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THÈSE
Pour l'obtention du diplôme de doctorat LMD

Spécialité: Équations aux Dérivées Partielles

Modèles d'ondes efficacement amorties avec mémoire non linéaire

Présentée Par : ALIMERINA Omar

Devant le jury composé de :

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THESIS
For obtaining the LMD doctorate degree
Specialty: Partial Differential Equations

Effectively damped wave models with nonlinear memory

Presented by: ALIMERINA Omar

In front of the jury composed of:

Belgacem Rachid	MCA, University of Chlef,	President.
Hadj Kaddour Tayeb	MCA, University of Chlef,	Supervisor.
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Benali Abdelkader	Prof, University of Chlef,	Guest.

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Dedication

I dedicate this work to:

My dear parents

All the teachers in the Department of Mathematics

To all Persons Those who have Encouraged me Throughout this Work

All my friends.

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Praise be to Allah, who granted me the strength, patience, and ability to complete this research. First, I would like to express my heartfelt gratitude and sincere appreciation to my family for their immense patience and unwavering support. Secondly, I extend my deepest thanks and gratitude to my supervisor, Professor. **Hadj Kaddour Tayeb**, for his continuous guidance, valuable advice, and constant support throughout my research journey. I also greatly value his patience and care, as he was always available to discuss and address my questions. I pray to Allah to bless him and reward him abundantly.

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ملخص

تُكرَّس هذه الأطروحة لدراسة وجود ووحدانية الحلول المحلية وظاهرة الانفجار لنماذج موجة مخمدة فعالة تتضمن ذاكرة غير خطية، ممثلة بعدم خطية كسرية من خلال مؤثر ريمان ليوفيل الكسري، استندنا بشكل رئيسي إلى مسألة كوشي التالية لنماذج موجة مخمدة فعالة بذاكرة غير خطية تتضمن نوعين من التخميد: الاحتكاك والمصطلحات اللزجة المرنة :

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s,\cdot)|^p\,ds, \;\; t>0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1}$$

حيث $p>1, \gamma \in (0,1)$ و $b=b(t)$ هو مصطلح تخميد فعال. قسمنا عملنا إلى جزأين رئيسيين. في الجزء الأول، نأخذ. $b(t)=1$ أي نعتبر مسألة كوشي للنموذج

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma}|u(s,\cdot)|^p\,ds, \;\; t>0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2}$$

حيث $p>1, \gamma \in (0,1)$. أثبتنا أن مصطلح الاحتكاك يهيمن على المصطلح اللزج المرن من خلال إثبات أنا مسألة كوشي (2) لها نفس الأس الأسي الحرج مع مسألة كوشي أخرى.

بالنسبة لكل نموذج، نثبت وجود و وحدانية الحلول المحلية باستخدام مبرهنة النقطة الصامدة. بالإضافة إلى ذلك، نحقق في نتائج الانفجار للحلول باستخدام طريقة دالة الاختبار للنموذج الأول وطريقة دالة الاختبار المعدلة للنموذج الثاني.

Abstract

This thesis is consecrated for the study of the existence and uniqueness of local solutions and the blow-up phenomenon, for certain effectively damped wave models involving non linear memory, represented by a fractional nonlinearity through the Riemann-Liouville fractional operator. Indeed, we mainly based on the following Cauchy problem of effectively damped wave models with non linear memory involving two types of damping, friction and viscoelastic terms:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1}$$

where $p > 1$, $\gamma \in (0,1)$ and $b = b(t)$ is effective in the sense of Definition 1.1. We split our work into two principle parts. In the first part, we take $b(t) = 1$ that is, we consider Cauchy problem for the model

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2}$$

where $p > 1$, $\gamma \in (0,1)$ where we proved that the damping term dominates the viscoelastic term by proving that Cauchy problem (4) has the same critical exponent with Cauchy problem (1.1) considered in [8] and [17]. The influence of the nonlinear memory on the critical exponent is still happening as we will explain in the historical overview. For each model, we establish the existence and uniqueness of local solutions using the fixed-point theorem. Additionally, we investigate blow-up results for the solutions, utilizing the test function method for the first model and the modified test function method for the second.

Keywords: Damped wave equation, Local existence, Global existence, Critical Fujita exponent, Test function method, Nonlinear memory, Blow-up.

Résumé

Cette thèse est consacrée à l'étude de l'existence et l'unicité de solutions locales, ainsi que le phénomène d'explosion (blow-up), pour certains modèles d'ondes amorties efficacement avec mémoire non linéaire, représentés par une non-linéarité fractionnaire via l'opérateur fractionnaire de Riemann-Liouville. En effet, nous avons basé principalement sur le problème de Cauchy suivant pour les modèles d'ondes amorties efficacement avec mémoire non linéaire, intervenant deux types d'amortissement: friction et viscoélastique :

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3}$$

où $p > 1$, $\gamma \in (0,1)$ et $b = b(t)$ est efficace au sens de la Définition 1.1. Nous avons divisé notre travail en deux parties principales. Dans la première partie, nous prenons $b(t) = 1$, c'est-à-dire que nous considérons le problème de Cauchy

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4}$$

où $p > 1$, $\gamma \in (0,1)$. Nous avons prouvé que le terme de friction domine le terme viscoélastique en démontrant que le problème de Cauchy (4) possède le même exposant critique que le problème de Cauchy (1.1) considéré dans [8] et [17]. L'influence de la mémoire non linéaire sur l'exposant critique reste pertinente, comme nous l'expliquerons dans l'aperçu historique. Pour chaque modèle, nous établissons l'existence et l'unicité de solutions locales en utilisant le théorème du point fixe. De plus, nous étudions les résultats d'explosion pour les solutions, en utilisant la méthode de la fonction test pour le premier modèle et une version modifiée de cette méthode pour le second.

Mots clés: Équation d'onde amortie, Existence locale, Existence globale, Exposant critique, Méthode de fonction test, Mémoire non linéaire, Explosion.

Notation

Symbols used in all parts of this thesis

$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$;
∂_x	The operator of partial differentiation with respect to x,
$\|\alpha\| = \alpha_1 + \cdots + \alpha_n$	Length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$;
$\\|v\\|_X$	The norm of a function $v \in X$;
∇, ∇_x	Gradient operator (with respect to) x;
Δ, Δ_x	Laplacian with respect to the spatial variables;
$\|D\|^\sigma, \|D_x\|^\sigma$	Pseudo-differential operator with symbol $\|\xi\|^\sigma$;
$u \lesssim v$	If there exists a positive constant$C$ such that $u \leq Cv$;
$u \approx v$	If there exist constants $C_1, C_2 > 0$ such that $C_1 v \leq u \leq C_2 v$;
$u \simeq v$	If $u = Cv$ for some constant $C > 0$;
$u * v$	Convolution of functions u and v;
$u *_{(x)} v$	Convolution between u and v with respect to the spatial variables;
$\mathscr{F}_{x\to\xi}(g)$	Fourier transform of g;
$\mathscr{F}^{-1}_{x\to\xi}(\hat{g})$	Inverse Fourier transform of $\hat{g}$;
$\mathrm{supp} f$	Support of the function f;

Functional spaces used throughout this thesis

$\mathscr{C}^k = \mathscr{C}^k(\mathbb{R}^n)$	Spaces of k-times continuously differentiable functions;
$\mathscr{C}^\infty = \mathscr{C}^\infty(\mathbb{R}^n)$	Spaces of infinitely continuously differentiable functions;
$\mathscr{C}_0^\infty = \mathscr{C}_0^\infty(\mathbb{R}^n)$	Spaces of infinitely continuously differentiable functions with compact support
$L^p = L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \leq p \leq \infty$;
$L^p_{loc} = L^p_{loc}(\mathbb{R}^n)$	Spaces of locally p-measurable functions, $1 \leq p < \infty$;
$W^m_p = W^{m,p}(\mathbb{R}^n)$	Sobolev spaces based on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty,\ m \in \mathbb{N}$;
$H^s = H^s(\mathbb{R}^n)$	Sobolev spaces based on $L^2(\mathbb{R}^n), s \in \mathbb{R}$;

Contents

Chapter 1 Introduction

1.1 Motivation

In the study of wave dynamics, some systems are devoted of behavior that has a finite life and characterized by conventional expectation, leading to phenomena where solutions do not only disappear but blow-up in finite time. In this thesis, we study some kind of these systems, namely, the effectively damped wave models with nonlinear memory. These models represent a topic of great significance for several reasons, making it an important and worthy subject for investigation. In the following we state some some events that modeled mathematically by this type of models

- **Real-World Applications:** ([1],[20], [41]) These models are crucial for understanding and predicting the behavior of various physical systems. They have applications in fields such as geophysics (seismic wave propagation), engineering (vibration analysis), and materials science (stress-strain relationships in viscoelastic materials).

- **Advanced Mathematical Techniques:**([34], [49]) Studying these models involves sophisticated mathematical tools and techniques. This includes fractional calculus, integral equations, and nonlinear analysis. This research can lead to the development of new mathematical methods and theories.

- **Memory Effects:**[12] Nonlinear memory effects are essential in many natural and engineered systems. These effects help describe how past states of a system influence its current and future behavior. Understanding these effects can lead to better modeling and prediction of complex systems.

- **Stability and Control:**([7], [54]) Investigating the stability and control of wave models with nonlinear memory is crucial for designing systems that can withstand disturbances. This has practical implications for structures and materials that need to remain stable under various conditions.

