. \end{cases}$$

$$= \begin{cases} \|(TT^{\sharp})^{\frac{k}{2}} Tu\|_A^2;\ if\ k\ even. \\ \|(T^{\sharp}T)^{\frac{k+1}{2}} u\|_A^2;\ if\ k\ odd. \end{cases}$$

alternatively,

$$\begin{aligned} \langle (T^{\sharp})^{k+1} T^{k+1} u|u\rangle_A =& \langle A \left(T^{\sharp}\right)^{k+1} T^{k+1} u|u\rangle \\ =& \langle T^{*k+1} A T^{k+1} u|u\rangle \\ =& \langle A T^{k+1} u | T^{k+1} u\rangle \\ =& \|T^{k+1} u\|_A^2 \end{aligned}$$

If k even,

$$\begin{aligned} \langle \left(T^{\sharp}\right)^{k+1} T^{k+1} - T^{\sharp}(TT^{\sharp})^k Tu|u\rangle_A \geq 0 \Leftrightarrow& \langle T^{\sharp} \left(T^{\sharp}\right)^{k+1} T^{k+1} u|u\rangle_A \geq \langle T^{\sharp}(TT^{\sharp})^k Tu|u\rangle_A \\ \Leftrightarrow& \|T^{k+1}u\|_A \geq \|T^{\sharp}(TT^{\sharp})^{\frac{k}{2}} T\|_A. \end{aligned}$$

If k odd,

$$\begin{aligned} \langle \left(T^{\sharp}\right)^{k+1} T^{k+1} - T^{\sharp}(TT^{\sharp})^k Tu|u\rangle_A \geq 0 \Leftrightarrow& \langle \left(T^{\sharp}\right)^{k+1} T^{k+1} u|u\rangle_A \geq \langle T^{\sharp}(TT^{\sharp})^k Tu|u\rangle_A \\ \Leftrightarrow& \|T^{k+1}u\|_A \geq \|T^{\sharp}(TT^{\sharp})^{\frac{k-1}{2}} T\|_A. \end{aligned}$$

Proof is complete.

Proposition 3.2.1 *Let $T \in \mathcal{B}_A(\mathcal{H})$ with $(TT^{\sharp})^k - (T^{\sharp}T)^k = T^k T^{\sharp k} - T^{\sharp k} T^k = 0$ and $\mathcal{N}(A)$ is an invariant subspace.At that point,*
An operator T belongs to Quasi Class $A_k^{\sharp}$ if and only if $T^{\sharp}$ also belongs to Quasi Class $A_k^{\sharp}$.

Proof 3.2.2 *Prove that if T is a Quasi Class $A_k^{\sharp}$,then $T^{\sharp}$ is a Quasi Class $A_k^{\sharp}$.We have*

$$\begin{aligned} & A\left[(T^{\sharp})^{\sharp k+1} T^{\sharp k+1} - (T^{\sharp})^{\sharp}(T^{\sharp}(T^{\sharp})^{\sharp})^k T^{\sharp}\right] \\ =& A\left[(\mathcal{P}_{\overline{R(A)}} T \mathcal{P}_{\overline{R(A)}})^{k+1} T^{\sharp k+1} - (\mathcal{P}_{\overline{R(A)}} T \mathcal{P}_{\overline{R(A)}})(T^{\sharp}(\mathcal{P}_{\overline{R(A)}} T \mathcal{P}_{\overline{R(A)}}))^k T^{\sharp}\right] \\ =& A\left[\mathcal{P}_{\overline{R(A)}}(T)^{k+1} T^{\sharp k+1} - (\mathcal{P}_{\overline{R(A)}} T \mathcal{P}_{\overline{R(A)}})\mathcal{P}_{\overline{R(A)}}(T^{\sharp}T)^k \mathcal{P}_{\overline{R(A)}} T^{\sharp}\right] \\ =& A\left[\mathcal{P}_{\overline{R(A)}} T^{k+1} T^{\sharp k+1} - \mathcal{P}_{\overline{R(A)}} T(T^{\sharp}T)^k T^{\sharp}\right] \\ =& A\left[T^{k+1} T^{\sharp k+1} - T(T^{\sharp}T)^k T^{\sharp}\right] \geq 0 \end{aligned}$$

hence $T^\sharp$ is a Quasi Class $A_k^\sharp$.
now,we can prove that $T^\sharp$ is a Quasi Class $A_k^\sharp$ implies T is a Quasi Class $A_k^\sharp$.We have

$$(T^\sharp)^{\sharp k}T^{\sharp k} \geq_A (T^\sharp(T^\sharp)^\sharp)^k$$

then,

$$\begin{aligned}(T^\sharp)^{\sharp k}T^{\sharp k} \geq_A (T^\sharp(T^\sharp)^\sharp)^k &\Longrightarrow P_{\overline{R(A)}}T^kP_{\overline{R(A)}}T^{\sharp k} \geq_A (T^\sharp P_{\overline{R(A)}}TP_{\overline{R(A)}})^k\\ &\Longrightarrow P_{\overline{R(A)}}T^kT^{\sharp k}P_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(T^\sharp T)^kP_{\overline{R(A)}}\\ &\Longrightarrow P_{\overline{R(A)}}T^{\sharp k}T^kP_{\overline{R(A)}} \geq_A P_{\overline{R(A)}}(\mathcal{T}\mathcal{T}^\sharp)^kP_{\overline{R(A)}}\\ &\Longrightarrow T^{\sharp k}T^k \geq_A (\mathcal{T}\mathcal{T}^\sharp)^k\end{aligned}$$

hence T is a Quasi Class $A_k^\sharp$operator.

Theorem 3.2.2 *If T is Quasi Class $A_k^\sharp$ and it has dense range.Then T is Class $A_k^\sharp$.*

Proof 3.2.3 *Assume that T is Quasi Class $A_k^\sharp$ and it has dense range (i,e:$\overline{\mathcal{R}(T)} = \mathcal{H}$).Then, for any $u \in \mathcal{H}$,one can find a sequence (x_n) in $\mathcal{H}$ satisfying $T\,(x_n) \longrightarrow_{n\longrightarrow\infty} u$.*

$$\begin{cases}\|T^{k+1}u\|_A \geq \|(TT^\sharp)^{\frac{k}{2}}Tu\|_A, & \text{if } k \text{ even number}\\ \|T^{k+1}u\|_A \geq \|(T^\sharp T)^{\frac{k+1}{2}}u\|_A, & \text{if } k \text{ odd number}\end{cases}$$

If k even number

$$\|T^{k+1}x_n\|_A^2 \geq \|\left(TT^\sharp\right)^{\frac{k}{2}}Tx_n\|_A^2, \quad \forall n \in \mathbb{N}$$

this yields

$$\|A^{\frac{1}{2}}T^{k+1}x_n\|^2 \geq \|A^{\frac{1}{2}}\left(TT^\sharp\right)^{\frac{k}{2}}Tx_n\|^2, \quad \forall n \in \mathbb{N}$$

we go to the lim

$$\|\lim_{n\to\infty} A^{\frac{1}{2}}T^{k+1}x_n\|^2 \geq \|\lim_{n\to\infty} A^{\frac{1}{2}}\left(TT^\sharp\right)^{\frac{k}{2}}Tx_n\|^2$$

thus

$$\|T^ku\|_A \geq \|\left(TT^\sharp\right)^{\frac{k}{2}}u\|_A$$

If k odd number

$$\|T^{k+1}x_n\|_A^2 \geq \|\left(T^\sharp T\right)^{\frac{k+1}{2}}x_n\|_A^2$$

this yields

$$\|A^{\frac{1}{2}}T^{k+1}x_n\|^2 \geq \|A^{\frac{1}{2}}T^\sharp\left(TT^\sharp\right)^{\frac{k-1}{2}}Tx_n\|^2$$

we go to the lim

$$\|\lim_{n\to\infty} A^{\frac{1}{2}}T^{k+1}x_n\|^2 \geq \|\lim_{n\to\infty} A^{\frac{1}{2}}T^\sharp\left(TT^\sharp\right)^{\frac{k-1}{2}}Tx_n\|^2$$

so

$$\|T^ku\|_A \geq \|T^\sharp\left(TT^\sharp\right)^{\frac{k-1}{2}}u\|_A$$

hence T is Class $A_k^\sharp$.

Proposition 3.2.2 *Let $T \in \mathcal{B}_A(\mathcal{H})$ with $(T^\sharp T)^{k+1} - (TT^\sharp)^{k+1} = 0$. Then,*

T is a Quasi Class $A_k^\sharp$ operator iff it is a Class $A_{k+1}^\sharp$.

Proof 3.2.4 *Suppose that T is a Quasi Class $A_k^\sharp$. Then,*

$$\begin{aligned} T^\sharp(T^{\sharp k}T^k - (TT^\sharp)^k)T \geq_A 0 &\Longrightarrow T^{\sharp k+1}T^{k+1} - T^\sharp(TT^\sharp)^k T \geq_A 0 \\ &\Longrightarrow T^{\sharp k+1}T^{k+1} - TT^\sharp(TT^\sharp)^k \geq_A 0 \\ &\Longrightarrow T^{\sharp k+1}T^{k+1} - (TT^\sharp)^{k+1} \geq_A 0. \end{aligned}$$

hence T is Class $A_{k+1}^\sharp$ operator.
Suppose that T is Class $A_{k+1}^\sharp$. Then,

$$\begin{aligned} T^{\sharp k+1}T^{k+1} - (TT^\sharp)^{k+1} \geq_A 0 &\Longrightarrow T^{\sharp k+1}T^{k+1} - TT^\sharp(TT^\sharp)^k \geq_A 0 \\ &\Longrightarrow T^\sharp(T^{\sharp k}T^k)T - T^\sharp(TT^\sharp)^k T \geq_A 0 \\ &\Longrightarrow T^\sharp(T^{\sharp k}T^k - (TT^\sharp)^k)T \geq_A 0. \end{aligned}$$

hence T is a Quasi Class $A_k^\sharp$ operator.

Proposition 3.2.3 *For $T \in \mathcal{B}_A(\mathcal{H})$ where T^{k+1} commutes with $T^{\sharp k+1}$ and $\mathcal{N}(A)$ invariant, we have*

T is a Quasi Class $A_k^\sharp$ iff $T^\sharp$ is a Class $A_{k+1}^\sharp$.

Proof 3.2.5 *Suppose that T is a Quasi Class $A_k^\sharp$ and we prove $T^\sharp$ is a Class $A_{k+1}^\sharp$. We have,*

$$\begin{aligned} T^\sharp(T^{\sharp k}T^k - (TT^\sharp)^k)T \geq_A 0 &\Longrightarrow T^{\sharp k+1}T^{k+1} - T^\sharp(TT^\sharp)^k T \geq_A 0 \\ &\Longrightarrow T^{\sharp k+1}T^{k+1} - (T^\sharp T)^{k+1} \geq_A 0 \\ &\Longrightarrow T^{k+1}T^{\sharp k+1} - (T^\sharp T)^{k+1} \geq_A 0. \end{aligned}$$

on other hand

$$\begin{aligned} \langle\Big((T^\sharp)^\sharp\Big)^{k+1}(T^\sharp)^{k+1} - (T^\sharp(T^\sharp)^\sharp)^{k+1}u|u\rangle_A &= \langle\Big(P_{\overline{R(A)}}TP_{\overline{R(A)}}\Big)^{k+1}(T^\sharp)^{k+1} - (T^\sharp(P_{\overline{R(A)}}TP_{\overline{R(A)}})^{k+1}u|u\rangle_A \\ &= \langle A\left[P_{\overline{R(A)}}T^{k+1}(T^\sharp)^{k+1} - P_{\overline{R(A)}}(T^\sharp T)^{k+1}\right]u|u\rangle \\ &= \langle A\left[T^{k+1}(T^\sharp)^{k+1} - (T^\sharp T)^{k+1}\right]u|u\rangle \\ &= \langle T^{k+1}(T^\sharp)^{k+1} - (T^\sharp T)^{k+1}u|u\rangle_A \\ &\geq_A 0. \end{aligned}$$

hence $T^\sharp$ is Class $A_{k+1}^\sharp$ operator.

Suppose that $T^\sharp$ is Class $A_{k+1}^\sharp$. Then,

$$A\left[\left((T^\sharp)^\sharp\right)^{k+1}(T^\sharp)^{k+1}-(T^\sharp(T^\sharp)^\sharp)^{k+1}\right]\geq 0 \Longrightarrow A\left[\left(P_{\overline{R(A)}}TP_{\overline{R(A)}}\right)^{k+1}(T^\sharp)^{k+1}-(T^\sharp(P_{\overline{R(A)}}TP_{\overline{R(A)}})^{k+1}\right]\geq 0$$
$$\Longrightarrow A\left[P_{\overline{R(A)}}T^{k+1}(T^\sharp)^{k+1}-P_{\overline{R(A)}}(T^\sharp T)^{k+1}\right]\geq 0$$
$$\Longrightarrow A\left[T^{k+1}(T^\sharp)^{k+1}-(T^\sharp T)^{k+1}\right]\geq 0$$
$$\Longrightarrow A\left[(T^\sharp)^{k+1}T^{k+1}-(T^\sharp T)^{k+1}\right]\geq 0$$
$$\Longrightarrow A\left[T^\sharp(T^{\sharp k}T^k-(TT^\sharp)^k)T\right]\geq 0.$$

hence T is a Quasi Class $A_k^\sharp$ operator.