- **Technological Innovation:** [58] The insights gained from this research can drive technological innovations. For example, in designing materials with specific damping properties or in creating new diagnostic tools in medical imaging and treatment.

1.2 Background overview

In this section we present the historical of damped wave models with a semi linear or non linear memory as a forcing term.

1.2.1 Cauchy problems for damped wave models with constant coefficients

Let us consider the following nonlinear Cauchy problem for the classical damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + u_t = f(u), \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

where f is a given function, null or depending on the solution u which represent the non linear term in the non trivial case. We distinguish between several cases regarding the function f.

A. Homogeneous equation

It corresponds to the case $f(u) = 0$. In this case, the solutions to linear Cauchy problem (1.1) represent wave propagation with friction, such as the telegraph equation and heat conduction with a finite propagation speed of perturbations.

The wave energy of Sobolev solutions to the linear Cauchy problem (1.1) is defined as follows:

$$E(u)(t) := \frac{1}{2}\big(\|\nabla_x u(t,\cdot)\|^2_{L^2} + \|u_t(t,\cdot)\|^2_{L^2}\big).$$

Here $E(u)(t)$ is monotonically decreasing. This result is obtained by differentiating the energy $E(u)(t)$ with respect to t and applying integration by parts. In other words, it can be expressed as follows:

$$\frac{d}{dt}E(u)(t) = \int_{\mathbb{R}^n} (u_t u_{tt} + \nabla u \cdot \nabla u_t)\, dx = -\int_{\mathbb{R}^n} u_t(t,x)^2\, dx \leq 0.$$

Therefore, we cannot expect energy conservation due to the presence of the damping term. This does not provide any information about whether the energy converges to zero or remains positive and so far to zero at all times. However, this raises a question regarding the behavior of the energy as $t \to \infty$. A particularly important question is whether the energy $E(u)(t)$ approaches to 0 as $t \to \infty$ or not? This behavior is known as *decay* in the PDE's literature.

In [42], precise decay estimates for solutions to the linear Cauchy problem (1.1) are established where the author demonstrated that the solutions to the damped wave equation (1.1) satisfy for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ the following estimates:

$$\|D_t^k D_x^\alpha u(t,x)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{|\alpha|}{2}-k}\big(\|f\|_{H^{k+|\alpha|}} + \|g\|_{H^{k+|\alpha|-1}} + \|(f,g)\|_{L^m}\big),$$

for all $t \geq 0$ and $m \in [1,2)$. In this estimate, f and g are the first and the second initial data respectively. In this context, the estimates are improved by introducing an additional L^m regularity, where $m \in [1,2)$. The associated $L^p - L^q$ decay estimates for the Cauchy problem (1.1) are presented as follows:

$$\|u_t(t,\cdot), \nabla_x u(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\left(\|f\|_{W_p^{N_p+1}} + \|g\|_{W_p^{N_p}}\right)$$

for $n \geq 2$, $p \in [1,2)$, $\frac{1}{p}+\frac{1}{q}=1$ and $N_p \geq n\left(\frac{1}{p}+\frac{1}{q}\right)$. In other words, these decay rates correspond to the estimates for the heat equation (see [50]). Therefore, this parabolic structure can be described in terms of the so-called *diffusion phenomenon*, and the asymptotic behavior of the solutions to the damped wave equation is related to the solutions of the heat equation.

B. Semi linear damped wave equation

For non vanishing f, the study becomes very delicate. We present only the interesting cases considered by the researchers in the last decades. The most important form is when

$f(u) = |u|^p$. This form is known as semi linear term or power-type non-linearity. In this case Cauchy problem (1.1) is called Cauchy problem for the semi linear damped wave equation. It reads as follows:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Many authors was interesting in this model where they investigated the global existence for small data solutions and local existence for any data. The main propriety for this type of models is the so-called "*critical exponent of Fujita*" denoted usually p^* or p_{Fuj}.

In paper [53], it is proved that the critical exponent of Fujita for Cauchy problem (1.2) is

$$p_{Fuj}(n) = 1 + \frac{2}{n} \tag{1.3}$$

and a global existence result for energy solutions is established too, under the condition compactly supported initial data, for $p > p_{Fuj}(n)$ and $p < p_{GN} := \frac{n}{n-2}$ if $n \geq 3$. The last requirement is caused by the use of Gagliardo-Nirenberg inequality and the parameter p_{GN} is called the Gagliardo-Nirenberg exponent. These results open the door for researchers to discover the connection between the classical damped wave equation and parabolic models. In addition, the same paper contains a blow-up result for $1 < p \leq p_{Fuj}(n)$, assuming that the data satisfies some integral sign conditions. Later, in [61] and [53], blow-up results for Cauchy problem (1.2) was refound by using what is known actually as the test function method. Specifically, the author extended the analysis to the broader framework of complete non compact Riemannian manifolds and demonstrated the blow-up of local solutions for $1 < p \leq p_{Fuj}(n)$, again under the same integral sign conditions as in [53].
Recently, in [32], the authors reduced the assumption of compact support for the data in the global (in time) existence result for 1.2.

Moreover, other cases where the power nonlinearity is substituted with the nonlinearity $\pm|u|^{p-1}$ in 1.2 have been widely studied in the context of the classical semi-linear damped wave equations. We refer the reader to check Eg. [29], [33], [45],[47], [48].

Comparing the results for 1.2 with those of the classical free wave model with the following power non-linearity:

$$\begin{cases} u_{tt} - \Delta u = |u|^p, \quad x \in \mathbb{R}^n, t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

We can immediately observe the beneficial effect generated by the presence of a lower order term, specifically the damping term u_t, since $p_0(n) > p_{Fuj}(n)$, where $p_0(n)$ represents the positive root of the quadratic equation $(n-1)p^2 - (n+1)p = 2$. In [52], it was proposed that $p_0(n)$ is the critical exponent for the Cauchy problem (1.4), and called the Strauss exponent. Recalling, as previously mentioned, the Fujita exponent acts as the critical exponent for the semilinear heat equation. The addition of the damping term significantly changes the characteristics of the model making it appears as "parabolic-like" from certain viewpoints. This phenomenon can be further elucidated by exploring two key aspects.

The first point is the so-called *diffusion phenomenon* observed between linear heat equations and linear classical damped wave equations, which particularly results in identical decay rates in the $L^p - L^q$ estimates for the solutions and their derivatives of both the linear heat equation and the linear classical damped wave equation. Therefore, in terms of decay estimates, the classical damped wave model can be considered "parabolic-like." The second noteworthy point is that the test function method was initially developed to derive blow-up results for semi linear parabolic models. Additionally, the precision of this method in determining the upper bound for p, which leads to a blow-up result (given appropriate assumptions for the data), further emphasizes the parabolic nature of the classical damped wave model. Nonlinearity plays a crucial role in the analysis of wave equations, evolution equations, and damped equations.

C. Damped wave equations with non linear memory

In certain cases, fractional nonlinearity is introduced through the Riemann-Liouville fractional operator and known as a nonlinear memory term. In this kind of models we based on the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = f(x), \ u_t(0,x) = g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

with $\gamma \in (0,1)$. This model is considered in [17] and more recent in [8]. The author of [17] focused on the global existence and blow-up phenomena together of solutions for the Cauchy problem described by (1.5). He utilized the weighted energy method to establish the existence and uniqueness of energy solutions, by considering a weight similar to that considered in [53] while he applied the test function method to study the of blow-up phenomenon. In particular, he proved that the critical exponent in Fujita sense is

$$p_{Fuj}(\gamma, n) = \max\left\{\frac{2(2-\gamma)}{\max\{n-2\gamma+2, 0\}}, \frac{1}{\gamma}\right\}. \tag{1.6}$$

Among the disadvantages of the weighted energy method is that creates some gaps. For this reason, the author of [8] thought to filling these gaps by changing the method. He used the method of linear estimates to establish global existence. He also, succeed to replace the condition of compactly supported of the data required in [17] by small L^2-norm of the data. The disadvantage of this method for this model is that it imposes hight regularity of the data and gives positive results in a space dimension smaller than six only. Finally, we mention that the author in [8] does not investigate blow-up results since they are done correctly in [17].

D. Semi linear damped wave models with viscoelastic term

Another remarkable model that has been extensively studied in several recent papers, we cite for instance [9], [30], [31] and [43], is the following Cauchy problem for the semi-linear wave equation with friction and viscoelastic damping, and power nonlinearity:

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = |u|^p, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.7}$$

Obviously, it is possible to explore Cauchy problem (1.7) by incorporating both friction and viscoelastic damping terms. The topic of the qualitative characteristics of solutions to (1.7) arises somehow. the understanding of the relationship between friction and viscoelastic damping is an interesting aspect. As a result, the authors in [30] demonstrated that the effect of frictional damping is more significant than that of viscoelastic damping in the asymptotic behavior as shown as $t \to \infty$. Recently, D'Abbicco [9] studied Cauchy problem

(1.7), assuming that the data (f, g) belong to the energy space with additional L^1 regularity, specifically $(H^1 \cap L^1) \times L^2 \cap L^1)$. The study demonstrated the global in time existence of small data energy solutions for the range of admissible exponents $1 + \frac{2}{n} < p \leq 1 + \frac{2}{n-1}$ for $n > 1$. One can see that the critical exponent of Fujita proposed for Cauchy problem (1.7) is as in (1.3). This explain why the friction term dominates the viscoelastic damping. In the other other hand, the authors of [43] characterized a generalized diffusion phenomenon and showed that the asymptotic behavior of solutions over a long period is a combination of diffusion and wave equation solutions.

E. viscoelastic damped wave models with non linear memory

In this direction, there is not a lot of results since this topic is new. The following Cauchy problem of viscoelastic damped wave equation with friction and nonlinear memory

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p d\tau, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = f(x), u_t(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

is considered in [14] where the authors proved the global existence of solutions for the Cauchy problem (1.8) with small data, but they did not study blow-up results for this problem. In particular, they proposed as critical exponent the parameter $p_{Fuj}(\gamma, n)$ defined by (1.6) (See Section 3.5). One can understand that the proposed parameter is just a threshold of the zone of global existence since the blow-up has not been investigated yet. In order to show that $p_{Fuj}(\gamma, n)$ is sharp one has to study what happens if $p \leq p_{Fuj}(\gamma, n)$. This is considered as an open subject of scientific research. As a consequence, our work is motivated by this open subject.

1.3 Connection between semi-linear and non linear memory terms

In fractional analysis, one can remark that, for any $\gamma \in (0, 1)$, $p > 1$ and $t > 0$, one has

$$\int_0^t (t-\tau)^{-\gamma} |u(\tau, \cdot)|^p \, d\tau = \Gamma(1-\gamma) I_{0|t}^{1-\gamma}(|u(t, \cdot)|^p),$$

where Γ is the well known Gamma function of Euler defined by (2.1) and $I^{1-\gamma}_{0|t}$ denotes for the right fractional Riemann–Liouville integral of order $1-\gamma$ defined by 2.6. Hence, we have, in distributional sense

$$\lim_{\gamma\to 1}\Gamma(1-\gamma)\int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^p\,d\tau = |u(t,\cdot)|^p. \tag{1.9}$$

Based on (1.9), one can see that when γ tends to 1, the non linear memory term converges to the semi linear term up to a constant. Consequently, Cauchy problem (1.5) converges to the Cauchy problem (1.4), it follows that Fujita exponent (1.6) approaches to Fujita exponent (1.3). The same turns true for Cauchy problems (1.8) and (1.7). One may interpret this remark as a continuity of Cauchy problems and Fujita exponents with respect to γ.

1.3.1 Cauchy problems for wave models with time-dependent coefficients

This kind of models is very delicate and require additional tools and based results. We select for example the main following difficulties:

- We can not express explicitly the solution of the corresponding Cauchy problem. As a consequence, we can not get directly the estimates of the L^2-norm of the solution and its gradient which complicates the study of whether the initial value problem is dissipative or not since we can not conclude that the energy decreasing to 0!
- We can not shift the fundamental solutions obtained by applying Fourier transform to express the solution of the non linear problem from the solution of the corresponding linear problem.
- Without some restrictions on the coefficients, we don't have always diffusion phenomenon.
- In the study of blow-up, the classical test function gives worth results, it requires to apply the modified method instead and this last one is more complicate.

Obviously, these difficulties turns the study hard somehow.

There is not a lot of results concerning these models. For this reason, we present, only, some models considered by some researchers in this direction. Let us start with the following most important Cauchy problem for the damped wave equation that involves time-dependent dissipation

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = 0, \quad x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.10}$$

The term $b(t)u_t$ is referred to the damping term. It hinders the wave effect and decreases its energy. The coefficient $b = b(t)$ measures the intensity of the damping. It is clear that, the solutions' asymptotic behavior and the evolution of their wave energy depend on the positive coefficient $b = b(t)$ in the damping term.

1.3.2 Classification of time-dependent dissipation

In the PhD thesis [60], the author summarizes his pioneering works by introducing a classification of time-dependent dissipation terms as follows:

- Scattering producing to the free wave equation,
- Non-effective dissipation,
- Effective dissipation,
- Over-damping producing.

If the solutions behave asymptotically like those of the wave equation, meaning that the damping term in (1.10) has no essential influence on the behavior of the solution, then the solution scatters to resemble that of the free wave equation as $t \to \infty$. This case is referred to as the scattering producing case. If the $L^p - L^q$ estimates for the solution to the Cauchy problem (1.10) are closely aligned with those of the solutions to the free wave equation, then the damping term is referred to as *non-effective*.
If the solution to the Cauchy problem (1.10) has the same decay behavior as the solution of the corresponding parabolic Cauchy problem

$$\begin{cases} u_t = \frac{1}{b(t)}\Delta u, \quad x \in \mathbb{R}^n, t > 0, \\ u(0,x) = f(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.11}$$

that is, if the damping term has a stronger influence, then the damping term is called *effective*. Finally, if there is no decay estimate for the energy of the solution, specifically, if the damping term exerts too strong an influence, then the damping term is referred to as *over-damping producing*.

In general, in cases of scattering or over-damping, there is no decrease in energy. Specifically, the effectiveness of the model (1.10) is defined as follows:

Definition 1.1. *(Effective dissipation) The damping term in (1.10) is considered effective if $b = b(t)$ satisfies the following properties:*

(1) $b(t) > 0$ *for any* $t > 0$.

(2) b *is monotone and* $tb(t) \to \infty$ *as* $t \to \infty$.

(3) $((1+t)^2 b(t))^{-1} \in L^1([0,\infty))$.

(4) $b \in \mathscr{C}^3(0,+\infty)$ *and*

$$|b^{(k)}(t)| \lesssim \frac{b(t)}{(1+t)^k}, \qquad \textit{for any } k = 1,2,3.$$

(5) $\dfrac{1}{b} \notin L^1(0,\infty)$.

For whom asking for examples of such $b(t)$ we refer him to check Eg. [11], [59], [60] and the references therein. In the case of effective dissipation, the $L^p - L^q$ decay estimates of the solution (1.10) and its derivatives were provided as follows in [59, 60]:

$$\|D_t^\kappa D_x^\alpha u(t,\cdot)\|_{L^q} \lesssim \frac{1}{b^\kappa(t)}\left(1+\int_0^t \frac{ds}{b(s)}\right)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{|\alpha|}{2}-\kappa}\left(\|f\|_{W_p^{Np+|\alpha|}} + \|g\|_{W_p^{Np+|\alpha|-1}}\right),$$

where $p \in [1,2]$, $\frac{1}{p}+\frac{1}{q}=1$ and the regularity $Np > \frac{n}{p}-\frac{n}{q}$. It is important to note that this decay rate aligns with the corresponding estimate for the heat equation.

Next, we examine the following Cauchy problem for the semi-linear time-dependent damped wave equation:

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.12}$$

In the case where $b(t) = \mu(1+t)^{-r}$ with $\mu \geq 0$ and $r \in \mathbb{R}$, We can classify the damping term into four cases based on the different values of r. When $r < -1$, the damping term is referred to as *over-damping*, and the solution does not approach to zero as $t \to \infty$. If $-1 \leq r < 1$, the solution behaves similarly to that of the heat equation, and we refer to the damping term as *effective*. Hence, the term u_{tt} in (1.10) does not affect the solution's behavior, and the $L^p - L^q$ decay estimates for the solution are nearly identical to those of the heat equation. In contrast, when $r > 1$, it is established that the solution behaves like that of the wave equation, indicating that the damping term in (1.10) does not affect the behavior of the solution. In this case, the solution scatters to that of the free wave equation as $t \to \infty$, and therefore, we refer to this phenomenon as scattering. Finally, when $r = 1$, the equation in (1.10) remains invariant under the scaling

$$v(x,t) = u(sx, s(1+t) - 1), \quad s > 0.$$

Thus, the damping term is referred to as *scale-invariant*. In this scenario, the behavior of the solution to (1.10) is observed to depend on the value of μ.
The classifications of the damping term in (1.10) are summarized in the table below.

Range of r	**Classification**
$r \in (-\infty, -1)$	over-damping
$r \in [-1, 1)$	effective
$r = 1$	scale-invariant
$r \in (1, \infty)$	scattering

Let us revisit problem (1.12), which adopts the aforementioned terminology and exhibits significantly different behaviors depending on the case. Let us compile some global existence and blow-up results in the following two tables.

Authors	**Range of** r	**Dimension** n	**Exponent** p
Ikeda,Wakasugi [28]	$r < -1$	$n \geq 1$	$p > 1$
Wakasugi [57]	$r = -1$	$n = 1, 2$	$p > p_F(n)$
		$n \geq 3$	$P_F(n) < p < \frac{n}{n-2}$
Todorova,Yordanov [53]	$r = 0$	$n = 1, 2$	$p > p_F(n)$
		$n \geq 3$	$P_F(n) < p \leq \frac{n}{n-2}$
D'Abbicco, Lucente,Reissig [11]	$-1 < r < 1$	$n = 1, 2$	$p > p_F(n)$
Nishihara [46] Lin, Nishihara,Zhai [38]	$r \neq 0$	$n \geq 3$	$P_F(n) < p < \frac{n+2}{n-2}$
Liu, Wang [40]	$r > 1$	$n = 3, 4$	$p > p_s(n)$

Table 1.1: Global-in-time existence for $r \neq 0$.