Proposition 3.2.4 *Let $T\in\mathcal{B}_A(\mathcal{H})$ is a Quasi Class $A_k^\sharp$.*
$(T^\sharp(T^{\sharp k}T^k-(TT^\sharp)^k)T)u\in\mathcal{N}(A)$ is equivalent to $\|T^{k+1}u\|_A=\begin{cases}\|(TT^\sharp)^{\frac{k}{2}}Tu\|_A, & \text{if } k \text{ even.}\\ \|(T^\sharp T)^{\frac{k+1}{2}}u\|_A, & \text{if } k \text{ odd.}\end{cases}$

Proof 3.2.6 *let T is a Quasi Class $A_k^\sharp$.*
suppose that

$$\begin{cases}\|T^{k+1}u\|_A=\|(TT^\sharp)^{\frac{k}{2}}Tu\|_A, & \text{if } k \text{ even number}\\ \|T^{k+1}u\|_A=\|(T^\sharp T)^{\frac{k+1}{2}}u\|_A, & \text{if } k \text{ odd number}\end{cases}$$
$$\Longrightarrow\begin{cases}\langle T^{k+1}u|T^{k+1}u\rangle_A=\langle (TT^\sharp)^{\frac{k}{2}}Tu|(TT^\sharp)^{\frac{k}{2}}Tu\rangle_A, & \text{if } k \text{ even number}\\ \langle T^{k+1}u|T^{k+1}u\rangle_A=\langle (T^\sharp T)^{\frac{k+1}{2}}u|(T^\sharp T)^{\frac{k+1}{2}}u\rangle_A, & \text{if } k \text{ odd number}\end{cases}$$
$$\Longrightarrow\begin{cases}\langle T^{k+1}T^{\sharp k+1}u|u\rangle_A=\langle T^\sharp(TT^\sharp)^kTu|u\rangle_A, & \text{if } k \text{ even number}\\ \langle T^{\sharp k+1}T^{k+1}u|u\rangle_A=\langle T^\sharp(T^\sharp T)^kTu|u\rangle_A, & \text{if } k \text{ odd number}\end{cases}$$
$$\Longrightarrow\langle T^\sharp\left[T^{\sharp k}T^k-(TT^\sharp)^k\right]Tu|u\rangle_A=0$$

applying the Cauchy-Schwarz inequality yields

$$\langle T^\sharp\left[T^{\sharp k}T^k-(TT^\sharp)^k\right]Tu|v\rangle_A=0|\quad\forall v\in\mathcal{H}$$

so

$$T^\sharp\left[T^{\sharp k}T^k-(TT^\sharp)^k\right]Tu\in\mathcal{N}(A)$$

Conversely if

$$AT^\sharp\left[T^{\sharp k}T^k-(TT^\sharp)^k\right]Tu=0$$

this implies that

$$\langle T^\sharp\left[T^{\sharp k}T^k-(TT^\sharp)^k\right]Tu|u\rangle_A=0$$

hence

$$\begin{cases}\|T^{k+1}u\|_A=\|(TT^\sharp)^{\frac{k}{2}}Tu\|_A, & \text{if } k \text{ even number}\\ \|T^{k+1}u\|_A=\|(T^\sharp T)^{\frac{k+1}{2}}u\|_A, & \text{if } k \text{ odd number}\end{cases}$$

Proof is complete.

Proposition 3.2.5 *Let $T \in \mathcal{B}_A(\mathcal{H})$ such that $TT^{\sharp k+1} - T^{\sharp k+1}T = 0$.*

If T belongs to Quasi Class $A_k^\sharp$ then it is belongs to Quasi Class $A_{k+1}^\sharp$.

Proof 3.2.7 *suppose that T is Class $A_k^\sharp$. Then,*

$$T^\sharp(T^{\sharp k}T^k - (\mathcal{T}\mathcal{T}^\sharp)^k)T \geq_A 0$$

we have

$$\begin{aligned} T^\sharp T(T^\sharp(T^{\sharp k}T^k - (\mathcal{T}\mathcal{T}^\sharp)^k T) \geq_A 0 &\Longrightarrow T^\sharp T(T^\sharp(T^{\sharp k}T^k)T) - T^\sharp T(T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^k T) \geq_A 0 \\ &\Longrightarrow T^\sharp \mathcal{T}\mathcal{T}^{\sharp k+1}T^{k+1} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+1}T \geq_A 0 \\ &\Longrightarrow T^{\sharp k+2}T^{k+2} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+1}T \geq_A 0 \\ &\Longrightarrow T^\sharp(T^{\sharp k+1}T^{k+1} - (\mathcal{T}\mathcal{T}^\sharp)^{k+1})T \geq_A 0 \end{aligned}$$

hence T is a Quasi Class $A_{k+1}^\sharp$operator.

Proposition 3.2.6 *Suppose $T \in \mathcal{B}_A(\mathcal{H})$ commutes with $T^{\sharp k+n}$ for all $n \in \mathbb{N}$. Then,*

$$\textit{Quasi Class } A_k^\sharp \subseteq \textit{Quasi Class } A_{k+1}^\sharp \subseteq \cdots \subseteq \textit{Quasi Class } A_{k+n}^\sharp.$$

Proof 3.2.8 *suppose that T is a Quasi Class $A_k^\sharp$. We have*

$$T^\sharp(T^{\sharp k}T^k - (\mathcal{T}\mathcal{T}^\sharp)^k)T \geq_A 0$$

then

$$\begin{aligned} (T^\sharp T)^n(T^\sharp(T^{\sharp k}T^k - (\mathcal{T}\mathcal{T}^\sharp)^k)T) \geq_A 0 &\Longrightarrow (T^\sharp T)^n T^\sharp(T^{\sharp k}T^k)T - (T^\sharp T)^n(T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^k)T) \geq_A 0 \\ &\Longrightarrow (T^\sharp T)^{n-1}T^\sharp\mathcal{T}\mathcal{T}^{\sharp k+1}T^{k+1} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+n}T \geq_A 0 \\ &\Longrightarrow (T^\sharp T)^{n-1}T^{\sharp k+2}T^{k+2} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+n}T \geq_A 0 \\ &\Longrightarrow (T^\sharp T)^{n-2}T^{\sharp k+3}T^{k+3} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+n}T \geq_A 0 \\ &\vdots \\ &\Longrightarrow T^{\sharp k+n+1}T^{k+n+1} - T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{k+n}T \geq_A 0 \\ &\Longrightarrow T^\sharp(T^{\sharp k+n}T^{k+n} - (\mathcal{T}\mathcal{T}^\sharp)^{k+n})T \geq_A 0 \end{aligned}$$

hence T is a Quasi Class $A_{k+n}^\sharp$operator for all positive integers number n.

Proposition 3.2.7 *Let $T \in \mathcal{B}_A(\mathcal{H})$ with $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$. The following are equivalent:*

1. *$T^\sharp$ is Quasi Class $A_{k+1}^\sharp$.*
2. *$\left[T^{k+2}T^{\sharp k+2} - (TT^\sharp)^{k+2}\right]$ is A-positive operator.*

Proof 3.2.9 *Assume $T^\sharp$ is Quasi Class $A_{k+1}^\sharp$ and $T(\mathcal{N}(A)) \subset \mathcal{N}(A)$. That is,*

$$(T^\sharp)^\sharp \left(\left(T^\sharp\right)^{\sharp k+1} T^{\sharp k+1} - (T^\sharp \left(T^\sharp\right)^\sharp)^{k+1} \right) T^\sharp \geq_A 0$$

therefore we have for all $u \in \mathcal{H}$

$$\begin{aligned}
&\left\langle (T^\sharp)^\sharp \left(\left(T^\sharp\right)^{\sharp k+1} T^{\sharp k+1} - (T^\sharp \left(T^\sharp\right)^\sharp)^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &\left\langle (T^\sharp)^\sharp \left(P_{\overline{\mathcal{R}(A)}} T^{k+1} P_{\overline{\mathcal{R}(A)}} T^{\sharp k+1} - (T^\sharp P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &\left\langle P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} \left(P_{\overline{\mathcal{R}(A)}} T^{k+1} T^{\sharp k+1} - P_{\overline{\mathcal{R}(A)}} (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &\left\langle P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} \left(T^{k+1} T^{\sharp k+1} - (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &\left\langle P_{\overline{\mathcal{R}(A)}} T \left(T^{k+1} T^{\sharp k+1} - (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &\left\langle A P_{\overline{\mathcal{R}(A)}} T \left(T^{k+1} T^{\sharp k+1} - (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle \geq 0 \\
\Longrightarrow &\left\langle A T \left(T^{k+1} T^{\sharp k+1} - (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle \geq 0 \\
\Longrightarrow &\left\langle T \left(T^{k+1} T^{\sharp k+1} - (T^\sharp T)^{k+1} \right) T^\sharp u \mid u \right\rangle_A \geq 0 \\
\Longrightarrow &T^{k+2} T^{\sharp k+2} \geq_A T (T^\sharp T)^{k+1} T^\sharp \\
\Longrightarrow &T^{k+2} T^{\sharp k+2} - (T T^\sharp)^{k+2} \geq_A 0.
\end{aligned}$$

on other hand

$$\begin{aligned}
T^{k+2} T^{\sharp k+2} \geq_A (T T^\sharp)^{k+2} &\Rightarrow T (T^{k+1} T^{\sharp k+1}) T^\sharp \geq_A T (T^\sharp T)^{k+1} T^\sharp \\
&\Rightarrow P_{\overline{\mathcal{R}(A)}} T (T^{k+1} T^{\sharp k+1}) T^\sharp P_{\overline{\mathcal{R}(A)}} \geq_A P_{\overline{\mathcal{R}(A)}} T (T^\sharp T)^{k+1} T^\sharp P_{\overline{\mathcal{R}(A)}} \\
&\Rightarrow P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} (P_{\overline{\mathcal{R}(A)}} T^{k+1} P_{\overline{\mathcal{R}(A)}} T^{\sharp k+1}) T^\sharp \geq_A P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}} (T^\sharp P_{\overline{\mathcal{R}(A)}} T P_{\overline{\mathcal{R}(A)}})^{k+1} T^\sharp \\
&\Rightarrow (T^\sharp)^\sharp ((T^\sharp)^{\sharp k+1} T^{\sharp k+1}) T^\sharp \geq_A (T^\sharp)^\sharp (T^\sharp (T^\sharp)^\sharp)^{k+1} T^\sharp \\
&\Rightarrow (T^\sharp)^{\sharp k+2} T^{\sharp k+2} \geq_A (T^\sharp)^\sharp (T^\sharp (T^\sharp)^\sharp)^{k+1} T^\sharp.
\end{aligned}$$

so $T^\sharp$ is Quasi Class $A_{k+1}^\sharp$.

Theorem 3.2.3 *For $T \in \mathcal{B}_A(\mathcal{H})$, the following are equivalent:*

1. *T is a Quasi Class $A_k^\sharp$.*
2. *The inequality $(\rho^2+1) T^{\sharp k+1} T^{k+1} \geq_A 2\rho (T^\sharp T)^{k+1}$ holds for every $\rho \in \mathbb{R}$.*

Proof 3.2.10 *Assume that T is a Quasi Class $A_k^\sharp$. We consider two cases based on the parity of k.*
If k is even number,

$$\begin{aligned}
\|T^{k+1}u\|_A^4 \geq \|(TT^\sharp)^{\frac{k}{2}} Tu\|_A^4 &\Longleftrightarrow 0 \geq \|(TT^\sharp)^{\frac{k}{2}} Tu\|_A^4 - \|T^{k+1}u\|_A^4 \\
&\Longleftrightarrow \rho^2 \|T^{k+1}u\|_A^2 + 2\rho \|(TT^\sharp)^{\frac{k}{2}} Tu\|_A^2 + \|T^{k+1}u\|_A^2 \geq 0 \\
&\Longleftrightarrow (\rho^2+1) < T^{k+1}u | T^{k+1}u >_A 0 + 2\rho < (\mathcal{T}\mathcal{T}^\sharp)^{\frac{k}{2}} Tu | (\mathcal{T}\mathcal{T}^\sharp)^{\frac{k}{2}} Tu >_A \geq 0 \\
&\Longleftrightarrow \langle (\rho^2+1) T^{\sharp k+1} T^{k+1} + 2\rho (T^\sharp T)^{k+1} u | u \rangle_A \geq 0 \\
&\Longleftrightarrow (\rho^2+1) T^{\sharp k+1} T^{k+1} \geq_A 2\rho (T^\sharp T)^{k+1}.
\end{aligned}$$

If k ***is odd number,***

$$\begin{aligned}\|T^{k+1}u\|_A^4 \geq \|T^\sharp(TT^\sharp)^{\frac{k-1}{2}}Tu\|_A^4 &\Longleftrightarrow 0 \geq \|T^\sharp(TT^\sharp)^{\frac{k-1}{2}}Tu\|_A^4 - \|T^{k+1}u\|_A^4\\ &\Longleftrightarrow \rho^2\|T^{k+1}u\|_A^2 + 2\rho\|T^\sharp(TT^\sharp)^{\frac{k-1}{2}}Tu\|_A^2\\ &\quad + \|T^{k+1}u\|_A^2 \geq 0\\ &\Longleftrightarrow \rho^2 < T^{k+1}u|T^{k+1}u >_A + 2\rho < T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{\frac{k-1}{2}}Tu|T^\sharp(\mathcal{T}\mathcal{T}^\sharp)^{\frac{k-1}{2}}Tu >_A 0\\ &\Longleftrightarrow \langle(\rho^2+1)T^{\sharp k+1}T^{k+1} + 2\rho(T^\sharp T)^{k+1}u|u\rangle_A \geq 0\\ &\Longleftrightarrow (\rho^2+1)T^{\sharp k+1}T^{k+1} \geq_A 2\rho(T^\sharp T)^{k+1}.\end{aligned}$$

we conclude that $(\rho^2+1)T^{\sharp k+1}T^{k+1} \geq_A 2\rho(T^\sharp T)^{k+1} \quad \forall \rho \in \mathbb{R}$.