Authors	**Range of** r	**Exponent** p
Fujiwara, Ikeda, Wakasugi [19] Ikeda, Inui[24]	$r = -1$	$1 < p < P_F(n)$ $p = p_F(n)$
Li, Zhou [39], Zhang [61] Todorova, Yordanov [53] Kirane, Qafsaoui [35] Ikeda, Ogawa [25]. Lai, Zhou [37] Ikeda, Wakasugi [27]. Nishihara [46] Fujiwara, Ikeda, Wakasugi [19]	$r = 0$	$1 < p < P_F(n)$ $p = p_F(n)$
Fujiwara, Ikeda, Wakasugi [19] Ikeda, Inui [24] Ikeda, Ogawa [25] Ikeda, Wakasugi [27]	$-1 < r < 1$ $r \neq 0$	$1 < p < P_F(n)$ $p = p_F(n)$
Lai, Takamura [36] Wakasa,Yordanov [55]	$r > 1$	$1 < p < P_s(n)$ $p = p_s(n)$

Table 1.2: Blow-up in finite time for $r \neq 0$.

D'Abbicco, Lucente, and Reissig [11] (see also [10]) broadened this finding to encompass more general forms of $b(t)$ that satisfy a monotonicity condition and exhibit polynomial-like behavior. Additionally, they relaxed the data assumptions to allow for an exponentially decaying condition. They also addressed initial data belonging to the class $L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \times L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ when $n \leq 4$, and the authors generalized this blow-up result to more general damping terms $b(t)u_t$ by using a modified test function method(cf.[10]). More precisely, the dissipation $b(t)u_t$, that is considered in [11], is effective according to the classification given in [59][60].

Next, we review some scientific studies that addressed wave equations with effective damping, where nonlinear memory appears as a forcing term

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.13}$$

is considered in [21] and [22] where the authors proved global existence of small data solution in [22] by using Matsumura type estimates method and blow-up in finite time of weak solution by the test function method in [22] for a family of effectively

$$b(t) \in \{(1+t)^r,\ r \in (-1,1)\}.$$

In this thesis, we focus on studying two types of damped wave models. The first type involves the Cauchy problem for damped waves with viscoelastic properties and nonlinear memory, while the second type is a model for damped waves where the damping depends on time-related parameters. Our study will examine cases where the damping is effective, accompanied by a viscoelastic term and nonlinear memory. The primary objectives of this research include exploring the relationship between the first and second models by proving blow-up results for local (in time) energy solutions and determining the critical exponent in the sense of Fujita for both models. To prove the local existence of solutions, we will apply the Banach Fixed Point Theorem along with well-known mathematical tools, such as the Gagliardo-Nirenberg inequality. For proving blow-up results for the solutions, we will use the test function method for the first model, and the modified test function method for the second model.

The importance of this study lies in its ability to provide several key results, including understanding the relationship between frictional damping and elastic damping in the first problem, and examining how these terms affect the critical exponent. As for the second model, which includes an effective damping term $b(t)u_t$, we aim to study how the size of

$b(t)$ influences the shape of the solution when t is sufficiently large. Since the models we are studying resemble the parabolic models in terms of energy decay estimates, and given that the damping is effective, we expect the solution to behave similarly to the solution of a parabolic equation. Therefore, the critical exponent is expected to be the same as the critical exponent for the damped wave equation with nonlinear memory, as well as for the heat equation with nonlinear memory.

1.4 Plan of the thesis

This thesis consists of four main chapters and an appendix. The first chapter provides an introduction that includes a brief historical overview of the most prominent damped wave models, as well as a discussion of results related to the global existence of solutions. It also highlights blow-up results, critical exponent values for these models, and their connection to nonlinear memory. Furthermore, the chapter reviews the methods and approaches used to study these models, whether they involve constant coefficients or time-dependent ones.

Chapter two is dedicated to defining some notations and preliminary concepts. In this chapter, we introduced some spaces that we work on in this thesis, such as Sobolev space and some related properties. We also defined the Fourier transform and its properties in these spaces. Additionally, we reviewed some techniques and results necessary to understand the rules of fractional calculus, focusing on the Riemann-Liouville concept. This chapter concludes with the concepts of diffusion and blow-up phenomena, where we discussed some methods used to study the blow-up phenomenon, such as the test function method.

The third chapter is dedicated to studying the damped wave model that includes a viscoelastic term and nonlinear memory. In this chapter, we focus on investigating the local existence of solutions by applying Banach's fixed-point theorem, along with utilizing mathematical tools such as the Gagliardo-Nirenberg inequality. Additionally, we explored the blow-up phenomena of solutions and determined the critical exponent using the test function method. This study generalizes the work of Cazenave [6] and Fino [17].

In Chapter Four, we will explore a different type of damped wave models, specifically the damped wave model with effective damping, accompanied by a viscoelastic term and non-linear memory. This study aims to establish the local existence (in time) of solutions for this model. Additionally, we will investigate the blow-up phenomenon under the condition of positive initial data, utilizing the modified test function method, which is considered a useful tool for addressing time-dependent damping terms, and in this chapter, we aim to highlight the impact of the damping term on the critical exponent.

This thesis concludes with a bibliography, which is an alphabetically arranged list of all the references used in its preparation.

Chapter

2 Preliminaries and Basic Concepts

In this chapter, we review some mathematical concepts that will be utilized throughout this thesis. In all this chapter, we denote by X for a bounded domain of $\mathbb{R}^n$.

2.1 Special functions

Special functions play an important role in the fractional calculus. In this thesis, we need to this kind of functions in the study of blow-up of our problem since the method used is the test function method which itself depends on the right choice of the test function. In our case we choose a test function defined via the right fractional differential operator of Riemann-Liouville. For this reason, we set the definition and some properties of the special functions used in this thesis.

2.1.1 Gamma function:

The Gamma function generalizes the notion of the factorial (defined only in $\mathbb{N}$) for any positive real number. It is defined for any $\kappa > 0$ by the following integral:

$$\Gamma(\kappa) = \int_0^\infty t^{\kappa-1} e^{-t} dt, \tag{2.1}$$

In the following lemma, we select among the proprieties of Γ that we need in this thesis :

Lemma 2.1. *For any κ and $\mu > 0$, one has*

- $\Gamma(\kappa + 1) = \kappa\Gamma(\kappa)$.
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- $\Gamma(\kappa)\Gamma(\mu) = \Gamma(\kappa + \mu)$.

In particular, the first propriety gives $\Gamma(\kappa) = \kappa!$ if $\kappa \in \mathbb{N}$. From this remark we can understand the generalization we are talking about.

2.1.2 Beta function:

The second special function is the Beta function which is defined, for any $\mu, \kappa > 0$, by:

$$\mathscr{B}(\mu, \kappa) = \int_0^1 t^{\mu-1}(1-t)^{\kappa-1}dt. \tag{2.2}$$

Recalling, that the Beta function is related to the Gamma function by the following useful identity:

$$\mathscr{B}(\kappa, \mu) = \frac{\Gamma(\kappa)\Gamma(\mu)}{\Gamma(\kappa+\mu)}. \tag{2.3}$$

for any $\mu, \kappa > 0$.

2.2 Functions Spaces

2.2.1 Lebesgue L^p Spaces

Definition 2.1. *[4] Let $p \in [1, +\infty)$, the L^p space is defined as :*

$$L^p(X) = \left\{ f : X \to \mathbb{R};\ f \ \textit{is measurable and} \int_X |f(x)|^p\, dx < +\infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^p} = \|f\|_p = \Big(\int_X |f(x)|^p dx \Big)^{1/p}.$$

Definition 2.2 ($p = \infty$). *[4] In the case $p = \infty$, the space $L^\infty(X)$ is called space of essentially bounded functions and defined by*

$$L^\infty(X) = \Big\{ f : X \to \mathbb{R}; f \ \textit{measurable and} \ \exists C > 0 \ \textit{suth that} \ |f(x)| \le C \ \textit{a.e. on} \ X \Big\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C;\ \ |f(x)| \le C \ \ \textit{a.e. on} \ X\}.$$

2.2.2 Space of test functions

Definition 2.3. *Let $X \subset \mathbb{R}^n$ be an open set and $\Phi : X \to \mathbb{R}$ be a function. The support of Φ is defined as*

$$\operatorname{supp}(\varPhi) = \overline{\{x \in X : \varPhi(x) \neq 0\}}.$$

Definition 2.4 (Test functions space)**.** *The space of infinitely differentiable functions with compact support in X is called space of test functions defined on X. It is denoted $\mathscr{D}(X)$ or $\mathscr{C}_0^\infty(X)$.*

Definition 2.5. *Let $X \subset \mathbb{R}^n$ be an open set. A measurable function $f : X \to \mathbb{R}$ is considered locally integrable if, for every compact set $K \subset X$,*

$$\int_K |u(x)|dx < \infty. \tag{2.4}$$

The space $L^1_{loc}(X)$ denotes for the space of locally integrable functions defined on X.