Proposition 3.2.8 *Consider two* T_1, T_2 *Quasi Class* $A_k^\sharp$ *operators such that* $T_1T_2 = T_2T_1 = T_2^\sharp T_1 = T_1^\sharp T_2 = 0$ *and* $T_{1,2}(\mathcal{N}(A)) \subset \mathcal{N}(A)$*. Then sum* $T_1 + T_2$ *is a Quasi Class* $A_k^\sharp$*.*

Proof 3.2.11 *Computing directly, we obtain,*

$$\begin{aligned}&(T_1+T_2)^\sharp((T_1+T_2)^{\sharp k}(T_1+T_2)^k - ((T_1+T_2)(T_1+T_2)^\sharp)^k)(T_1+T_2)\\ &= (T_1^\sharp + T_2^\sharp)(T_1^{\sharp k} + T_2^{\sharp k} + \sum_{i=1}^{k-1}\binom{i}{k}T_1^{\sharp k}T_2^{\sharp k-i})(T_1^k + T_2^k + \sum_{i=1}^{k-1}\binom{i}{k}T_1^kT_2^{k-i})\\ &- (T_1T_1^\sharp + T_1T_2^\sharp + T_2T_1^\sharp + T_2T_2^\sharp)^k)(T_1+T_2)\\ &= (T_1^\sharp + T_2^\sharp)((T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_1^k + T_1^{\sharp k}T_2^k + T_2^{\sharp k}T_2^k) - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k)(T_1+T_2)\\ &= (T_1^\sharp + T_2^\sharp)(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))(T_1+T_2)\\ &= T_1^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k\\ &+ (T_2T_2^\sharp)^k))T_1 + T_1^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))T_2\\ &+ T_2^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))T_1\\ &+ T_2^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))T_2\\ &= T_1^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))T_1\\ &+ T_2^\sharp(T_1^{\sharp k}T_1^k + T_2^{\sharp k}T_2^k - ((T_1T_1^\sharp)^k + (T_2T_2^\sharp)^k))T_2\\ &= T_1^\sharp(T_1^{\sharp k}T_1^k - ((T_1T_1^\sharp)^k)T_1 + T_2^\sharp(T_2^{\sharp k}T_2^k - (T_2T_2^\sharp)^k)T_2\\ &\geq_A 0.\end{aligned}$$

Thus $T_1 + T_2$ *is Quasi Class* $A_k^\sharp$*.*

Proposition 3.2.9 *Let* T_1 *is Quasi Class* $A_k^\sharp$ *and* T_2 *in* $\mathcal{B}_A(\mathcal{H})$ *such that* $T_1T_2 - T_2T_1 = T_1^\sharp T_2 - T_2T_1^\sharp = 0$ *and* T_1 *is Quasi Class* $A_k^\sharp$*. We have,*

1. *If* T_2 *is A-selfadjoint, then* T_2T_1 *is a Quasi Class* $A_k^\sharp$*.*
2. *If* T_2 *is A-normal, then* T_2T_1 *is a Quasi Class* $A_k^\sharp$*.*

Proof 3.2.12 *Assume that* T_1 *is Quasi Class* $A_k^\sharp$*. Then,* *1. Using the properties of A-selfadjoint,*

$$\begin{aligned}
&T_1^{\sharp k+1}T_1^{k+1} \geq_A T_1^\sharp (T_1T_1^\sharp)^k T_1\\
\Longrightarrow &T_2^{k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} \geq_A T_2^{k+1}T_1^\sharp (T_1T_1^\sharp)^k T_1 T_2^{k+1}\\
\Longrightarrow &T_2^{k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} \geq_A T_2T_1^\sharp T_2^k (T_1T_1^\sharp)^k T_2^k T_1 T_2\\
\Longrightarrow &(T_2T_1^\sharp)^{k+1}(T_1T_2)^{k+1} \geq_A T_2T_1^\sharp (T_2T_1T_1^\sharp T_2)^k T_1T_2\\
\Longrightarrow &(T_2^\sharp T_1^\sharp)^{k+1}(T_1T_2)^{k+1} \geq_A T_2^\sharp T_1^\sharp (T_2T_1T_1^\sharp T_2^\sharp)^k T_1T_2\\
\Longrightarrow &(T_1T_2)^{\sharp k+1}(T_1T_2)^{k+1} \geq_A (T_1T_2)^\sharp (T_1T_2(T_1T_2)^\sharp)^k T_1T_2
\end{aligned}$$

so T_2T_1 Quasi Class $A_k^\sharp$.
2.Let T_2 is A-normal operator that is $T_2T_2^\sharp = T_2^\sharp T_2$ and T is a Quasi Class $A_k^\sharp$.Then,

$$\langle T_1^\sharp \left[T_1^{\sharp k}T_1^k - (T_1^\sharp T_1)^k\right] T_1 u|u\rangle_A \geq 0 \quad \forall u \in \mathcal{H}$$

we get that

$$\begin{aligned}
&\langle (T_1T_2)^\sharp \left[(T_1T_2)^{\sharp k}(T_1T_2)^k - \Big((T_1T_2)^\sharp (T_1T_2)\Big)^k\right] (T_1T_2)u|u\rangle_A\\
&= \langle T_1^\sharp T_2^\sharp \left[T_2^{\sharp k}T_1^{\sharp k}T_1^kT_2^k - T_2^{\sharp k}(T_1^\sharp T_1)^k T_2^k\right] (T_1T_2)\mu|\mu\rangle_A\\
&= \langle AT_1^\sharp \left[T_1^{\sharp k}T_1^k - (T_1^\sharp T_1)^k\right] T_1T_2^{k+1}\mu|T_2^{k+1}u\rangle\\
&= \langle T_1^\sharp \left[T_1^{\sharp k}T_1^k - (T_1^\sharp T_1)^k\right] T_1T_2^{k+1}\mu|T_2^{k+1}u\rangle_A \geq 0
\end{aligned}$$

so T_2T_1 Quasi Class $A_k^\sharp$.

Proposition 3.2.10 *Let $T_1;T_2 \in \mathcal{B}_A(\mathcal{H})$ such that $T_1T_2 - T_2T_1 = T_1^\sharp T_2 - T_2T_1^\sharp = T_1T_1^\sharp T_2T_2^\sharp - T_2T_2^\sharp T_1T_1^\sharp = 0$. If T_1, T_2 are Quasi Class $A_k^\sharp$ operators,then T_1T_2 belongs to Quasi Class $A_k^\sharp$.*

Proof 3.2.13 *Suppose T_1 and T_2 are a Quasi Class $A_k^\sharp$ operators.*
If k is even number,

$$\begin{aligned}
\|(T_2T_1)^{k+1}u\|_A &= \|T_1^{k+1}T_2^{k+1}u\|_A\\
&\geq_A \|(T_1T_1^\sharp)^{\frac{k}{2}}T_1T_2^{k+1}u\|_A\\
&= \|T_2^{k+1}(T_1T_1^\sharp)^{\frac{k}{2}}T_1u\|_A\\
&\geq_A \|(T_2T_2^\sharp)^{\frac{k}{2}}T_2(T_1T_1^\sharp)^{\frac{k}{2}}T_1u\|_A\\
&= \|(T_2T_1T_1^\sharp T_2^\sharp)^{\frac{k}{2}}T_1T_2u\|_A\\
&= \|(T_2T_1(T_2T_1)^\sharp)^{\frac{k}{2}}T_1T_2u\|_A
\end{aligned}$$

so T_2T_1 Quasi Class $A_k^\sharp$.

If k is odd number,

$$\begin{aligned}\|(T_2T_1)^{k+1}u\|_A &= \|T_1^{k+1}T_2^{k+1}u\|_A \\ &\geq_A \|(T_1^\sharp T_1)^{\frac{k+1}{2}}T_2^{k+1}u\|_A \\ &= \|T_2^{k+1}(T_1^\sharp T_1)^{\frac{k+1}{2}}u\|_A \\ &\geq_A \|(T_2^\sharp T_2)^{\frac{k+1}{2}}(T_1^\sharp T_1)^{\frac{k+1}{2}}u\|_A \\ &= \|(T_2^\sharp T_2 T_1^\sharp T_1)^{\frac{k+1}{2}}u\|_A \\ &= \|(T_2^\sharp T_1^\sharp T_2 T_1)^{\frac{k+1}{2}}u\|_A \\ &= \|((T_2T_1)^\sharp T_2T_1)^{\frac{k+1}{2}}u\|_A\end{aligned}$$

so T_2T_1 Quasi Class $A_k^\sharp$.

3.3 Tensor Product of Class $A_k^\sharp$ Operators in Semi-Hilbertian Spaces

While the tensor product operation preserves many properties of its constituent operators,it is important to note that this preservation does not extend universally.This illustrates how properties are selectively preserved under tensor product.
In this section,we rigorously examine the tensor product structure,delineate its fundamental properties,and conduct a detailed analysis of its behavior within this class.We present a theorem that characterizes when the tensor product $T_1 \otimes T_2$ belongs to class $(A \otimes A')_k^\sharp$.

Theorem 3.3.1 *Let A, A' be positive operators.The following are equivalent:*

1. *T_1 and T_2 is element of Quasi Class $A_k^\sharp$ and Quasi Class $A_k'^\sharp$,respectively.*
2. *The tensor product falls within Quasi Class $(A \otimes A')_k^\sharp$.*

Proof 3.3.1 *Assume that T_1 is Quasi Class $A_k^\sharp$ and T_2 is Quasi Class $A_k'^\sharp$ operators.We obtain*

$$\begin{aligned}\left[(T_1 \otimes T_2)^\sharp\right]^{k+1}(T_1 \otimes T_2)^{k+1} &= \left(T_1^{\sharp k+1} \otimes T_2^{\sharp k+1}\right)\left(T_1^{k+1} \otimes T_2^{k+1}\right) \\ &= T_1^{\sharp k+1}T_1^{k+1} \otimes T_2^{\sharp k+1}T_2^{k+1} \\ &\geq_{A\otimes A'} T_1^\sharp(T_1T_1^\sharp)^kT_1 \otimes T_2^\sharp(T_2T_2^\sharp)^kT_2 \\ &= (T_1^\sharp \otimes T_2^\sharp)((T_1T_1^\sharp)^k \otimes (T_2T_2^\sharp)^k)(T_1 \otimes T_2) \\ &= (T_1^\sharp \otimes T_2^\sharp)(T_1T_1^\sharp \otimes T_2T_2^\sharp)^k(T_1 \otimes T_2) \\ &= (T_1^\sharp \otimes T_2^\sharp)\left[(T_1 \otimes T_2)(T_1^\sharp \otimes T_2^\sharp)\right]^k(T_1 \otimes T_2) \\ &= (T_1^\sharp \otimes T_2^\sharp)\left[(T_1 \otimes T_2)(T_1 \otimes T_2)^\sharp\right]^k(T_1 \otimes T_2).\end{aligned}$$

Consequently,$T_1 \otimes T_2$ belongs to class $(A \otimes A')_k^\sharp$.

For the converse suppose $T_1 \otimes T_2$ belongs to Quasi Class $(A \otimes A')_k^\sharp$. This yields,

$$\begin{aligned}
&\left((T_1 \otimes T_2)^\sharp\right)^{k+1} (T_1 \otimes T_2)^{k+1} \geq_{A\otimes A'} (T_1 \otimes T_2)^\sharp((T_1 \otimes T_2)(T_1 \otimes T_2)^\sharp)^k)(T_1 \otimes T_2)^\sharp \\
\Longrightarrow &\left(T_1^{\sharp k+1} \otimes T_2^{\sharp k+1}\right) (T_1^{k+1} \otimes T_2^{k+1}) \geq_{A\otimes A'} (T_1 \otimes T_2)^\sharp (T_1T_1^\sharp \otimes T_2T_2^\sharp)^k (T_1 \otimes T_2) \\
\Longrightarrow &T_1^{\sharp k+1}T_1^{k+1} \otimes T_2^{\sharp k+1}T_2^{k+1} \geq_{A\otimes A'} T_1^\sharp (T_1T_1^\sharp)^k T_1 \otimes T_2^\sharp (T_2T_2^\sharp)^k T_2 \\
\Longrightarrow &T_1^{\sharp k+1}T_1^{k+1} \otimes T_2^{\sharp k+1}T_2^{k+1} \geq_{A\otimes A'} \rho T^\sharp (T_1T_1^\sharp)^k T_1 \otimes \rho^{-1} T_2^\sharp (T_2T_2^\sharp)^k T_2
\end{aligned}$$

applying Proposition 2.3.1,we obtain $0 < \rho$ satisfying,

$$\rho T_1^{\sharp k+1}T_1^{k+1} \geq_A \rho T_1^\sharp (T_1T_1^\sharp)^k T_1 \quad \text{and} \quad \rho^{-1} T_2^{\sharp k+1}T_2^{k+1} \geq_A' \rho^{-1} T_2^\sharp (T_2T_2^\sharp)^k T_2$$

Canceling ρ and ρ^{-1} yields

$$T_1^{\sharp k+1}T_1^{k+1} \geq_A T_1^\sharp (T_1T_1^\sharp)^k T_1 \quad \text{and} \quad T_2^{\sharp k+1}T_2^{k+1} \geq_A' T_2^\sharp (T_2T_2^\sharp)^k T_2$$

as a result T_1 is Quasi Class $A_k^\sharp$ and T_2 is Quasi Class $A_k'^\sharp$.

Proposition 3.3.1 *Let T_1 and T_2 are Quasi Class $A_k^\sharp$ such that $T_1T_2 = T_2T_1$. The following statements hold:*

1. *If $T_1^\sharp T_2 - T_2T_1^\sharp = 0$ then both $T_1T_2 \otimes T_1$ and $T_1T_2 \otimes T_2$ belong to Quasi Class $(A \otimes A)_k^\sharp$.*
2. *If $T_1T_2^\sharp - T_2^\sharp T_1 = 0$ then both $T_2T_1 \otimes T_1$ and $T_2T_1 \otimes T_2$ are Quasi Class $(A \otimes A)_k^\sharp$.*

Proof 3.3.2

1. *Suppose $T_1^\sharp T_2 = T_2T_1^\sharp$ a simple calculation,*

$$\begin{aligned}
(T_1T_2 \otimes T_1)^{\sharp k+1}(T_1T_2 \otimes T_1)^{k+1} &= ((T_1T_2)^{\sharp k+1} \otimes T_1^{\sharp k+1})((T_1T_2)^{k+1} \otimes (T_1)^{k+1}) \\
&= \left(T_2^{\sharp k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} \otimes T_1^{\sharp k+1}T_1^{k+1}\right).
\end{aligned}$$

together this

$$\begin{aligned}
T_2^{\sharp k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} &\geq_A T_2^{\sharp k+1}T_1^\sharp (T_1T_1^\sharp)^k T_1 T_2^{k+1} \\
&= T_2^{\sharp k+1}T_2^{k+1}T_1^\sharp (T_1T_1^\sharp)^k T_1 \\
&\geq_A T_2^\sharp (T_2T_2^\sharp)^k T_2 T_1^\sharp (T_1T_1^\sharp)^k T_1 \\
&= T_2^\sharp (T_2T_2^\sharp)^k T_2 T_1^\sharp (T_1T_1^\sharp)^k T_1 \\
&= (T_2T_1)^\sharp (T_2T_1(T_2T_1)^\sharp)^k (T_2T_1).
\end{aligned}$$

thus

$$\begin{cases}
T_2^{\sharp k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} \geq_A (T_2T_1)^\sharp (T_2T_1(T_2T_1)^\sharp)^k T_2T_1 \geq_A 0. \\
\text{and} \\
T_1^{\sharp k+1}T_1^{k+1} \geq_A T_1^\sharp (T_1T_1^\sharp)^k T_1 \geq_A 0.
\end{cases}$$

yields that

$$\left(T_2^{\sharp k+1}T_1^{\sharp k+1}T_1^{k+1}T_2^{k+1} \otimes T_1^{\sharp k+1}T_1^{k+1}\right) \geq_A (T_2T_1)^\sharp (T_2T_1(T_2T_1)^\sharp)^k T_2T_1 \otimes T_1^\sharp (T_1T_1^\sharp)^k T_1.$$

so

$$(T_1T_2 \otimes T_1)^{\sharp k+1}(T_1T_2 \otimes T_1)^{k+1} \geq_A (T_1T_2 \otimes T_1)^\sharp \left((T_1T_2 \otimes T_1)(T_1T_2 \otimes T_1)^\sharp\right)^k (T_1T_2 \otimes T_1).$$

Consequently $(T_1T_2 \otimes T_1)$ *belongs to Quasi Class* $(A \otimes A)_k^\sharp$.
In the same way,we may deduce the Quasi Class $(A \otimes A)_k^\sharp$ *of* $T_1T_2 \otimes T_2$.