2.2.3 The space of distributions $\mathscr{D}'(X)$

Definition 2.6. *$\mathscr{D}'(X)$ is the vector space of linear maps $T : \mathscr{C}_0^\infty(X) \to \mathbb{C}$ satisfying the following condition: For any compact set $K \subset X$, there exist constants $C > 0$ and $k \in \mathbb{N}$ such that:*

$$|\langle T, \varphi\rangle| \leq C \sum_{|\alpha|\leq k} \sup_K |\partial^\alpha \varphi|, \quad \forall \varphi \in \mathscr{C}_0^\infty(K). \tag{1.1}$$

2.2.4 The Schwartz space $\mathcal{S}(\mathbb{R}^n)$

Definition 2.7. *$\mathcal{S}(\mathbb{R}^n)$ is the vector space of $\mathscr{C}^\infty-$functions $f : \mathbb{R}^n \to \mathbb{C}$ which satisfy:*

$$\forall \alpha, \beta \in \mathbb{N}^n,\ \exists\, C_{\alpha,\beta} > 0 \text{ such that } |x^\alpha \partial^\beta u(x)| \leq C_{\alpha,\beta},\ \forall x \in \mathbb{R}^n.$$

2.2.5 Sobolev Spaces

Let $X \subset \mathbb{R}^n$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 2.8. *Let $m \in \mathbb{N}$. Then the Sobolev spaces $W_p^m(\mathbb{R}^n)$ are defined as*

$$W_p^m(\mathbb{R}^n) := \Big\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W_p^m} := \sum_{|\alpha| \le m} \|\partial_x^\alpha f\|_{L^p} < \infty \Big\}.$$

Definition 2.9. *Let $s > 0$ be not an integer number. By m we denote the integer part of s. Then the Sobolev-Slobodeckij spaces $W_p^s(\mathbb{R}^n)$ are defined as*

$$W_p^s(\mathbb{R}^n) := \Big\{ f \in W_p^m(\mathbb{R}^n) : \|f\|_{W_p^s} < \infty \Big) \Big\},$$

where the norm $\|\cdot\|$ in $W_p^s(\mathbb{R}^n)$ is defined by

$$\|f\|_{W_p^s} := \|f\|_{W_p^m} + \Big(\sum_{|\alpha|=m} \Big(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\partial_x^\alpha f(x) - \partial_y^\alpha f(y)|^p}{|x-y|^{n+(s-m)p}} \, dx \, dy \Big)^{1/p}.$$

In the particular case $p = 2$, the space $W_2^s(\mathbb{R}^n)$ is simply denoted $W^s(\mathbb{R}^n)$.

Definition 2.10. *Let $s \in \mathbb{R}$. Then the Sobolev spaces of fractional order $H_p^s(\mathbb{R}^n)$ are defined as*

$$H_p^s(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H_p^s} := \|\mathscr{F}^{-1}(\langle \xi \rangle^s \mathscr{F}(f))\|_{L^p} < \infty \right\}.$$

Again, when $p = 2$, we also use the notation $H^s(\mathbb{R}^n)$ instead of $H_2^s(\mathbb{R}^n)$. Where $\mathcal{S}'(\mathbb{R}^n)$ we denote the topological dual to $\mathcal{S}(\mathbb{R}^n)$.

Definition 2.11. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then the homogeneous Sobolev spaces of fractional order $\dot{H}_p^s(\mathbb{R}^n)$ are defined as*

$$\dot{H}_p^s(\mathbb{R}^n) := \Big\{ f \in \mathcal{Z}'(\mathbb{R}^n) : \|f\|_{\dot{H}_p^s} := \|\mathcal{F}^{-1}(|\xi|^s \mathcal{F}(f))\|_{L^p} < \infty \Big\}.$$

Based on that, we define the function space

$$\mathcal{Z}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}(\mathbb{R}^n) : D_\xi^\alpha \mathcal{F}(f)(\xi) = 0 \text{ for all } \alpha \in \mathbb{N}^n \right\}.$$

By $\mathcal{Z}'(\mathbb{R}^n)$ we denote the topological dual to $\mathcal{Z}(\mathbb{R}^n)$.

Remark 2.1. *The property of homogeneity is an important tool to prove embeddings between the spaces $\dot{H}_2^s(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ (see [5]). Among other things, we have the following embeddings:*

- *If $s \in \big[0, \frac{n}{2}\big)$, then the space $\dot{H}_2^s(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ if and only if $q = \frac{2n}{n-2s}$. Consider the following embedding for non-homogeneous spaces: if $s \in \big[0, \frac{n}{2}\big)$, then the space $H_2^s(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ if and only if $q \in [2, \frac{2n}{n-2s}]$.*
- *If $p \in (1, 2]$, then $L^p(\mathbb{R}^n)$ is continuously embedded in $\dot{H}_2^s(\mathbb{R}^n)$ with $s = n\left(\frac{1}{2} - \frac{1}{p}\right)$.*

2.2.6 Basics of Fourier Transformation

This section focuses on providing the principles of Fourier transform, which is a key tool in analyzing the global existence of solutions.
The Fourier transform is one of main types of integral transformation. It is defined on $L^1(\mathbb{R}^n)$ using the classical definition as follows:

$$\mathscr{F}(u)(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} u(x)dx, \quad \xi \in \mathbb{R}^n. \tag{2.5}$$

The inverse Fourier transform is defined, formally, using the classical definition as follows:

$$\mathscr{F}^{-1}(v)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\xi} v(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

Some proprieties of $\mathscr{F}$ used in the sequel

Among the proprieties of $\mathscr{F}$ needed in particular, in the proof and some estimates of global existence of small data solution for Cauchy problem (3) types we select

Proposition 2.1. *Let $u \in L^1(\mathbb{R}^n)$. Then, $\mathscr{F}(u)$ belongs to $L^\infty(\mathbb{R}^n)$. Moreover, the following identity of convolution holds for all $u, v \in L^1(\mathbb{R}^n)$:*

$$\mathscr{F}(u * v)(\xi) = \mathscr{F}(u)(\xi)\mathscr{F}(v)(\xi), \quad \xi \in \mathbb{R}^n.$$

In the space L^2, Fourier transform is defined by density technique. We recapitulate the main proprieties of $\mathscr{F}$ in L^2 the following theorem

Theorem 2.1. *[16] Let $\mathscr{F}$ the Fourier transform operator defied by (2.5). Then the following assertions hold*

1. *The Fourier transformation is a unitary operator on $L^2(\mathbb{R}^n)$.*
2. *The Fourier transformation $\mathscr{F}$ satisfies the well known Parseval-Plancherel formula*

$$\langle \mathscr{F}(u), \mathscr{F}(v)\rangle_{L^2} = \langle u, v\rangle_{L^2} \ \textit{for any} \ \ u, v \in L^2(\mathbb{R}^n).$$

3. *The Fourier transformation $\mathscr{F}$ is isometric from $L^2(\mathbb{R}^n)$ into it self satisfying for any $u \in \ L^2(\mathbb{R}^n)$*

$$\|u\|_{L^2} = \|\mathscr{F}(u)\|_{L^2}.$$

Application to H^s Spaces

We will define the space $H^m(\mathbb{R}^n)$; $m \in \mathbb{N}$. This space consists of functions given by

$$H^m(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \|u\|_{H^m}^2 = \int_{\mathbb{R}^n} |\mathcal{F}(u)(\xi)|^2 (1 + |\xi|^2)^m d\xi < \infty \right\}.$$

This space is norm-equivalent to the Sobolev space $W_2^m(\mathbb{R}^n)$. In fact, by applying *Parseval-Plancherel's* formula and utilizing the properties of the Fourier transform, we obtain

$$\|\partial_x^\alpha u\|_{L^2} = \|\xi^\alpha \mathcal{F}(u)\|_{L^2} \quad \text{for all } |\alpha| \leq m.$$

It can be generalized to all real $s \in \mathbb{R}$

Theorem 2.2. *[16] The Fourier inversion formula holds on $H^s(\mathbb{R}^n)$, that is,*

$$\mathcal{F}^{-1}(\mathcal{F}(u)) = u \quad \textit{for} \quad u \in \ H^s(\mathbb{R}^n).$$

Theorem 2.3. *[15] The Fourier transform of a smooth function is a smooth function.*

2.3 Functional analysis

2.3.1 Green's formula

Theorem 2.4. ***(Green's formula)****[44] Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $\mathscr{C}^1$ and η denotes its outward unit normal. If $u, v \in \mathscr{C}^2(\overline{\Omega})$, then*

$$\int_\Omega v\Delta u \, dx = \int_\Omega u\Delta v \, dx + \int_{\partial\Omega} \Big(v\frac{\partial u}{\partial \eta} - u\frac{\partial v}{\partial \eta} \Big) d\sigma$$

where σ is the surface measure on $\partial\Omega$.

2.3.2 Banach fixed point theory

We present one of the fixed point theorems, which plays a fundamental role in proving the existence and uniqueness of solution to nonlinear equations and systems.

Theorem 2.5 (Contraction mapping principle). *[23] Let $\big(\Omega, \|\cdot\|\big)$ be a Banach space and let K be a closed convex set in Ω. If N is a mapping defined on K such that* *i)* $Nu \in K$ *for all* $u \in K$,

ii) $\exists \kappa \in (0,1)$, *such that* $\|Nu - Nv\| \leq \kappa \|u - v\|$ *for any* $u, v \in K$,

then N *has a unique fixed point* $\bar{u}$ *satisfying* $N(\bar{u}) = \bar{u}$ *in* K.

The following corollary that the proof is based on Theorem 2.5, is used, in particular, to prove the results of Section 3.5.

Corollary 2.1. *[13] The operator* N *maps* X *into itself and has one and only one fixed point* $u \in X$ *if the following inequalities hold:*

$$\begin{aligned} \|Nu\|_X &\leq C_0(t) \|(u_0, u_1)\|_{X_0} + C_1(t) \|v\|_X^p \\ \|Nu - Nv\|_X &\leq C_2(t) \|u - v\|_X \left(\|u\|_X^{p-1} + \|v\|_X^{p-1} \right) \end{aligned}$$

where $(u_0, u_1) \in X_0,\ C_1(t), C_2(t) \longrightarrow 0$ *for* $t \longrightarrow 0_+$.

The proof of this corollary can be found in [13].