Chapter 4

Class $A_k^\star$ and Quasi Class $A_k^\star$ Operators

4.1 introduction

In this chapter,we examine the Hilbert space analogue of the class introduced in Chapter 2 We present examples,observations,and foundational results.This classes that develops and expands upon earlier studies conducted by several researchers([27],[29]).

4.2 Class $A_k^\star$ Operators

Definition 4.2.1 *Let $T \in B(\mathcal{H})$. We say that T a Class $A_k^\star$ if there exists a positive integer k such that*

$$T^{\star k}T^k \geq (TT^\star)^k \qquad i.e: \qquad \left|T^k\right|^2 - |T^\star|^{2k} \geq 0$$

Remark 4.2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. The following observations hold:*

1. *If T is Class $A_1^\star$,then T is a hyponormal.*
2. *If T,$T^\star$ are Class $A_1^\star$,then T is a normal.*
3. *If T is Class $A_2^\star$,then T is $\star$-Class A.*

Theorem 4.2.1 *Let $T \in \mathcal{B}(\mathcal{H})$. Then,*

The operator T is a Class $A_k^\star$ operator if and only if $\begin{cases} \|T^k v\| \geq \| (TT^\star)^{\frac{k}{2}} v\|, & \textit{if } k \textit{ even} \\ \|T^k v\| \geq \|T^\star (TT^\star)^{\frac{k-1}{2}} v\|, & \textit{if } k \textit{ odd} \end{cases}$

Proof 4.2.1 *Assume that T is Class $A_k^\star$*

If k even,

$$\begin{aligned}\langle |T^\star|^{2k}\, v|v\rangle &= \langle\underbrace{(TT^\star)\cdots(TT^\star)}_{k\ time}\, v|v\rangle\\ &= \langle\underbrace{(TT^\star)\cdots(TT^\star)}_{k-1\ time}\, v|TT^\star v\rangle\\ &\ \vdots\\ &= \langle\underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time}\, v|\underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time}\, v\rangle\\ &= \|\,(TT^\star)^{\frac{k}{2}}\, v\|^2.\end{aligned}$$

other hand,

$$\begin{aligned}\langle \left|T^k\right|^2 v|v\rangle &= \left\langle T^k v|T^k v\right\rangle\\ &= \|T^k v\|^2.\end{aligned}$$

as a result,

$$\begin{aligned}\langle \left[\left|T^k\right|^2 - |T^\star|^{2k}\right] v|v\rangle \geq 0 &\Longleftrightarrow \left\langle (T^\star)^k\, T^k v|v\right\rangle \geq \left\langle (TT^\star)^k\, v|v\right\rangle\\ &\Longleftrightarrow \|T^k v\| \geq \|\,(TT^\star)^{\frac{k}{2}}\, v\|\end{aligned}$$

If k odd,

$$\begin{aligned}\langle |T^\star|^{2k}\, v|v\rangle =&\langle\underbrace{(TT^\star)\cdots(TT^\star)}_{k-1\ time}\, v|TT^\star v\rangle\\ =&\langle\underbrace{(TT^\star)\cdots(TT^\star)}_{k-1\ time}\, v|TT^\star v\rangle\\ &\ \vdots\\ =&\langle T^\star\underbrace{TT^\star\cdots TT^\star}_{\frac{k-1}{2}\ time}\, v|T^\star\underbrace{TT^\star\cdots TT^\star}_{\frac{k-1}{2}\ time}\, v\rangle\\ =&\|T^\star\,(TT^\star)^{\frac{k-1}{2}}\, v\|^2.\end{aligned}$$

we conclude,

$$\begin{aligned}\langle \left[\left|T^k\right|^2 - |T^\star|^{2k}\right] v|v\rangle \geq 0 &\Longleftrightarrow \left\langle (T^\star)^k\, T^k v|v\right\rangle \geq \left\langle (TT^\star)^k\, v|v\right\rangle\\ &\Longleftrightarrow \|T^k v\| \geq \|T^\star\,(TT^\star)^{\frac{k-1}{2}}\, v\|.\end{aligned}$$

Proof is complete.

Proposition 4.2.1 *Let $T \in \mathcal{B}(\mathcal{H})$ such that $TT^{\star k+n} = T^{\star k+n}T$ for all positive integers number n.Then,*
Class $A_k^\star \subseteq$ Class $A_{k+n}^\star$ $\quad \forall n \in \mathbb{N}$.

Proof 4.2.2 *Suppose that T is Class $A_k^\star$.Then,*

$$\begin{aligned}TT^\star\left(T^{\star k}T^k - (TT^\star)^k\right) =&T^{\star k+1}TT^k - (TT^\star)^{k+1}\\ =&(T^{\star k}T^k - (TT^\star)^k)TT^\star\end{aligned}$$

so

$$TT^\star\left(T^{\star k}T^k-(TT^\star)^k\right)\geq 0$$

we have

$$\begin{aligned}TT^\star\left(T^{\star k}T^k-(TT^\star)^k\right)\geq 0 &\Longrightarrow TT^{\star k+1}T^k-TT^\star(TT^\star)^k\geq 0\\ &\Longrightarrow T^{\star k+1}T^{k+1}-(TT^\star)^{k+1}\geq 0.\end{aligned}$$

thus Class $A_k^\star\subseteq$ *Class* $A_{k+1}^\star$
Assume that Class $A_k^\star\subseteq$ *Class* $A_{k+n}^\star$ *i.e:*

$$T^{\star k}T^k-(TT^\star)^k\geq 0\Longrightarrow T^{\star k+n}T^{k+n}-(TT^\star)^{k+n}\geq 0.$$

now we prove that Class $A_k^\star\subseteq$ *Class* $A_{k+n+1}^\star$ *i.e:*

$$T^{\star k}T^k-(TT^\star)^k\geq 0\Longrightarrow T^{\star k+n+1}T^{k+n+1}-(TT^\star)^{k+n+1}\geq 0.$$

we have

$$\begin{aligned}TT^\star\geq 0\wedge T^{\star k}T^k-(TT^\star)^k\geq 0 &\Longrightarrow TT^\star\geq 0\wedge T^{\star k+n}T^{k+n}-(TT^\star)^{k+n}\geq 0\\ &\Longrightarrow T^{\star k+n+1}T^{k+n+1}-(TT^\star)^{k+n+1}\geq 0.\end{aligned}$$

hence Class $A_k^\star\subseteq$ *Class* $A_{k+n+1}^\star$.
We conclude that Class $A_k^\star\subseteq$ *Class* $A_{k+n}^\star$,*for all* $n\in\mathbb{N}$.

Theorem 4.2.2 *Let* $T\in\mathcal{B}(\mathcal{H})$. *Then,*

$$T \text{ is Class } A_k^\star \text{ if only if } \quad \rho^2T^kT^k+2\rho\,(TT^\star)^k+T^{\star k}T^k\geq 0,\forall\rho\in\mathbb{R}$$

Proof 4.2.3 *Assume that* $T\in\mathcal{B}(\mathcal{H})$.
If k is even number,

$$\begin{aligned}\left\|T^ku\right\|^4\geq\left\|(TT^\star)^{\frac{k}{2}}u\right\|^4 &\Longleftrightarrow 0\geq\left\|(TT^\star)^{\frac{k}{2}}Tu\right\|^4-\left\|T^ku\right\|^4\\ &\Longleftrightarrow\rho^2\left\|T^ku\right\|^2+2\rho\left\|(TT^\star)^{\frac{k}{2}}Tu\right\|^2+\left\|T^{k+1}u\right\|^2\geq 0\\ &\Longleftrightarrow\rho^2<T^ku|T^ku>_A+2\rho<(TT^\star)^{\frac{k}{2}}u|\,(TT^\star)^{\frac{k}{2}}u>\\ &\qquad+<T^ku|T^ku>_A\geq 0\\ &\Longleftrightarrow<\left(\rho^2T^{\star k}T^k+2\rho\,(TT^\star)^kT+T^{\star k}T^k\right)u|u>\geq 0\\ &\Longleftrightarrow\rho^2T^{\star k}T^k+2\rho\,(TT^\star)^k+T^{\star k}T^k\geq 0.\end{aligned}$$

If k is odd number,

$$\begin{aligned}\left\|T^k u\right\|^4 \geq \left\|T^\star (TT^\star)^{\frac{k-1}{2}} u\right\|^4 &\Longleftrightarrow 0 \geq \left\|T^\star (TT^\star)^{\frac{k-1}{2}} Tu\right\|^4 - \left\|T^k u\right\|^4 \\ &\Longleftrightarrow \rho^2 \left\|T^k u\right\|^2 + 2\rho \left\|T^\star (TT^\star)^{\frac{k-1}{2}} u\right\|^2 + \left\|T^k u\right\|^2 \geq 0 \\ &\Longleftrightarrow \rho^2 < T^k u | T^k u >_A + 2\rho < T^\star (TT^\star)^{\frac{k-1}{2}} u | T^\star (TT^\star)^{\frac{k-1}{2}} u > \\ &\quad + < T^k u | T^k u > \geq 0 \\ &\Longleftrightarrow < \left(\rho^2 T^{\star k} T^k + 2\rho (TT^\star)^k + T^{\star k} T^k\right) u | u > \geq 0 \\ &\Longleftrightarrow \rho^2 T^{\star k} T^k + 2\rho (TT^\star)^k + T^{\star k} T^k \geq 0.\end{aligned}$$

The desired result is attained.

Proposition 4.2.2 *Let $T_1, T_2 \in \mathrm{B}(\mathcal{H})$.If T_2 is a Class $A_k^\star$ operator and T_1 is unitary equivalent to T_2 then T_1 is a Class $A_k^\star$ operator.*

Proof 4.2.4 *T_2 is Class $A_k^\star$ operator.Then,*

$$\begin{aligned}\left|T_2^k\right|^2 - |T_2^\star|^{2k} \geq 0 &\Longrightarrow \left|(UT_1U^\star)^k\right|^2 - |(UT_1U^\star)^\star|^{2k} \geq 0 \\ &\Longrightarrow \left|UT_1^k U^\star\right|^2 - |UT_1^\star U^\star|^{2k} \geq 0 \\ &\Longrightarrow U \left|T_1^k\right|^2 U^\star - U \left|T_1^\star\right|^{2k} U^\star \geq 0 \\ &\Longrightarrow U \left(\left|T_1^k\right|^2 - |T_1^\star|^{2k}\right) U^\star \geq 0 \\ &\Longrightarrow \left|T_1^k\right|^2 - |T_1^\star|^{2k} \geq 0\end{aligned}$$

hence T_1 is Class ${A_k}^\star$ operator.

Proposition 4.2.3 *Let T_1, T_2 are Class $A_k^\star$ operator such that $T_1T_2 = T_1T_2 = T_2^\star T_1 = T_1^\star T_2 = 0$,then $T_1 + T_2$ is Class $A_k^\star$ operator.*

Proof 4.2.5 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are Class $A_k^\star$ operator. Then,*

$$\begin{aligned}
\left|(T_1+T_2)^k\right|^2 - |(T_1+T_2)^\star|^{2k} &= \left|(T_1+T_2)^k\right|^2 - |(T_1+T_2)^\star|^{2k} \\
&= \left|T_1^k + T_2^k + \sum_{i=1}^{k-1}\binom{i}{k-1} T_1^k T_2^{k-i}\right|^2 \\
&- ((T_1+T_2)(T_1+T_2)^\star)^k \\
&= \left|T_1^k + T_2^k + \sum_{i=1}^{k-1}\binom{i}{k-1} T_1^k T_2^{k-i}\right|^2 \\
&- (T_1T_1^\star + T_2T_1^\star + T_2T_2^\star + T_1T_2^\star)^k \\
&= \left|T_1^k + T_2^k\right|^2 - (T_1T_1^\star + T_2T_2^\star)^k \\
&= \left(T_1^{\star k} + T_2^{\star k}\right)\left(T_1^k + T_2^k\right) \\
&- \left((T_1T_1^\star)^k + (T_2T_2^\star)^k + \sum_{i=1}^{k-1}\binom{i}{k-1}(T_1T_1^\star)^k (T_2T_2^\star)^{k-i}\right) \\
&= T_1^{\star k}T_1^k + T_2^{\star k}T_1^k + T_2^{\star k}T_2^k + T_1^{\star k}T_2^k - (T_1T_1^\star)^k - (T_2T_2^\star)^k \\
&= T_1^{\star k}T_1^k + T_2^{\star k}T_2^k - (T_1T_1^\star)^k - (T_2T_2^\star)^k \geq 0.
\end{aligned}$$

so $T_1 + T_2$ is Class $A_k^\star$ operator.

Example 4.2.1 *Let* $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$
then

$$\begin{aligned}
\left|T_1^k\right|^2 - |T_1^\star|^{2k} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^k - \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)^k \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0
\end{aligned}$$

thus T_1 is Class $A_k^\star$ operator.
and $T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$

then

$$\left|T_2^k\right|^2 - |T_2^\star|^{2k} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^k - \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right)^k$$

$$= \begin{pmatrix} \frac{(-1)^k+1}{2} & 0 & \frac{-(-1)^k+1}{2} \\ 0 & 0 & 0 \\ \frac{-(-1)^k+1}{2} & 0 & \frac{(-1)^k+1}{2} \end{pmatrix}\begin{pmatrix} \frac{(-1)^k+1}{2} & 0 & \frac{-(-1)^k+1}{2} \\ 0 & 0 & 0 \\ \frac{-(-1)^k+1}{2} & 0 & \frac{(-1)^k+1}{2} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^k$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0.$$

hence T_2 is Quasi Class $A_k^\star$ operator.

And $\mathrm{T}_1\mathrm{T}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$

$$\left|(T_1T_2)^k\right|^2 - |(T_1T_2)^\star|^{2k} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}^k \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^k - \left[\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right]^k$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq 0.$$

so T_1T_2 is not Class $A_k^\star$ operator.

Proposition 4.2.4 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are doubly commute.If T_2 is normal operator and T_1 is Class $A_k^\star$ operator,then T_1T_2 is Class $A_k^\star$ operator.*

Proof 4.2.6 *T_1 is Class $A_k^\star$ operator and T_2 is a normal operator.Then,*

$$T_1^{\star k}T_1^k - (T_1T_1^\star)^k \geq 0 \quad \text{and} \quad T_2T_2^\star = T_2^\star T_2$$

after our calculation

$$\begin{aligned} &(T_1T_2)^{\star k}(T_1T_2)^k - (T_1T_2(T_1T_2)^\star)^k \\ &= \left(T_2^{\star k}T_1^{\star k}\right)\left(T_1^kT_2^k\right) - (T_2^\star(T_1T_1^\star)T_2)^k \\ &= T_2^{\star k}\left(T_1^{\star k}T_1^k\right)T_2^k - T_2^{\star k}(T_1T_1^\star)^kT_2^k \\ &= T_2^{\star k}\left(T_1^{\star k}T_1^k - (T_1T_1^\star)^k\right)T_2^k \geq 0. \end{aligned}$$

therefore T_1T_2 is Class $A_k^\star$.

Proposition 4.2.5 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are doubly commute.If T_1 and T_2 are class $\mathrm{A}_k^\star$ operator,then T_1T_2 is class $\mathrm{A}_k^\star$ operator.*

Proof 4.2.7 *Let T_1 and T_2 are Class $A_k^\star$ operator.Then,*

$$T_1^{\star k}T_1^k - (T_1T_1^\star)^k \geq 0 \quad \text{and} \quad T_2^{\star k}T_2^k - (T_2T_2^\star)^k \geq 0$$

therefore

$$\begin{aligned}(T_1T_2)^{\star k}(T_1T_2)^k &= T_2^{\star k}\left(T_1^{\star k}T_1^k\right)T_2^k \\ &\geq T_2^{\star k}(T_1T_1^\star)^k T_2^k \\ &= T_2^{\star k}T_2^k(T_1T_1^\star)^k \\ &\geq (T_2T_2^\star)^k(T_1T_1^\star)^k \\ &= (T_1T_2(T_1T_2)^\star)^k .\end{aligned}$$

thus T_1T_2 is Class $A_k^\star$ operator.

Lemma 4.2.1 *(Holder-McCarthy's inequality) (see[5])Let $T \geq 0$.The following assertions hold:*
(1) $\langle T^r v, v\rangle \geq \langle Tv, v\rangle^r \|v\|^{2(1-r)}, r\rangle 1$ and for all $v \in \mathcal{H}$.
(2) $\langle T^r v, v\rangle \leq \langle Tv, v\rangle^r \|v\|^{2(1-r)}, 0 \leq r \leq 1$ and for all $v \in \mathcal{H}$.

Proof 4.2.8 *let $T \geq 0$,by the spectral theorem.It has a spectral decomposition*

$$T = \sum_i \langle \lambda_i e_i, e_i\rangle$$

where $\lambda_i \geq 0$ are eigenvalues and $\{e_i\}$ from an orthonormal basis.Then,

$$T^r = \sum_i \langle \lambda_i^r e_i, e_i\rangle$$

expand v in eigenbasis $v = \sum_i c_i e_i$ with $\sum_i |c_i|^2 = 1$.Compute

$$\langle Tv, v\rangle = \sum_i \lambda_i |c_i|^2 \quad \text{and} \quad \langle T^r v, v\rangle = \sum_i \lambda_i^r |c_i|^2$$

the function $f(t) = t^r$ is concave $1 \geq r \geq 0$ by jensen's inequalty for the concave function f:

$$\left(\sum_i \lambda_i |c_i|^2\right)^r \geq \sum_i \lambda_i^r |c_i|^2$$

this directly gives

$$(\langle Tv, v\rangle)^r \geq \langle T^r v, v\rangle$$

assume $v \neq 0$,let $v = \frac{v}{\|v\|}$ so $\|v\| = 1$. therefore,

$$\left(\langle T\frac{v}{\|v\|}, \frac{v}{\|v\|}\rangle\right)^r \geq \langle T^r\frac{v}{\|v\|}, \frac{v}{\|v\|}\rangle$$

hence

$$\left(\frac{1}{\|v\|^2}\langle Tv, v\rangle\right)^r \geq \frac{1}{\|v\|^2}\langle T^r v, v\rangle$$

then

$$\langle Tv, v\rangle^r \|v\|^{2(1-r)} \geq \langle T^r v, v\rangle \quad \forall v \in \mathcal{H}$$

(2) if $r' \geq 1$ onpose $r = \frac{1}{r'}$ and $\mathcal{J} = T^r$. Then,

$$\langle Tv, v\rangle^r \|v\|^{2(1-r)} \geq \langle T^r v, v\rangle \quad \forall v \in \mathcal{H}$$

therefore

$$\langle \mathcal{J}^{\frac{1}{r}} v, v\rangle^r \|v\|^{2(1-r)} \geq \langle \mathcal{J} v, v\rangle \quad \forall v \in \mathcal{H}$$

hence

$$\langle \mathcal{J}^{r'} v, v\rangle^{\frac{1}{r'}} \|v\|^{2(1-\frac{1}{r'})} \geq \langle \mathcal{J} v, v\rangle \quad \forall v \in \mathcal{H}$$

then

$$\langle \mathcal{J}^{r'} v, v\rangle \|v\|^{2(r'-1)} \geq \langle \mathcal{J} v, v\rangle^{r'} \quad \forall v \in \mathcal{H}$$

therefore

$$\langle \mathcal{J}^{r'} v, v\rangle \geq \langle \mathcal{J} v, v\rangle^{r'} \|v\|^{2(1-r')} \quad \forall v \in \mathcal{H}$$

Proposition 4.2.6 *Let T is Class $A_k^\star$.*
If T is Class $A_k^\star$ operator then $\|T^k v\| \geq \|T^\star v\|^k \|v\|^{1-k} \quad \forall v \in \mathcal{H}$.

Proof 4.2.9 *Let T is Class $A_k^\star$. Then,*

$$T^{\star k} T^k - (TT^\star)^k \geq 0$$

and let $v \in \mathcal{H}$ we have

$$\begin{aligned}\langle (TT^\star)^k v | v\rangle &\geq \langle TT^\star v | v\rangle^k \|v\|^{2(1-k)} \\ &= \langle T^\star v | T^\star v\rangle^k \|v\|^{2(1-k)} \\ &= \|T^\star v\|^{2k} \|v\|^{2(1-k)}\end{aligned}$$

on other hand

$$\langle T^{\star k} T^k v | v\rangle = \|T^k v\|^2$$

therefore

$$\|T^k v\| \geq \|T^\star v\|^k \|v\|^{1-k}$$

4.3 Quasi Class $A_k^\star$ Operator

Definition 4.3.1 *An operator $T \in \mathcal{B}(\mathcal{H})$ T is called a Quasi Class $A_k^\star$ when there is a positive integer k for which*

$$T^\star \left(\left|T^k\right|^2 - |T^\star|^{2k} \right) T \geq 0 \quad \text{i.e:} \quad T^{\star k+1} T^{k+1} \geq T^\star \left((TT^\star)^k \right) T$$

Theorem 4.3.1 *Let $T \in \mathcal{B}(\mathcal{H})$.*
T is Quasi Class $A_k^\star$ for some positive integer k iff T satisfying the following condition:

$$\begin{cases} \|T^{k+1} v\|^2 \geq \| (TT^\star)^{\frac{k}{2}} Tv\|, & \text{if } k \text{ even number} \\ \|T^{k+1} v\|^2 \geq \| (T^\star T)^{\frac{k+1}{2}} v\|, & \text{if } k \text{ odd number} \end{cases}$$

Proof 4.3.1 *If k even number,*

$$\begin{aligned}
\left\langle T^\star (TT^\star)^k Tu|u \right\rangle &= \langle \underbrace{(TT^\star)\cdots(TT^\star)}_{k\ time} Tu|Tu\rangle \\
&= \langle \underbrace{(TT^\star)\cdots(TT^\star)}_{k-1\ time} Tu|TT^\star Tu\rangle \\
&\vdots \\
&= \langle \underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time} Tu|\underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time} Tu\rangle \\
&= \left\langle (TT^\star)^{\frac{k}{2}} Tu|\, (TT^\star)^{\frac{k}{2}} Tu \right\rangle \\
&= \left\| (TT^\star)^{\frac{k}{2}} Tu \right\|^2 .
\end{aligned}$$

other hand that

$$\begin{aligned}
\left\langle (T^\star)^{k+1} T^{k+1}u|u \right\rangle &= \left\langle T^{k+1}u|T^{k+1}u \right\rangle \\
&= \left\| T^{k+1}u \right\|^2 .
\end{aligned}$$

therefore

$$\begin{aligned}
\left\langle T^\star \left[(T^\star)^k T^k - (TT^\star)^k\right] Tu|u \right\rangle \geq 0 &\Longleftrightarrow \left\langle (T^\star)^{k+1} T^{k+1}u|u \right\rangle \geq \left\langle T^\star (TT^\star)^k Tu|u \right\rangle \\
&\Longleftrightarrow \left\| T^{k+1}u \right\| \geq \left\| (TT^\star)^{\frac{k}{2}} Tu \right\| .
\end{aligned}$$

If k odd number,

$$\begin{aligned}
\left\langle T^\star (TT^\star)^k Tu|u \right\rangle =& \langle \underbrace{(TT^\star)\cdots(TT^\star)}_{k\ time} Tu|Tu\rangle \\
=& \langle \underbrace{(TT^\star)\cdots(TT^\star)}_{k-1\ time} Tu|TT^\star Tu\rangle \\
&\vdots \\
&= \langle T^\star \underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time} Tu|T^\star \underbrace{TT^\star\cdots TT^\star}_{\frac{k}{2}\ time} Tu\rangle \\
&= \left\langle T^\star (TT^\star)^{\frac{k-1}{2}} u|T^\star (TT^\star)^{\frac{k-1}{2}} Tu \right\rangle \\
&= \left\| T^\star (TT^\star)^{\frac{k-1}{2}} Tu \right\|^2 \\
&= \left\| (T^\star T)^{\frac{k+1}{2}} u \right\|^2 .
\end{aligned}$$

therefore

$$\left\langle T^\star \left[(T^\star)^k T^k - (TT^\star)^k \right] Tu|u \right\rangle \geq 0 \Longleftrightarrow \left\langle (T^\star)^{k+1} T^{k+1} u|u \right\rangle \geq \left\langle T^\star (TT^\star)^k Tu|u \right\rangle$$
$$\Longleftrightarrow \left\| T^{k+1} u \right\| \geq \left\| (T^\star T)^{\frac{k+1}{2}} u \right\|.$$

The proof is complete.

Proposition 4.3.1 *If T is Quasi Class $A_k^\star$ and it has dense range.Then T is Class $A_k^\star$.*

Proof 4.3.2 *Let T is Quasi Class $A_k^\star$.Then,*

$$\begin{cases} \|T^{k+1}v\| \geq \| (TT^\star)^{\frac{k}{2}} Tv\|, & \text{if } k \text{ even number} \\ \|T^{k+1}v\| \geq \| (T^\star T)^{\frac{k+1}{2}} v\|, & \text{if } k \text{ odd number} \end{cases}$$

so there exists a sequence (x_n) in $\mathcal{H}$ such that $T(x_n) \longrightarrow v$ as $n \longrightarrow \infty$.
therefore,for all $n \in \mathbb{N}$

$$\begin{cases} \|T^{k+1}x_n\|^2 \geq \| (TT^\star)^{\frac{k}{2}} Tx_n\|, & \text{if } k \text{ even number.} \\ \|T^{k+1}x_n\|^2 \geq \| (T^\star T)^{\frac{k+1}{2}} x_n\|, & \text{if } k \text{ odd number.} \end{cases}$$

hence

$$\begin{cases} \| \lim\limits_{n\to\infty} T^{k+1}x_n\| \geq \| \lim\limits_{n\to\infty} (TT^\star)^{\frac{k}{2}} Tx_n\|, & \text{if } k \text{ even number.} \\ \| \lim\limits_{n\to\infty} T^{k+1}x_n\| \geq \| \lim\limits_{n\to\infty} T^\star (TT^\star)^{\frac{k-1}{2}} Tx_n\|, & \text{if } k \text{ odd number.} \end{cases}$$

Then

$$\begin{cases} \|T^k v\| \geq \| (TT^\star)^{\frac{k}{2}} v\|, & \text{if } k \text{ even number.} \\ \|T^k v\| \geq \|T^\star (TT^\star)^{\frac{k-1}{2}} v\|, & \text{if } k \text{ odd number.} \end{cases}$$

hence T is Class $A_k^\star$.

Proposition 4.3.2 *Let $T \in \mathcal{B}(\mathcal{H})$ such that $TT^{\star k+n} = T^{\star k+n}T$ for all positive integers number n.Then,*
Quasi Class $A_k^\star \subseteq$ Quasi Class $A_{k+1}^\star \subseteq \ldots \subseteq$ Quasi Class $A_{k+n}^\star$.

Proof 4.3.3 *Suppose that T is Quasi Class $A_k^\star$.Then,*

$$T^\star \left(T^{\star k} T^k - (TT^\star)^k \right) T \geq 0$$

we have,

$$(TT^\star)^n T^\star \left(T^{\star k} T^k - (TT^\star)^k \right) T \geq_A 0 \Longrightarrow (TT^\star)^{n-1} T^\star \left(TT^\star T^{\star k} T^k - TT^\star (TT^\star)^k \right) T \geq 0$$
$$\Longrightarrow (TT^\star)^{n-1} T^\star \left(TT^{\star k+1} T^k - (TT^\star)^{k+1} \right) T \geq 0$$
$$\Longrightarrow (TT^\star)^{n-1} T^\star \left(T^{\star k+1} T^{k+1} - (TT^\star)^{k+1} \right) T \geq 0$$
$$\vdots$$
$$\Longrightarrow T^\star \left(T^{\star k+n} T^{k+n} - (TT^\star)^{k+n} \right) T \geq 0.$$

hence T is a Quasi Class $A_{k+n}^\star$ operator for all positive integers number n.

Theorem 4.3.2 *Let $T \in \mathcal{B}(\mathcal{H})$.*
T is Class $A_k^\star$ if and only if $(\rho^2+1)T^{k+1}T^{k+1} + 2\rho T^\star (TT^\star)^k T \geq 0, \forall \rho \in \mathbb{R}$.