2.3.3 Gronwal type lemma

In this section we announce two main inequalities regarding there importance in this thesis. The first one, is known as *Gronwal lemma.* it permits to extract an estimate for a function defined via a integral recurrent relation somehow. It reads as follow

Lemma 2.2. *Let* $a,\ b \in \mathbb{R},\ a < b$ *and* $f = f(t)$ *be a continuous real valued function on the interval* $[a, b]$ *and let* $\ell \in \mathbb{R}$. *Suppose that* $k(t)$ *is a continuous nonnegative function on* $[a, b]$ *satisfying*

$$\frac{1}{2} f^2(t) \leq \frac{1}{2} \ell^2 + \int_a^t k(s) f(s) ds$$

for any $t \in [a, b]$. *Then*

$$|f(t)| \leq |\ell| + \int_a^t k(s) ds$$

holds for any $t \in [a, b]$.

Proof. Put, for $t \in [a, b]$ and $\varepsilon > 0$,

$$F_\varepsilon(t) = \frac{1}{2}(\ell^2 + \varepsilon^2) + \int_a^t k(s) f(s) ds.$$

Then $F_\varepsilon(t)$ is strictly positive and satisfies

$$F'_\varepsilon(t) \leq k(t)|f(t)| \leq (2F_\varepsilon(t))^{\frac{1}{2}} k(t),$$

which means that

$$\frac{F'_\varepsilon(t)}{(2F_\varepsilon(t))^{\frac{1}{2}}} \leq k(t).$$

Integrating over $[a,t]$ we get

$$\big(2F_\varepsilon(t)\big)^{\frac{1}{2}} \leq \big(2F_\varepsilon(a)\big)^{\frac{1}{2}} + \int_a^t k(s)ds.$$

From the inequality

$$\big(2F_\varepsilon(a)\big)^{\frac{1}{2}} \leq (\ell^2 + \varepsilon^2)^{\frac{1}{2}} \leq |\ell| + \varepsilon$$

we obtain according to the assumption,

$$\begin{aligned} |f(t)| &\leq \big(2F_\varepsilon(t)\big)^{\frac{1}{2}} \\ &\leq \big(2F_\varepsilon(a)\big)^{\frac{1}{2}} + \int_a^t k(s)ds \\ &\leq |\ell| + \varepsilon + \int_a^t k(s)ds. \end{aligned}$$

Finally, by taking the limit as $\varepsilon \to 0_+$, we derive the required estimate. □

2.4 Tools from fractional analysis

The second member of the model we consider can be expressed by a fractional integral of Riemann-Liouville up to a constant (see Section 1.3). Then is better to present the main tools needed to treat this member and the results in connection. First we present the definitions

2.4.1 Fractional integration and differentiation

In this section, we will review essential definitions and useful lemmas concerning fractional derivatives and integration, which will be utilized in this thesis. we restrict ourselves on Rimann-Lioville sense. First

Definition 2.12. *[51](**Absolutely continuous functions**): A function $h : [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$, is absolutely continuous if and only if there exists a Lebesgue measurable function $\psi \in L^1[a, b]$ such that*

$$h(t) = h(a) + \int_a^t \psi(s)ds.$$

The space of these functions is denoted by $\mathcal{AC}[a, b]$. Furthermore,

$$\mathcal{AC}^k[a, b] = \Big\{ f : [a, b] \longrightarrow \mathbb{R};\ f^{(k)} \in \mathcal{AC}[a, b],\ for\, all\, k \in \mathbb{N} \Big\}.$$

Definition 2.13. *[51](**Riemann-Liouville fractional integrals**): Let $g \in L^1(0, T)$ with $T > 0$.The Riemann-Liouville left- and right-sided fractional integrals of order $\alpha \in (0, 1)$ are, respectively, defined by*

$$I^{\alpha}_{0|t} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s)ds \qquad for \quad t > 0, \tag{2.6}$$

and

$$I^{\alpha}_{t|T} g(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha-1} g(s)ds \qquad for \quad t < T,$$

where $\Gamma(\cdot)$ is the Euler's gamma function, defined by (2.1).

Definition 2.14. *[51](**Riemann-Liouville fractional derivatives**): Let $g \in \mathcal{AC}[0; T]$ with $T > 0$. The Riemann-Liouville left- and right-sided fractional derivatives of order $\alpha \in (0, 1)$ are, respectively, defined by*

$$D^{\alpha}_{0|t} g(t) = \frac{d}{dt} I^{1-\alpha}_{0|t} g(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t - s)^{-\alpha} g(s)ds \quad for\ t > 0, \tag{2.7}$$

and

$$D^{\alpha}_{t|T} g(t) = \frac{d}{dt} I^{1-\alpha}_{t|T} g(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T (s - t)^{-\alpha} g(s)ds \quad for\ t < T. \tag{2.8}$$

We state some properties as preliminary results

2.4.2 Some proprieties

The fractional version of integration by parts formula is given in the following proposition

Proposition 2.2. *[51]: Let $T > 0$ and $\alpha \in (0, 1)$. The fractional integration by parts formula is given by*

$$\int_0^T g(t) D^{\alpha}_{t|T} h(t)dt = \int_0^T \big(D^{\alpha}_{0|t} g(t) \big) h(t)dt, \tag{2.9}$$

holds for every $g \in I^{\alpha}_{t|T}(L^p(0,T))$, $h \in I^{\alpha}_{0|t}(L^q(0,T))$, *where* $1+\alpha \geq \frac{1}{p}+\frac{1}{q}$ *with* $p,q>1$ *and*

$$I^{\alpha}_{0|t}\big(L^q(0,T)\big) = \Big\{g = I^{\alpha}_{0|t}f \ for \ f \in L^q(0,T)\Big\},$$
$$I^{\alpha}_{t|T}\big(L^p(0,T)\big) = \Big\{g = I^{\alpha}_{t|T}f \ for \ f \in L^p(0,T)\Big\}.$$

The following proposition will show us that the composition of right hand sided fractional integral with right fractional differential operator gives the identical operator

Proposition 2.3. *Let* $T>0$ *and* $\alpha \in (0,1)$. *The following identities are valid:*

$$\big(D^{\alpha}_{0|t} \circ I^{\alpha}_{0|t}\big)g(t) = g(t) \ for \ all \ g \in L^q([0,T]), \tag{2.10}$$

and

$$(-1)^n d^n_t D^{\alpha}_{t|T} g(t) = D^{\alpha+n}_{t|T} g(t), \quad n \in \mathbb{N}, \ for \, all \ g \in \mathcal{AC}^{n+1}[0,T], \tag{2.11}$$

where $1 \leq q \leq \infty$, $n \in \mathbb{N}$ *and* d^n_t *is the ordinary* n^{th} *derivative.*

2.4.3 Typical example

In the following proposition, we state an example where we explain how the operator (2.8) operates. We also consider it as a preparatory result for Section 3.4.

Proposition 2.4. *Given* $\beta > 1$, *let* φ_2 *be the function defined by*

$$\varphi_2(t) = \Big(1-\frac{t}{T}\Big)^{\beta}_{+}. \tag{2.12}$$

Then, for all $\alpha \in (0,1)$ *and* $j=0,1,2$ *we have*

$$D^{\alpha+j}_{t|T}\varphi_2(t) = C_j T^{-\alpha-j}\Big(1-\frac{t}{T}\Big)^{\beta-\alpha-j}_{+}, \tag{2.13}$$

where

$$C_j = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1-j)}.$$

Proof. The proof of Proposition (2.4) depends on straight forward calculations. Indeed, we have by definition (2.14)

$$D^{\alpha}_{t|T}\varphi_2(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\varphi_2(s)}{(s-t)^{\alpha}}ds.$$

Then, using the Euler's change of variable

$$s \mapsto y = \frac{s-t}{T-t},$$

we have for $j = 0$ after using formula (2.3),

$$\begin{aligned} D^{\alpha}_{t|T}\varphi_2(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_t^T \frac{\left(1-\frac{s}{T}\right)^{\beta}}{(s-t)^{\alpha}}ds \\ &= \frac{T^{-\beta}}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\left((T-t)^{\beta-\alpha+1}\int_0^1 y^{-\alpha}(1-y)^{\beta}dy\right) \\ &= \frac{(\beta-\alpha+1)\,\mathscr{B}(1-\alpha,\beta+1)}{\Gamma(1-\alpha)}T^{-\beta}\,(T-t)^{\beta-\alpha} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\beta}(T-t)^{\beta-\alpha}, \end{aligned}$$

where $\mathscr{B}$ is the *Beta function* defined by (2.2).

For $j = 1, 2$ we apply directly formula (2.11), we get after using Lemma (2.1),

$$\forall t \in [0, T] : D^{\alpha+1}_{t|T}\varphi_2(t) = -d_t D^{\alpha}_{t|T}\varphi_2(t) \quad \text{and} \quad D^{\alpha+2}_{t|T}\varphi_2(t) = d_t^2 D^{\alpha}_{t|T}\varphi_2(t).$$

Hence the result is concluded. □

Corollary 2.2. *Let φ_2 be the function defined by (2.12). Then for all $\alpha \in (0, 1)$ and $j = 0, 1, 2$ one has*

$$D^{\alpha+j}_{t|T}\varphi_2(0) = C_j T^{-\alpha-j},$$

where the constants C_j are as in Proposition 2.4.