Proof 4.3.4 *If k is even number,*

$$\begin{aligned}
\|T^{k+1}v\|^4 \geq \| (TT^\star)^{\frac{k}{2}} Tv\|^4 &\Longleftrightarrow 0 \geq \| (TT^\star)^{\frac{k}{2}} Tv\|^4 - \|T^{k+1}v\|^4 \\
&\Longleftrightarrow \rho^2\|T^{k+1}v\| + 2\rho\| (TT^\star)^{\frac{k}{2}} Tv\| + \|T^{k+1}v\| \geq 0 \\
&\Longleftrightarrow (\rho^2+1)T^{\star k+1}T^{k+1} + 2\rho T^\star (TT^\star)^k T \geq 0.
\end{aligned}$$

If k is odd number,

$$\begin{aligned}
\|T^{k+1}v\|^4 \geq \|T^\star (TT^\star)^{\frac{k-1}{2}} Tv\|^4 &\Longleftrightarrow 0 \geq \|T^\star (TT^\star)^{\frac{k-1}{2}} Tv\|^4 - \|T^{k+1}v\|^4 \\
&\Longleftrightarrow (\rho^2+1)\|T^{k+1}v\| + 2\rho\|T^\star (TT^\star)^{\frac{k-1}{2}} Tv\|^2 \geq 0 \\
&\quad + \langle T^{k+1}v|T^{k+1}v\rangle \geq 0 \\
&\Longleftrightarrow (\rho^2+1)T^{\star k+1}T^{k+1} + 2\rho T^\star (TT^\star)^k T \geq 0.
\end{aligned}$$

Proposition 4.3.3 *If T is a Quasi Class $A_k^\star$ operator then,*

$$\|T^{k+1}v\| \geq \|T^\star Tv\|^k \|Tv\|^{1-k} \quad \forall v \in \mathcal{H}.$$

Proof 4.3.5 *Let T is Quasi Class $A_k^\star$. Then,*

$$\begin{aligned}
\langle T^\star (TT^\star)^k Tv|v\rangle &= \langle (TT^\star)^k Tv|Tv\rangle \\
&\geq \langle TT^\star Tv|Tv\rangle^k \|Tv\|^{2(1-k)} \\
&= \langle T^\star Tv|T^\star Tv\rangle^k \|Tv\|^{2(1-k)} \\
&= \|T^\star Tv\|^{2k}\|Tv\|^{2(1-k)}.
\end{aligned}$$

and

$$\begin{aligned}
\langle T^{\star k+1}T^{k+1}v|v\rangle &= \langle T^{k+1}v|T^{k+1}v\rangle \\
&= \|T^{k+1}v\|^2.
\end{aligned}$$

as a result,

$$\|T^{k+1}v\| \geq \|T^\star Tv\|^k \|Tv\|^{1-k}.$$

Theorem 4.3.3 *If T is Quasi Class $A_k^\star$ and it doesn't have dense range. Then,*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$$

in which $T_1 = T_{\overline{T(\mathcal{H})}}$ is a Class $A_k^\star$.

Proof 4.3.6 *Since T doesn't have dense range,*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$$

we shall show that T_1 is an Class $A_k^\star$ operator.Since T is Quasi Class $A_k^\star$,there is a positive real number k such that

$$T^\star\left(T^{\star k}T^k-(TT^\star)^k\right)T\geq 0$$

hence

$$<T^\star\left(T_1^{\star k}T_1^k-(T_1T_1^\star)^k\right)Tv,v>\geq 0\quad \forall v\in\mathcal{H}$$

it follows that

$$\langle\left(T_1^{\star k}T_1^k-(T_1T_1^\star)^k\right)Tv,Tv\rangle\geq 0\quad \forall v\in\mathcal{H}.$$

Consequently T_1 is a Class $A_k^\star$.

4.4 Tensor Product of Class $A_k^\star$ operators

Theorem 4.4.1 *Let $T_1\in\mathcal{B}(\mathcal{H}_1)$ and $T_2\in\mathcal{B}(\mathcal{H}_2)$. Then,*

T_1 and T_2 are a Quasi Class $A_k^\star$ if only if $T_1\otimes T_2$ is a Quasi Class $A_k^\star$.

Proof 4.4.1 *Suppose that T_1 and T_2 are a Quasi Class $A_k^\star$ operators. We've,*

$$T_1^{\star k+1}T_1^{k+1}\geq T_1^\star(T_1T_1^\star)^kT_1\quad \textit{and}\quad T_2^{\star k+1}T_2^{k+1}\geq T_2^\star(T_2T_2^\star)^kT_2$$

then

$$\begin{aligned}(T_1\otimes T_2)^{\star k+1}(T_1\otimes T_2)^{k+1}&=(T_1^\star\otimes T_2^\star)^{k+1}(T_1\otimes T_2)^{k+1}\\&=\left(T_1^{\star k+1}\otimes T_2^{\star k+1}\right)\left(T_1^{k+1}\otimes T_2^{k+1}\right)\\&=T_1^{\star k+1}T_1^{k+1}\otimes T_2^{\star k+1}T_2^{k+1}\\&\geq T_1^\star(T_1T_1^\star)^kT_1\otimes T_2^\star(T_2T_2^\star)^kT_2\\&=(T_1^\star\otimes T_2^\star)(T_1T_1^\star)^k\otimes T_2^\star(T_2T_2^\star)^k(T_1\otimes T_2)\\&=(T_1\otimes T_2)^\star(T_1T_1^\star\otimes T_2T_2^\star)^k(T_1\otimes T_2)\\&=(T_1\otimes T_2)^\star((T_1\otimes T_2)(T_1^\star\otimes T_2^\star))^k(T_1\otimes T_2)\\&=(T_1\otimes T_2)^\star((T_1\otimes T_2)(T_1\otimes T_2)^\star)^k(T_1\otimes T_2).\end{aligned}$$

so $T_1\otimes T_2$ is a Quasi Class $A_k^\star$.
Suppose $T_1\otimes T_2$ is a Quasi Class $A_k^\star$. We've,

$$\begin{aligned}&(T_1\otimes T_2)^{\star k+1}(T_1\otimes T_2)^{k+1}\geq(T_1\otimes T_2)^\star((T_1\otimes T_2)(T_1\otimes T_2)^\star)^{k+1}(T_1\otimes T_2)\\&\Longrightarrow\left(T_1^{\star k+1}T_1^{k+1}\right)\otimes\left(T_2^{\star k+1}T_2^{k+1}\right)\geq T_1^\star(T_1T_1^\star)^kT_1\otimes T_2^\star(T_2T_2^\star)^kT_2\\&\Longrightarrow\left(T_1^{\star k+1}T_1^{k+1}\right)\otimes\left(T_2^{\star k+1}T_2^{k+1}\right)\geq v(T_1T_1^\star)^k\otimes v^{-1}(T_2T_2^\star)^k.\end{aligned}$$

we deduce from proposition 2.3.1 there exists a constant $0<\rho$ such that

$$\begin{cases}\rho T_1^{\star k+1}T_1^{k+1}\geq\rho T_1^\star(T_1T_1^\star)^kT_1.\\ \rho^{-1}T_2^{\star k+1}T_2^{k+1}\geq\rho^{-1}T_2^\star(T_2T_2^\star)^kT_2.\end{cases}$$

hence

$$T_1^{\star k+1}T_1^{k+1} \geq (T_1T_1^\star)^k \quad \wedge \quad T_2^{\star k+1}T_2^{k+1} \geq (T_2T_2^\star)^k$$

therefore T_1, T_2 *are a Quasi Class* $A_k^\star$ *operators.*

Theorem 4.4.2 *Let* $T_1 \in \mathcal{B}(\mathcal{H}_1)$ *and* $T_2 \in \mathcal{B}(\mathcal{H}_2)$ *such that* T_1 *doubly commute with* T_2*. We have,*

If T_1 *(resp.*T_2*) belongs to Quasi Class* $A_k^\star$ *then* $T_1T_2 \otimes T_1$*,*$T_2T_1 \otimes T_1$ *(resp.*$T_1T_2 \otimes T_2$*,*$T_2T_1 \otimes T_2$*) also belong to Quasi Class* $A_k^\star$*.*

Proof 4.4.2 *(1) Suppose* T_1 *is a Quasi Class* $A_k^\star$*. Then,*

$$T_1^{\star k+1}T_1^{k+1} \geq T_1^\star (T_1T_1^\star)^k T_1$$

so

$$\begin{aligned}
(T_1T_2 \otimes T_1)^{\star k+1} (T_1T_2 \otimes T_1)^{k+1} &= (T_2^\star T_1^\star \otimes T_1^\star)^{k+1} (T_1T_2 \otimes T_1)^{k+1} \\
&= \left((T_2^\star T_1^\star)^{k+1} \otimes T_1^{\star k+1}\right) \left((T_1T_2)^{k+1} \otimes T_1^{k+1}\right) \\
&= (T_2^\star T_1^\star)^{k+1} (T_1T_2)^{k+1} \otimes T_1^{\star k+1}T_1^{k+1} \\
&= T_2^{\star k+1}T_1^{\star k+1}T_1^{k+1}T_2^{k+1} \otimes T_1^{\star k+1}T_1^{k+1} \\
&\geq T_2^{\star k+1}T_1^\star (T_1T_1^\star)^k T_1T_2^{k+1} \otimes T_1^\star (T_1T_1^\star)^k T_1 \\
&= T_2^\star T_1^\star \otimes T_1^\star (T_1T_2T_2^\star T_1^\star \otimes T_1T_1^\star)^k (T_1T_2 \otimes T_1) \\
&= (T_1T_2 \otimes T_1)^\star ((T_1T_2 \otimes T_1)(T_1T_2 \otimes T_1)^\star)^k (T_1T_2 \otimes T_1).
\end{aligned}$$

thus $T_1T_2 \otimes T_1$ *is a Quasi Class* $A_k^\star$*.*

By an analogous reasoning, we conclude that $T_2T_1 \otimes T_1$ *belongs to Quasi* $A_k^\star$*.*

(2) In the same way, we may deduce the $T_1T_2 \otimes T_2$ *and* $T_2T_1 \otimes T_2$ *are a Quasi Class* $A_k^\star$*.*

Chapter 5

The Quasi Totally Class $A_k^\star$ Operators in Hilbert Space

5.1 Introduction

In this present chapiter,we will introduce a new class of operators that we call Quasi Totally Class $A_k^\star$ operator in Hilbert spaces.It is a generalization of some previous studies in the field of classes of operators,especially for this especially for a Quasi Class $A_k^\star$.We will study some properties,provide example and discuss tensor product of this class of operators.

5.2 Totally Class $A_k^\star$ and Quasi Totally Class $A_k^\star$ Operator

Definition 5.2.1 *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be of Totally Class $A_k^\star$ if there exists a positive integer k such that*

$$\left|T^k - \eta\right|^2 \geq |T - \eta|^{2k} \quad \forall \eta \in \mathbb{C}.$$

$$i.e \quad (T^k - \eta)^\star (T^k - \eta) \geq ((T - \eta)(T - \eta)^\star)^k \quad \forall \eta \in \mathbb{C}.$$

Definition 5.2.2 *Let $T \in \mathcal{B}(\mathcal{H})$.An operator T is said to be Quasi Totally Class $A_k^\star$ if there exists a positive integer k such that*

$$T^\star \left(\left|T^k - \eta\right|^2 - |(T - \eta)^\star|^{2k} \right) T \geq 0 \quad \forall \eta \in \mathbb{C}.$$

In particular (choose $\eta = 0$) an operator T is called Quasi Class $A_k^\star$.

In general,the following implication holds:

$$\begin{aligned} \text{Hyponormal operator } &\Longrightarrow \text{Class } A_k^\star \text{ operator} \\ &\Longrightarrow \text{ Totally Class } A_k^\star \text{ operator} \\ &\Longrightarrow \text{ Quasi Totally Class } A_k^\star \text{ operator.} \end{aligned}$$

Example 5.2.1 *Let* $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *Then,*

$$I_2^\star(\left|I_2^k - \eta\right|^2 - |(I_2-\eta)^\star|^{2k})I_2 = \begin{pmatrix} |1-\eta|^2 - |1-\eta|^{2k} & 0 \\ 0 & |1-\eta|^2 - |1-\eta|^{2k} \end{pmatrix}.$$

For $k=1$ then I_2 is Quasi Totally Class $A_k^\star$ operator.

The following example shows that T_1 and T_2 are a Quasi Totally Class $A_k^\star$ operator but the sum T_1+T_2 isn't a Quasi Totally Class $A_k^\star$ operator.

Example 5.2.2 *Let* $T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *Then,*

$$T_1^\star(\left|T_1^k - \eta\right|^2 - |(T_1-\eta)^\star|^{2k})T_1 = \begin{pmatrix} |1-\eta|^2 - |1-\eta|^{2k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

for $|1-\eta|^2 \geq |1-\eta|^{2k}$ so T_1 is a Quasi Totally Class $A_k^\star$ operator.
and let $T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$

$$T_2^\star(\left|T_2^k - \eta\right|^2 - |(T_2-\eta)^\star|^{2k})T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |\eta|^{2k} - |\eta|^2 - 1 \end{pmatrix}.$$

for $|\eta|^{2k} \geq |\eta|^2 + 1$ then T_2 is Quasi Totally Class $A_k^\star$ operator.
and $T_1 + T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ *then*

$$(T_1+T_2)^\star\left(\left|(T_1+T_2)^{\mathrm{k}} - \eta\right|^2 - |(T_1+T_2-\eta)^\star|^{2k}\right)(T_1+T_2)$$
$$= \begin{pmatrix} |\eta|^2 - |1-\eta|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & |\eta|^{2k} - |\eta|^2 - 1 \end{pmatrix}.$$

for $\eta = 0$

$$(T_1+T_2)^\star\left(\left|(T_1+T_2)^{\mathrm{k}} - \eta\right|^{\frac{2}{\mathrm{k}}} - |(T_1+T_2-\eta)^\star|^{2k}\right)(T_1+T_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

so T_1+T_2 isn't Quasi Totally Class $A_k^\star$ operator.

Proposition 5.2.1 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are Quasi Totally Class $A_k^\star$ operator such that $T_1(T_2-\eta) = T_2(T_1-\eta) = (T_2-\eta)^\star T_1 = (T_1-\eta)^\star T_2 = 0$ for $\eta \in \mathbb{C}$ then T_1+T_2 is Quasi Totally Class $A_k^\star$ operator.*

Proof 5.2.1 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are Quasi Totally Class $A_k^\star$ operator.Then,*

$$T_1^\star\left(\left|T_1^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}\right)T_1\geq 0 \quad \textit{and} \quad T_2^\star\left(\left|T_2^k-\eta\right|^2-|(T_2-\eta)^\star|^{2k}\right)T_2\geq 0$$

so

$$\begin{aligned}
&(T_1+T_2)^\star\left(\left|(T_1+T_2)^k-\eta'\right|^2-\left|(T_1+T_2-\eta')^\star\right|^{2k}\right)(T_1+T_2)\\
&=(T_1+T_2)^\star\left(\left|T_1^k+T_2^k+\sum_{i=1}^{k-1}\binom{i}{k-1}T_1^kT_2^{k-i}-\eta-\eta\right|^2-|T_1^\star+T_2^\star-\bar{\eta}-\bar{\eta}|^{2k}\right)(T_1+T_2)\\
&=(T_1+T_2)^\star\left(\left|T_1^k-\eta\right|^2+\left|T_2^k-\eta\right|^2-|T_1^\star-\bar{\eta}|^{2k}-|T_2^\star-\bar{\eta}|^{2k}\right)(T_1+T_2)\\
&=(T_1+T_2)^\star\left(\left|T_1^k-\eta\right|^2+\left|T_2^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}-|(T_2-\eta)^\star|^{2k}\right)(T_1+T_2)\\
&=T_1^\star\left(\left|T_1^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}\right)T_1+T_2^\star\left(\left|T_2^k-\eta\right|^2-|(T_2-\eta)^\star|^{2k}\right)T_2\\
&\geq 0.
\end{aligned}$$

therefore T_1+T_2 is Quasi Totally Class $A_k^\star$ operator.