2.5 Tools from harmonic analysis

In all this section $p \in [1, +\infty]$ and q is its harmonic conjugate that is

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then we have :

Proposition 2.5. ***(Young's inequality)****[4] Let $1 < p, q < \infty$ be real conjugates, then*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}. \qquad \textit{For all } a, b \in \mathbb{R}.$$

As a consequence, if $p = q = 2$, we have

$$|ab| \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \quad \forall \varepsilon > 0.$$

It is sometimes convenient to use the form $ab \le \varepsilon a^p + C_\varepsilon b^q$ with $C_\varepsilon = \varepsilon^{-1/(p-1)}$.

Proposition 2.6. ***(Minkowski's Inequality).****[4] Assume $1 \le p \le \infty$ and $u, v \in L^p(\Omega)$. Then*

$$\|u + v\|_{L^p} \le \|u\|_{L^p} + \|v\|_{L^p}.$$

Proposition 2.7. ***(Hölder's inequality)*** *[4]*
Let Ω be an open subset of $\mathbb{R}^n$. Suppose that $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$ where $1 \le p \le \infty$. Then, $uv \in L^1$ and

$$\int |uv| \le \|u\|_p \|v\|_q .$$

Proposition 2.8. ***(Cauchy-Schwarz inequality)****[4] Let X be a Hilbert space and $\langle \cdot, \cdot \rangle$ be its inner product, then*

$$|\langle f, g \rangle| \le \langle f, f \rangle^{\frac{1}{2}} \langle g, g \rangle^{\frac{1}{2}}, \quad \forall f, g \in X.$$

Gagliardo-Nirenberg inequality types are used, in general, to estimate the L^p-norms of a given function u by its L^r and L^q norms of u and its derivatives. It is announced as follows:

Proposition 2.9. *[16] Let $j, m \in \mathbb{N}$ with $j < m$ and let $u \in \mathscr{C}_0^m(\mathbb{R}^n)$. Let $\theta \in [\frac{j}{m}, 1]$ and let $p, q, r \in [1, \infty]$ be such that*

$$j - \frac{m}{q} = \Big(m - \frac{n}{r}\Big)\theta - \frac{n}{p}(1 - \theta).$$

Then,

$$\left\|D^j u\right\|_{L^q} \lesssim \|D^m u\|_{L^r}^{\theta} \|u\|_{L^p}^{1-\theta}$$

On condition that

$$\Big(m - \frac{n}{r}\Big) - j \in \mathbb{N}, \text{ that is, } \frac{n}{r} + j > m \text{ or } \frac{n}{r} \notin \mathbb{N}.$$

If

$$(m - \frac{n}{r}) - j \in \mathbb{N},$$

then Gagliardo-Nirenberg inequality holds provided that $\theta \in [\frac{j}{m}, 1)$.

In the particular case $j = 0$, $m = 1$ and $r = p = 2$, the Gagliardo-Nirenberg inequality reduces to the classical Gagliardo-Nirenberg inequality which used frequently to estimate the L^p-norm of a given function u via L^2-norms of u and the gradient of u. Classical Gagliardo-Nirenberg inequality is summarized in the following corollary

Corollary 2.3.

$$|u\|_{L^q} \lesssim \|\nabla u\|_{L^2}^{\theta(q)} \|u\|_{L^2}^{1-\theta(q)}$$

Where $\theta(q)$ is defined by the equation

$$\theta(q) = \frac{n}{2}\Big(1 - \frac{1}{q}\Big).$$

Fubini theorem is a a fundamental result in integral calculus that used to evaluate the integral of an integrable function of two variables expressed in the separate form.

Theorem 2.6. ***(Fubini)*** *If $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, then:*

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} f(x,y)\, d(x,y) = \int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^m} f(x,y)\, dy\Big) dx = \int_{\mathbb{R}^m} \Big(\int_{\mathbb{R}^n} f(x,y)\, dx\Big) dy, \tag{2.14}$$

provided the inner integral exists.

The dominate convergence theorem is one of the useful in analysis. it allows to interchanging the limit with the integral of a functional sequence. The version below is a corollary of dominated convergence theorem of Lebesgue.

Theorem 2.7. ***(Dominated Convergence Theorem)***
Let (f_n) be a sequence of measurable functions on $\mathbb{R}^n$ converging simply almost everywhere to a function f. If there exists a function $g \in L^1(\mathbb{R}^n)$ such that $|f_n(x)| \leq g(x)$ for all n and a.e. $x \in \mathbb{R}^n$, then $f \in L^1(\mathbb{R}^n)$, and

- $\|f_n - f\| \to 0.$
- $\lim\limits_{n\to\infty} \int_{\mathbb{R}^n} f_n(x)\, dx = \int_{\mathbb{R}^n} f(x)\, dx.$

2.6 Diffusion phenomena

The diffusion phenomenon between classical damped wave models and the heat equation is one of the main reasons for studying solution estimates in equation (1.5). The key result is the remarkable relationship between the solutions of the heat equation and the damped wave equation. In other words, solutions to damped waves seem to behave more like solutions of the heat equation at large times, which allows us to use estimates of solutions to heat models to understand some properties of the solutions to equation (1.5). It is worth noting that the critical exponent obtained for damped wave equations with nonlinear memory is exactly the same as the critical exponent for the heat equation with nonlinear memory

$$\begin{cases} u_t - \Delta u = \int_0^t (t-\tau)^{-\gamma} |u(\tau,\cdot)|^p d\tau, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = f(x), \quad x \in \mathbb{R}^n. \end{cases}$$

2.7 Concepts of Blow-Up

Introduction. In the theory of nonlinear evolution equations, a solution is said to be global in time if it is defined for all positive times. Conversely, if a solution exists only on a bounded time interval of the $[0,T)$, it is called a local (in time) solution. In the last case, and when the maximal existence time is associated with a blow-up alternative, that is the solution grows without to be bound within a finite period of time or over a limited region of space the solution is said to blow-up in finite time. However, to give a precise meaning to the notion of finite time blow-up, it is necessary to clearly specify the functional space in which the solution is considered and the norm used to "measure" it. This phenomenon arises in different types of PDEs, such as parabolic, hyperbolic, and elliptic equations.
Based on this remark, we may conclude the following definition:

Definition 2.15. *Let $X \subset \mathbb{R}^n$ and $v = v(t,x)$ be a solution of a given evolution PDE on the set $X = [0,T] \times A$. We say that v blows up in finite time T if such that*

$$\lim_{t \to T^-} |v(t,x)| = +\infty.$$

In this case one has

$$\sup_{x\in X}|v(t,x)| = +\infty.$$

and T is called the time of Blow-up.

A referential example of the blow-up phenomenon in partial differential equations. The blow-up phenomenon arises particularly when the unknown function in the studied problem depends on both time and the spatial variable, especially in reaction-diffusion problems or wave propagation evolution. The notion of blow-up showed the light for the first time in 1966 by the Japanese mathematician H. Fujita in his pioneering works especially in the study of Cauchy problem (2.15) below.

Critical exponent of Fujita

Let us examine the following Cauchy problem associated with Fujita's equation

$$\begin{cases} u_t - \Delta u = u^p, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = f(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.15}$$

This equation was studied by Fujita in 1966 in his pioneering work [18]. In particular, he showed that if $p \in (1, 1 + \frac{2}{n})$, then all solutions within a given class blow up in finite time.

Definition 2.16. *The upper bound $1 + \frac{2}{n}$, of the parameter p, is called the exponent of Fujita, it is denoted $p_{Fuj}(n)$. That is $p_{Fuj}(n) = 1 + \frac{2}{n}$ It is characterized by the following:*

- *If $1 < p < p_{Fuj}(n)$ then all solution blows up in finite time.*
- *If $p > p_{Fuj}(n)$ all solution is global in time, that is defined on $(0, \infty) \times X$, $X \subset \mathbb{R}^n$.*
- *If $p = p_{Fuj}(n)$ this is a critical case!*

To prove the blow-up of solutions in partial differential equations (PDEs), particularly in nonlinear damped wave equations, *the test function method* and *the modified test function method* are commonly utilized. Below is a comprehensive explanation of each method, how they function, and the differences between them:

1) **Standard Test Function Method**

Definition 2.17. *The test function method involves the use of a smooth, compactly supported function (the test function) to derive inequalities that provide information about the behavior of the solution to a PDE. This method often relies on energy estimates or integral inequalities to show that a solution cannot remain bounded for all time.*

The basic steps of how this method works:

- **Selection of Test Functions:** Choose a suitable test function $\varphi = \varphi(t, x)$ that is smooth and has compact support within a domain of interest.
- **Multiplication and Integration:** Multiply the differential equation by the test function and integrate over the relevant domain. This step often leads to inequalities that provide insights into the solution's behavior.
- **Estimation of Growth:** By analyzing the resulting integral inequalities, one can derive conditions under which solutions may blow up in finite time.

2) **Modified Test Function Method**

Definition 2.18. *The modified test function method is an extension of the traditional test function method, incorporating adjustments to improve its applicability and effectiveness in proving blow-up results. This method often involves altering the choice of test functions or adding additional terms to better capture the dynamics of the solutions.*

The basic steps of how this method works :

- **Improved Test Functions:** Choose or create modified test functions that may incorporate parameters or extra terms to capture the specific features of the problem under investigation.
- **Integration with Modifications:** Like the original test function method, the differential equation is multiplied by the modified test functions and then integrated. However, this step may include more complex interactions or extra terms that account for nonlinearity or other contributing factors.