Proposition 5.2.2 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$.If T_2 is a Quasi Totally Class $A_k^\star$ operator and T_1 is unitary equivalent to T_2 then T_1 is Quasi Totally Class $A_k^\star$ operator.*

Proof 5.2.2 *T_2 is Quasi Totally Class $A_k^\star$ operator.Then,*

$$\begin{aligned}
&T_2^\star\left(\left|T_2^k-\eta\right|^2-|(T_2-\eta)^\star|^{2k}\right)T_2\geq 0\\
&\Rightarrow UT_1^\star U^\star\left(\left|UT_1^kU^\star-\eta UU^\star\right|^2-|UT_1^\star U^\star-\bar{\eta}UU^\star|^{2k}\right)UT_1U^\star\geq 0\\
&\Rightarrow UT_1^\star U^\star\left(U\left|T_1^k-\eta\right|^2U^\star-U\left|(T_1-\eta)^\star\right|^{2k}U^\star\right)UT_1U^\star\geq 0\\
&\Rightarrow UT_1U^\star\left(U\left|T_1^k-\eta\right|^2-U\left|(T_1-\eta)^\star\right|^{2k}\right)T_1U^\star\geq 0\\
&\Rightarrow UT_1^\star\left(\left|T_1^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}\right)T_1U^\star\geq 0\\
&\Rightarrow T_1^\star\left(\left|T_1^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}\right)T_1\geq 0.
\end{aligned}$$

so T_1 is a Quasi Totally Class $A_k^\star$ operator.

Proposition 5.2.3 *Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$ are doubly commutes.*
If T_1 is a Quasi Totally Class $A_k^\star$ operator such that $T_1T_2 = I_{\mathcal{H}}$,then T_1T_2 is a Quasi Totally Class $A_k^\star$ operator.

Proof 5.2.3 *Let T_1 is Quasi Totally Class $A_k^\star$ operator,we have that*

$$T_1^\star\left(\left|T_1^k-\eta\right|^2-|(T_1-\eta)^\star|^{2k}\right)T_1\geq 0\quad\forall\eta\in\mathbb{C}$$

then

$$\begin{aligned}&(T_1T_2)^\star\left(\left|(T_1T_2)^k-\eta\right|^2-|(T_1T_2-\eta)^\star|^{2k}\right)(T_1T_2)\\&=(T_1T_2)^\star\left(\left|T_1^kT_2^k-\eta T_1^kT_2^k\right|^2-|(T_1T_2-\eta T_1T_2)^\star|^{2k}\right)T_1T_2\\&=(T_1T_2)^\star\left(T_2^{\star k}\left|T_1^k-\eta T_1^k\right|^2T_2^k-T_2^{\star k}\left|(T_1-\eta T_1)^\star\right|^{2k}T_2^k\right)T_1T_2\\&=T_2^{\star k+1}T_1^\star\left(\left|T_1^k-\eta T_1^k\right|^2-|(T_1-\eta T_1)^\star|^{2k}\right)T_1T_2^{k+1}\\&\geq 0.\end{aligned}$$

hence T_1T_2 is a Quasi Totally Class $A_k^\star$ operator.

Proposition 5.2.4 *If T is Totally Class $A_k^\star$ operator.Then*

$$\|(T^k-\eta)v\|\geq\|(T-\eta)^\star v\|^k\|v\|^{1-k}\quad\forall v\in\mathcal{H};\forall\eta\in\mathbb{C}$$

Proof 5.2.4 *Suppose that T is Totally Class $A_k^\star$.We have*

$$(T^k-\eta)^\star(T^k-\eta)-((T-\eta)(T-\eta)^\star)^k\geq 0\quad\forall\eta\in\mathbb{C}.$$

then

$$\begin{aligned}\|(T^k-\eta)v\|^2&=\langle(T^k-\eta)v,(T^k-\eta)v\rangle\\&=\langle(T^k-\eta)^\star(T^k-\eta)v,v\rangle\\&\geq\langle((T-\eta)(T-\eta)^\star)^k\,v|v\rangle\\&\geq\langle(T-\eta)(T-\eta)^\star v|v\rangle^k\|v\|^{2-2k}\\&=\langle(T-\eta)^\star v|(T-\eta)^\star v\rangle^k\|v\|^{2-2k}\\&=\|(T-\eta)^\star v\|^{2k}\|v\|^{2-2k}.\end{aligned}$$

therefore

$$\|(T^k-\eta)v\|\geq\|(T-\eta)^\star v\|^k\|v\|^{1-k}\quad\forall v\in\mathcal{H};\forall\eta\in\mathbb{C}$$

Proposition 5.2.5 *If T is Quasi Totally Class $A_k^\star$ operator.Then*

$$\|(T^k-\eta)Tv\|\geq\|(T-\eta)^\star T\|^k\|Tv\|^{1-k}\quad\forall v\in\mathcal{H},\quad\forall\eta\in\mathbb{C}$$

Proof 5.2.5 *Suppose that T is Quasi Totally Class $A_k^\star$.We have*

$$T^\star\left((T^k-\eta)^\star(T^k-\eta)\right)T\geq T^\star\left((T-\eta)(T-\eta)^\star\right)^kT\quad\forall\eta\in\mathbb{C}.$$

then

$$\begin{aligned}\|(T^k-\eta)Tv\|^2 &= \langle (T^k-\eta)Tv, (T^k-\eta)Tv\rangle \\ &= \langle (T^k-\eta)^\star(T^k-\eta)Tv, Tv\rangle \\ &= \langle T^\star(T^k-\eta)^\star(T^k-\eta)Tv, v\rangle \\ &\geq \langle T^\star\left((T-\eta)(T-\eta)^\star\right)^k Tv, v\rangle \\ &= \langle \left((T-\eta)(T-\eta)^\star\right)^k Tv, Tv\rangle \\ &\geq \langle (T-\eta)(T-\eta)^\star Tv, Tv\rangle^k \|Tv\|^{2-2k} \\ &= \|(T-\eta)^\star Tv\|^{2k}\|Tv\|^{2-2k}.\end{aligned}$$

therefore

$$\|(T^k-\eta)Tv\| \geq \|(T-\eta)^\star Tv\|^k\|Tv\|^{1-k} \quad \forall v\in\mathcal{H}, \forall \eta\in\mathbb{C}$$

Proposition 5.2.6 *Let T is a Quasi Totally Class $A_k^\star$.Then $\mathcal{N}(T-\alpha)\subseteq\mathcal{N}((T-\alpha^k)^\star)$ for each $\alpha\neq 0$.*

Proof 5.2.6 *Suppose T is Quasi Totally Class $A_k^\star$ .It follows from proposition 5.2.5 that*

$$\|(T^k-\eta)Tv\| \geq \|(T-\eta)^\star Tv\|^k\|Tv\|^{1-k} \quad \forall v\in\mathcal{H}, \forall \eta\in\mathbb{C}$$

and we have $v\in\mathcal{N}(T-\alpha)$ then $Tv=\alpha v$.In particular($\eta=\alpha^k$)

$$\|(T^k-\alpha^k)Tv\| \geq \|(T-\alpha^k)^\star Tv\|^k\|Tv\|^{1-k}$$

since

$$0 \geq \|(T-\alpha^k)^\star Tv\|$$

.
then $\alpha\neq 0$,

$$\|(T-\alpha^k)^\star v\| = 0$$

therefore

$$v\in\mathcal{N}((T-\alpha^k)^\star)$$

hence $\mathcal{N}(T-\alpha)\subseteq\mathcal{N}(T-\alpha^k)^\star$ for each $\alpha\neq 0$.

Proposition 5.2.7 *If T is Quasi Totally Class $A_k^\star$ and it has dense range then T is Totally Class $A_k^\star$.*

Proof 5.2.7 *Let T is Quasi Totally Class $A_k^\star$.Then,*

$$\|(T^k-\eta)Tv\| \geq \|(T-\eta)^\star Tv\|^k\|Tv\|^{1-k} \quad \forall v\in\mathcal{H}, \forall \eta\in\mathbb{C}.$$

since T has dense range ($\overline{T(\mathcal{H})}=\mathcal{H}$) then there exists a sequence (x_n) in $\mathcal{H}$ such that $T(x_n)\longrightarrow v$ as $n\longrightarrow\infty$.
in particular,

$$\|(T^k-\eta)Tx_n\| \geq \|(T-\eta)^\star Tx_n\|^k\|Tx_n\|^{1-k} \quad \forall n\in\mathbb{N}, \forall \eta\in\mathbb{C}.$$

therefore

$$\begin{aligned}\|(T^k-\eta)Tx_n\| &= \|\lim_{n\to\infty}(T^k-\eta)Tx_n\| \\ &= \lim_{n\to\infty}\|(T^k-\eta)Tx_n\| \\ &\geq \lim_{n\to\infty}\|(T-\eta)^\star Tx_n\|^k\|Tx_n\|^{1-k} \\ &= \|\lim_{n\to\infty}(T-\eta)^\star Tx_n\|^k\|\lim_{n\to\infty}Tx_n\|^{1-k} \\ &= \|(T-\eta)^\star v\|^k\|v\|^{1-k}.\end{aligned}$$

hence T is Totally Class $A_k^\star$.

Theorem 5.2.1 *Let T is Quasi Totally Class $A_k^\star$ it doesn't have dense range.Then,*

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & 0\end{pmatrix}$$

where $T_1 = T_{|\overline{T(\mathcal{H})}}$ is a Totally Class $A_k^\star$ operator.

Proof 5.2.8 *Let T is Quasi Totally Class $A_k^\star$ and T doesn't have dense range.We can represent T as matrix follows:*

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & 0\end{pmatrix} \quad on\ \mathcal{H}=\overline{T(\mathcal{H})}\oplus\mathcal{N}(T^\star)$$

therefore

$$T^\star\left((T_1^k-\eta)^\star(T_1^k-\eta)-((T_1-\eta)(T_1-\eta)^\star)^k\right)T\geq 0 \quad \forall\eta\in\mathbb{C}$$

hence

$$\left\langle\left((T_1^k-\eta)^\star(T_1^k-\eta)-((T_1-\eta)(T_1-\eta)^\star)^k\right)Tv,Tv\right\rangle\geq 0 \quad \forall v\in\mathcal{H}$$

so

$$\left\langle\left((T_1^k-\eta)^\star(T_1^k-\eta)-((T_1-\eta)(T_1-\eta)^\star)^k\right)u,u\right\rangle\geq 0 \quad \forall u\in\mathcal{H}.$$

then T_1 is a Totally Class $A_k^\star$ operator.

5.3 Tensor Product of Quasi Totally Class $A_k^\star$ operators

Theorem 5.3.1 *Let $T_1\in\mathcal{B}(\mathcal{H})$ and $T_2\in\mathcal{B}(\mathcal{K})$ such that $T_1\otimes I_\mathcal{K}+I_\mathcal{H}\otimes T_2=0$.*

If T_1,T_2 are a Quasi Totally Class $A_k^\star$ operator then $T_1\otimes T_2$ is a Quasi Totally Class $A_k^\star$.

Proof 5.3.1 *Let T_1,T_2 are Quasi Totally Class $A_k^\star$ operator.We have*

$$\left\langle T_1^\star(\left|(T_1^k-\eta I_\mathcal{H})\right|^2-|(T_1-\eta I_\mathcal{H})^\star|^{2k})T_1v,v\right\rangle\geq 0 \quad \forall v\in\mathcal{H}.$$

and

$$\left\langle T_2^\star(\left|(T_2^k-\eta I_\mathcal{H})\right|^2-|(T_2-\eta I_\mathcal{H})^\star|^{2k})T_2u,u\right\rangle\geq 0 \quad \forall u\in\mathcal{K}.$$

then

$$(T_1 \otimes T_2)^\star \left(\left|(T_1 \otimes T_2)^k - \eta'\right|^2 - \left|(T_1 \otimes T_2)^\star - \eta'\right|^{2k} \right) (T_1 \otimes T_2)$$
$$= (T_1 \otimes T_2)^\star(\left|T_1^k \otimes T_2^k - \eta T_1^k \otimes I_{\mathcal{K}} - \eta I_{\mathcal{H}} \otimes T_2^k - \eta^2 I_{\mathcal{H}} \otimes I_{\mathcal{K}}\right|^2$$
$$- \left|T_1^\star \otimes T_2^\star - \eta T_1^\star \otimes I_{\mathcal{K}} - \eta I_{\mathcal{H}} \otimes T_2^\star - \eta^2 I_{\mathcal{H}} \otimes I_{\mathcal{K}}\right|^{2k})(T_1 \otimes T_2)$$
$$= (T_1 \otimes T_2)^\star(\left|T_1^k \otimes (T_2^k - \eta I_{\mathcal{K}}) - \eta I_{\mathcal{H}} \otimes (T_2^k - \eta I_{\mathcal{K}})\right|^2$$
$$- |T_1^\star \otimes (T_2^\star - \eta I_{\mathcal{K}}) - \eta I_{\mathcal{H}} \otimes (T_2^\star - \eta I_{\mathcal{H}})|^{2k})(T_1 \otimes T_2)$$
$$= (T_1^\star \otimes T_2^\star)(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k})(T_1 \otimes T_2)$$
$$= (T_1^\star \otimes T_2^\star)(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k})(T_1 \otimes T_2)$$
$$= (T_1^\star \otimes T_2^\star)(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2$$
$$+ |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k})(T_1 \otimes T_2)$$
$$= (T_1^\star \otimes T_2^\star)(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k}) \otimes \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2$$
$$+ |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k} \otimes (\left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k}))(T_1 \otimes T_2)$$
$$= (T_1^\star(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k}) \otimes T_2^\star \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2$$
$$+ T_1^\star(|(T_1 - \eta I_{\mathcal{H}})^\star|^{2k}) \otimes T_2^\star(\left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k}))(T_1 \otimes T_2)$$
$$= T_1^\star(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k})T_1 \otimes T_2^\star \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 T_2$$
$$+ T_1^\star(|(T_1 - \eta I_{\mathcal{H}})^\star|^{2k})T_1 \otimes T_2^\star(\left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k})T_2.$$

therefore for ever $v \in \mathcal{H}$ *and* $u \in \mathcal{K}$

$$\left\langle T_1^\star(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k})T_1 v, v \right\rangle \left\langle T_2^\star \left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 T_2 u, u \right\rangle$$
$$+ \left\langle T_1^\star(|(T_1 - \eta I_{\mathcal{H}})^\star|^{2k})T_1 v, v \right\rangle \left\langle T_2^\star(\left|(T_2^k - \eta I_{\mathcal{K}})\right|^2 - |(T_2 - \eta I_{\mathcal{K}})^\star|^{2k})T_2 u, u \right\rangle \geq 0$$
$$= \left\langle T_1^\star(\left|(T_1^k - \eta I_{\mathcal{H}})\right|^2 - |(T_1 - \eta I_{\mathcal{H}})^\star|^{2k})T_1 v, v \right\rangle \|(T_2^k - \eta I_{\mathcal{K}})T_2 u\|^2$$
$$+ \|(T_1 - \eta I_{\mathcal{K}})^k T_1 v\|^2 \left\langle T_2^\star(|(T_2 - \eta I_{\mathcal{H}})|^2 - |(T_2 - \eta I_{\mathcal{H}})^\star|^{2k})T_2 u, u \right\rangle$$
$$\geq 0.$$

then

$$(T_1 \otimes T_2)^\star \left[\left|(T_1 \otimes T_2)^k - \eta'\right|^2 - \left|(T_1 \otimes T_2)^\star - \eta'\right|^{2k} \right] (T_1 \otimes T_2) \geq 0.$$

therefore $T_1 \otimes T_2$ *is Quasi Totally Class* $A_k^\star$ *operator.*

Bibliography

[1] Putnam, Calvin R. "On normal operators in Hilbert space." American Journal of Mathematics 73.2 (1951): 357-362.

[2] A. Brown, On a class of operators, Proc. Amer. Math.Soc,4 (1953),723-728

[3] N. Dunford & J. T. Schwartz, Linear Operators (Interscience, New York, 1958), Vol. I; (1963), Vol. II.

[4] J.G. Stampfli. "Hyponormal operators." (1962): 1453-1458.

[5] C.A. McCarthy. "C_p", Isr. J. Math. 5 (1967): 249-271.

[6] I.H. Sheth. " Quasi-hyponormal operators", Rev. Roumaine Math. Pures Appl.19 (1974), 1049–1053.

[7] Radjabalipour, M. "An extension of Putnam-Fuglede theorem for hyponormal operators." Mathematische Zeitschrift 194.1 (1987): 117-120.

[8] Kreyszig, Erwin. Introductory functional analysis with applications. John Wiley and Sons, 1991.

[9] H. Jinchuan. "On the tensor products of operators." Acta Math. Sinica, 9(2) (1993): 195-202.

[10] Furuta, Ito, Yamazaki, A subclass of paranormal operators including class of log-hyponormal and several related classes, Sci. Math. 1 (1998), 389–403.

[11] J. Han, H. Lee and W.Y. Lee. "Invertible completions of 2×2 upper triangular operator matrices." Proc. Amer. Math. Soc., 128.1 (2000): 119-123.

[12] M. Chō and T. Yamazaki, "On the class A and the p-hyponormal operators," *Scientiae Mathematicae Japonicae*, 53(2):245–254, 2001.

[13] P. Shanmugapriya, P.M. Naik and A. Benali. "Kronecker Product and spectral properties of Quasi M-Class A_k^* Operators", Theory , Quantum Physics- arXiv:quant-ph/0104019v1.ss

[14] J. M. Steele, *The Cauchy-Schwarz Master Class*, Cambridge University Press, 2004.

[15] Arias, María Laura, Gustavo Corach, and Maria Celeste Gonzalez. "Metric properties of projections in semi-Hilbertian spaces." (2008).

[16] M.L. Arias, G. Corach, M.C. Gonzalez, Partial isometries in semi-Hilbertian spaces, Linear Alge bra Appl. 428 (7) (2008) 1460-1475.

[17] Suciu, Laurian. "Quasi-isometries in semi-Hilbertian spaces." Linear algebra and its applications 430.8-9 (2009): 2474-2487.

[18] B. P. Duggal, "Spectral properties of a class of operators," *Linear Algebra and its Applications*, 433(1):200–206, 2010.

[19] Y. Han and J.H. Son. " On quasi-M-hyponormal operators ", Filomat, 25(1) (2011), 37-52.

[20] M. S. Moslehian and S. M. Bakar, "Norm inequalities for operators on semi-Hilbertian spaces," *Linear and Multilinear Algebra*, 59(11):1265–1272, 2011.

[21] O. A. M. Sid Ahmed, "Some properties of A-hyponormal operators," *Linear Algebra and its Applications*, 435(9):2233–2243, 2011.

[22] Jordan, Thomas F. Linear operators for quantum mechanics. Courier Corporation, 2012.

[23] S. Panayappan and N. Sivamani, "A-Quasi normal operators in semi-Hilbertian space," *Linear Algebra and its Applications*, 436(10):3930–3942, 2012.

[24] S.Panayappan and N. Sivamani,A-Quasi Normal Operators in Semi Hilbertian Space, Linear Algebra and its Applications 436 (2012) 3930-3942 .

[25] S. Panayappan, N. Jayanthi and D. Sumathi. "Weyl' s theorem and tensor product for Class A_k operators." Pure Mathematical Sciences 1.1 (2012): 13-23.

[26] O. A. Mahmoud Sid Ahmed and A. SADDI, A-m-Isomertic operators in semi Hilbertian spaces, Linear Algebra and its Applications, 436 (2012), pp. 3930-3942.

[27] S. Panayappan and N. Jayanthi. "Weyl and Weyl type theorems for class $A_k^\star$ and quasi class $A_k^\star$ operators." Int. J. math. Anal.(Russia) 7 (2013): 683-698.

[28] Jah, Sidi Hamidou, and Ould Ahmed Mahmoud Sid Ahmed. "Positive-normal operators in semi-Hilbertian spaces." International Mathematical Forum. Vol. 9. No. 11. 2014.

[29] P. Shanmugapriya and P.M. Naik. "A study on Spectral properties of extended class $A_k^\star$ operator on a complex Hilbert Space" in National Conference "COMIST 19" ,during 15.02.2019-16.02.2019 at Bannari Amman Institute of Technology, Sathyamangalam.

[30] P. Shanmugapriya and P.M. Naik. "Spectral properties of m- class $A_k^\star$ operator", Int. J. Math. Appl., 7 (2) (2019), 171–176.

[31] M. W. Alomari, "The generalized Schwarz inequality for semi-Hilbertian space operators," *arXiv preprint arXiv:2007.01701*, 2020.

[32] Alomari, Mohammad W. "The generalized Schwarz inequality for semi-Hilbertian space operators and Some A-numerical radius inequalities." arXiv preprint arXiv:2007.01701 (2020).

[33] K. Feki, "On the A-norm of semi-Hilbertian space operators," *arXiv preprint arXiv:2006.05201*, 2020.

[34] Guesba, Messaoud, and Sid Ahmed Ould Ahmed Mahmoud. "Paranormal operators in semi-Hilbertian spaces." Afrika Matematika 31.7 (2020): 1409-1428.

[35] S. S. Rana and J. K. Nayak, "Generalization of class A operators in Semi-Hilbertian spaces," *Journal of Pseudo-Differential Operators and Applications*, 12(3):42, 2021.

[36] M. A. Shebrawi, "Lower bounds for the A-numerical radius of certain operators," *Linear Algebra and its Applications*, 611:1–14, 2021.

[37] K. Feki, "A note on the A-spectral radius of weighted conditional expectation operators," *Journal of Pseudo-Differential Operators and Applications*, 13(1):1–18, 2022.

[38] K. Feki, "Spectral radius of semi-Hilbertian space operators," *Banach Journal of Mathematical Analysis*, 16(2):1–25, 2022.

[39] M. Mabrouk, "Further results on A-quasi-hyponormal operators," *Extracta Mathematicae*, 37(1):45–58, 2022.

[40] R. K. Nayak, "Bounds for the A-numerical radius of operators on semi-Hilbertian spaces," *Linear and Multilinear Algebra*, 70(18):3450–3465, 2022.

[41] H. Baklouti, "Characterization of A-m-hyponormal operators on semi-Hilbertian spaces," *Complex Analysis and Operator Theory*, 17(4):56, 2023.

[42] P. Bhunia and K. Paul, "Bounds for the A-numerical radius of operators," *Mathematical Inequalities and Applications*, 26(1):145–158, 2023.

[43] R. Gupta, "On the A-Quasi-Class A_k operators," *Operators and Matrices*, 17(2):315–330, 2023.

[44] A. Benali, "Spectral mapping theorems for quasi-class A_k^* operators," *Functional Analysis and Its Applications*, 58(4):210–225, 2024.

[45] K. Feki, "New inequalities for the A-numerical radius of certain operators," *Journal of Mathematical Inequalities*, 18(1):85–102, 2024.

[46] P. Ghosh and S. S. Rana, "On a new subclass of A-quasi-operator categories," *Journal of Mathematical Analysis and Applications*, 530(2):127–144, 2024.

[47] M. Saini, "Higher-order A-quasi-hyponormality and its generalizations," *Annals of Functional Analysis*, 15(2):45–60, 2024.

[48] MAADANI Mustapha , BENALI Abdekader, and NASLI BAKIR Aissa. "ON QUASI TOTALLY m-CLASS A k* OPERATORS." Journal of Computational Analysis & Applications 34.2 (2025).

[49] S. Yadav, "New perspectives on power bounded operators in semi-Hilbertian spaces," *Operator Theory: Advances and Applications*, 295:301–318, 2025.

[50] Steele, J. Michael. "THE CAUCHY–SCHWARZ MASTER CLASS."

Abstract

This thesis investigates novel class of operators acting on both Hilbert spaces and semi-Hilbertian spaces.Building upon and extending previously studied operator class,we introduce and analyze three new class: Class $A_k^\sharp$,Quasi Class $A_k^\sharp$,and Quasi Totally Class $A_k^\star$ .We systematically examine their structural properties and establish the conditions under which these properties are preserved under various algebraic operations.Chapter 1 provides the necessary preliminaries,reviewing fundamental concepts in Hilbert spaces and presenting essential results on operator theory in semi-Hilbertian spaces,with particular attention to A-adjoint operators and existing operator class.In Chapter 2,we introduce Class $A_k^\sharp$ in the semi-Hilbertian setting,exploring its properties and providing illustrative examples,along with a discussion of its behavior under tensor products.Chapter 3 is dedicated to Quasi Class $A_k^\sharp$operators in semi-Hilbertian spaces,where we establish key results concerning their structure and stability under direct sums and tensor products.Chapter 4 shifts the focus to Hilbert spaces,where we study Class $A_k^\star$ (the Hilbert space analogue of Class $A_k^\sharp$) and Quasi Class $A_k^\star$,comparing their properties with those of related operator class.Finally,Chapter 5 introduces Quasi Totally Class $A_k^\star$operators in Hilbert spaces,proving several structural theorems and examining their behavior under tensor product operations.

ملخص

تقوم هذه الأطروحة بدراسة فئات مبتكرة من المؤثرات العاملة في كل من فضاءات هلبرت وفضاءات شبه-هلبرت.وانطلاقاً من فئات مؤثرات تمت دراستها سابقاً وتوسيعاً لها، نقوم بتقديم وتحليل ثلاث فئات جديدة: $ClassA_k^\sharp$ و $QuasiClassA_k^\sharp$ و $QuasiTotallyClassA_k^*$.ندرس بشكل منهجي خصائصها البنيوية ونحدد الشروط التي يتم في ظلها الحفاظ على هذه الخصائص تحت تأثير العمليات الجبرية المختلفة.يقدم الفصل الأول الأساسيات الضرورية، من خلال استعراض المفاهيم التأسيسية في فضاءات هلبرت وعرض النتائج الأساسية لنظرية المؤثرات في فضاءات شبه-هلبرت، مع إيلاء اهتمام خاص للمؤثرات المرافقة-A وفئات المؤثرات الموجودة.في الفصل الثاني، نقدم $ClassA_k^\sharp$ في سياق شبه-هلبرت، ونستكشف خصائصه مع تقديم أمثلة توضيحية، بالإضافة إلى مناقشة سلوكه تحت تأثير حاصل الضرب التنسوري.الفصل الثالث مخصص لدراسة مؤثرات$QuasiClassA_k^\sharp$ في فضاءات شبه-هلبرت، حيث نثبت نتائج رئيسية تتعلق ببنيتها واستقرارها تحت تأثير المجموع المباشر وحاصل الضرب التنسوري.ينتقل الفصل الرابع إلى التركيز على فضاءات هلبرت، حيث ندرس $ClassA_k^*$ (المماثل في فضاء هلبرت لـ $ClassA_k^\sharp$) و $QuasiClassA_k^*$، ونقارن خصائصها مع خصائص فئات المؤثرات الأخرى ذات الصلة.أخيراً، يقدم الفصل الخامس مؤثرات $QuasiTotallyClassA_k^*$ في فضاءات هلبرت، مُثبتين العديد من المبرهنات البنيوية ومُتفحصين سلوكها تحت عمليات حاصل الضرب التنسوري.

Résumé

Cette thèse étudie de nouvelles class d'opérateurs agissant sur les espaces de Hilbert et les espaces semi-hilbertiens.En nous appuyant sur des class d'opérateurs précédemment étudiées et en les étendant,nous introduisons et analysons trois nouvelles class: la classe $A_k^\sharp$,la quasi-class $A_k^\sharp$ et la quasi-class totalement $A_k^\star$ .Nous examinons systématiquement leurs propriétés structurelles et établissons les conditions de leur préservation sous diverses opérations algébriques.Le chapitre1 présente les notions préliminaires nécessaires,en rappelant les concepts fondamentaux des espaces de Hilbert et en exposant les résultats essentiels de la théorie des opérateurs dans les espaces semi-hilbertiens,avec une attention particulière portée aux opérateurs A-adjoints et aux class d'opérateurs existantes.Dans le chapitre2,nous introduisons la classe $A_k^\sharp$ dans le cadre semi-hilbertien,en explorant ses propriétés et en fournissant des exemples illustratifs,ainsi qu'une discussion de son comportement sous l'effet des produits tensoriels.Le chapitre 3 est consacré aux opérateurs de quasi-class $A_k^{\sharp}$ dans les espaces semi-hilbertiens.Nous y établissons des résultats clés concernant leur structure et leur stabilité sous l'effet de sommes directes et de produits tensoriels.Le chapitre 4 se concentre sur les espaces de Hilbert et étudie la classe $A_k^{\star}$(l'analogue hilbertien de la classe $A_k^{\sharp}$) et la quasi-class $A_k^{\star}$,en comparant leurs propriétés à celles de class d'opérateurs apparentées.Enfin,le chapitre 5 introduit les opérateurs de quasi-totalement classe $A_k^{\star}$ dans les espaces de Hilbert,en démontrant plusieurs théorèmes de structure et en examinant leur comportement sous l'effet de produits tensoriels.