- **Refined Analysis:** The modifications allow for a more nuanced analysis of how solutions behave as they approach blow-up, often leading to sharper results regarding lifespan estimates and conditions for blow-up.

The modified test function method can yield more robust results than the standard approach by uncovering finer details of the solution's behavior near critical thresholds.

3) **Differences Between the two Methods**

A) **Complexity:** The modified test function method is, in general, more complex than the standard test function method because it involves a family of functions that can adapt to the blow-up behavior, whereas the standard test function method typically uses a single, fixed function.

B) **Adaptability:** The modified test function method is more flexible and can capture a wider range of blow-up scenarios by adjusting the test functions based on the solution's behavior. The standard test function method may not be as effective in cases where the solution's growth is not uniform or predictable.

C) **Results:** The modified test function method can often yield sharper results or conditions for blow-up, especially in cases where the blow-up is influenced by the structure of the solution itself.

In the context of blow-up phenomena for damped wave equations, the choice between classic and modified test functions can significantly impact the results. The classic method may provide a broad understanding of when blow-up occurs, but it might not capture the nuances of how the blow-up develops over time or under specific initial conditions. The modified method allows for a more detailed analysis by incorporating additional information about the damping terms or the geometry of the domain into the test functions. This can lead to more precise criteria for blow-up or even reveal new types of blow-up behavior that were previously unnoticed.

Chapter 3 Non influence of the visco-elastic term on the blow-up results

3.1 Introduction

The results presented in this chapter were published in [2].
In this chapter, we focus on the study of the viscoelastic damped wave model incorporating two types of damping and a nonlinear memory term as well. The first damping, which is assumed to be effective, is a friction that measures the dissipation in the PDE literature while the second is viscoelastic. Our aim is to get blow-up results and to find the critical exponent in Fujita sense for the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \displaystyle\int_0^t (t-s)^{-\gamma} |u(s,\cdot)|^p \, ds, \ t > 0, \\ u(0,x) = f(x), u_t(0,x) = g(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $\gamma \in (0,1)$ and $p > 1$. Before starting the study of blow-up, it is necessary, first, to ensure that Cauchy problem (3.1) is, really, well posed in some appropriate space, typically, the energy space. This what we will do in Section 3.3. For this reason, we begin by proving a local well posedness result for Cauchy problem (3.1). In other hand, in order to clarify our aim, we see that is better to add a motivation of our goal.

3.2 Motivation

Cauchy problem (4) is considered in [14] where the authors investigate global (in time) existence for small data solution and proposed as a critical exponent the parameter (1.6). In particular, they proved that if $p > p_{Fuj}(\gamma, n)$ then Cauchy problem (4) is dissipative and has a unique global energy solution, but they don't show what happens if $1 < p < p_{Fuj}(\gamma, n)$.

The researchers in this domain consider this as a gap. In fact, for this values of p the solutions of (4) blow-up in finite time. This what we will discover in Section 3.4. By filling this gap, one can understand the interplay between effective damping, viscoelasticity and nonlinear memory terms, which play a significant role in the qualitative behavior of solutions and show that the proposed parameter (1.6) is sharp.

3.3 Local existence

The aim of this section is to establish a local (in time) well-posedness result for the Cauchy problem 3.1 with assuming that the data (f, g) belongs to the energy space $H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. To do so, let us, first, turn for $T > 0$ to the linear Cauchy problem for non homogeneous viscoelastic damped wave equation

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = F(t,x), (t,x) \in (0,T)\times\mathbb{R}^n, \\ u(0,x) = f(x),\ u_t(0,x) = g(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{3.2}$$

The key result from PDE analysis related to the Cauchy problem 3.2 and needed in this section is given in the following lemma:

Lemma 3.1. *[56] Let $(f, g) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and $F \in \mathscr{C}\big([0,T], L^2(\mathbb{R}^n)\big)$ for all $T > 0$. Then, there exists a unique energy solution $u \in \mathscr{C}\big([0,T], H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T], L^2(\mathbb{R}^n)$ to (3.2). Moreover, the energy solution $u = u(t,x)$ satisfies to the following estimates for any t in the interval $(0,T)$:*

$$\begin{aligned} &\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \le \|\nabla f\|_{L^2} + \|g\|_{L^2} + \int_0^t \|F(s,\cdot)\|_{L^2}\, ds, \\ &\|u(t,\cdot)\|_{L^2} \le \|f\|_{L^2} + t\big(\|\nabla f\|_{L^2} + \|g\|_{L^2}\big) + \int_0^t \Big(\int_0^s \|F(\tau,\cdot)\|_{L^2}\, d\tau\Big)\, ds. \end{aligned}$$

Proof. By multiplying equation $u_{tt} - \Delta u + u_t - \Delta u_t = 0$ by u_t and integrating over $\mathbb{R}^n$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\big(|u_t|^2 + |\nabla u|^2\big)dx - \int_{\mathbb{R}^n}\nabla.(u_t\nabla u)dx + \int_{\mathbb{R}^n} u_t^2 dx - \int_{\mathbb{R}^n} u_t\Delta u_t dx = \int_{\mathbb{R}^n} F(t,x)u_t dx.$$

Next, we employ the divergence theorem, and using the fact that $u_t\nabla u \in L^1$, we find after using Green's formula

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\big(|u_t|^2+|\nabla u|^2\big)dx &\le C\int_{\mathbb{R}^n}F(t,x)u_t(t,x)dx\\ &\le C\|F(t,\cdot)\|_{L^2}\|u_t(t,\cdot)\|_{L^2}.\end{aligned}$$

Utilizing the Cauchy-Schwarz inequality, and then integrating over the interval $[0,t]$, We derive

$$\frac{1}{2}\big(\|u_t(t,\cdot)\|_{L^2}^2+\|\nabla u(t,\cdot)\|_{L^2}^2\big)\le\frac{1}{2}\big(\|g\|_{L^2}+\|\nabla f\|_{L^2}\big)^2+C\int_0^t\|F(s,\cdot)\|_{L^2}\|u_t(s,\cdot)\|_{L^2}ds.$$

By applying Gronwall-type Lemmas precisely Lemma 2.2, we achieve at

$$\|u_t(t,\cdot)\|_{L^2}+\|\nabla u(t,\cdot)\|_{L^2}\le C\big(\|g\|_{L^2}+\|\nabla f\|_{L^2}+\int_0^t\|F(s,\cdot)\|_{L^2}ds\big). \tag{3.3}$$

For the second estimate, we may deduce it directly from the above estimate and formula

$$u(t,\cdot)=f+\int_0^t u_t(s,\cdot)ds. \tag{3.4}$$

□

Now, we are able to present the main result of local existence and uniqueness of energy solution for Cauchy problem (3.1) in the follwing theorem.

Theorem 3.1. *Let $\gamma\in(0,1)$ and $(f,g)\in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. If $p>1$ for $n=1,2$ or $1\le p\le\frac{n}{n-2}$ for $n\ge 3$, then the Cauchy problem*

$$\begin{cases}u_{tt}-\Delta u+u_t-\Delta u_t=\displaystyle\int_0^t(t-\tau)^{-\gamma}|u(\tau,\cdot)|^p\,d\tau, & (t,x)\in(0,\infty)\times\mathbb{R}^n,\\ u(0,x)=f(x),\quad u_t(0,x)=g(x), & x\in\mathbb{R}^n,\end{cases}$$

has a unique local (in time) energy solution

$$u\in\mathscr{C}\big([0,T_{max}),H^1(\mathbb{R}^n)\big)\cap\mathscr{C}^1\big([0,T_{max}),L^2(\mathbb{R}^n)\big),$$

where $T_{max}\le\infty$.

The proof of Theorem 3.1 is based on the Banach fixed point theorem. In order to Lemma 3.1, we have to ensure that the function

$$h(s,v(s,x))=\int_0^s(s-\tau)^{-\gamma}|v(\tau,x)|^p\,d\tau. \tag{3.5}$$

satisfy the requirement of Lemma 3.1. This what we will do in the following lemma.

Lemma 3.2. *Let $T > 0$ and h be the function defined by (3.5). Then, the L^2-norm of $h(t,\cdot)$ fulfils the following estimate for all $t \in (0,T)$:*

$$\|h(t,v)\|_{L^2} \lesssim t^{1-\gamma} \|v\|^p_{L^\infty((0,T),H^1)}.$$

In particular, $h \in \mathscr{C}\big([0,T], L^2(\mathbb{R}^n)\big)$.

Proof. Applying Gagliardo-Nirenberg inequality !!, we obtain

$$\begin{aligned} \|h(t,v)\|_{L^2} &\lesssim \int_0^t (t-s)^{-\gamma} \|v(s,\cdot)\|^p_{L^{2p}}\, ds \\ &\lesssim \int_0^t (t-s)^{-\gamma} \|v(s,\cdot)\|^{p(1-\theta)}_{L^2} \|\nabla v(s,\cdot)\|^{p\theta}_{L^2}\, ds, \end{aligned}$$

with $\theta = \frac{n}{2} - \frac{n}{2p} \in (0,1]$ if and only if

$$p > 1 \quad \text{for} \quad n = 1,2 \quad \text{and} \quad 1 < p \leq \frac{n}{n-2} \quad \text{for } n \geq 3. \tag{3.6}$$

Noting that $0 < \theta < 1$, we may derive by applying Young's inequality the following estimate:

$$\begin{aligned} \|h(t,v)\|_{L^2} &\lesssim \int_0^t (t-s)^{-\gamma} \big(\|v(s,\cdot)\|^p_{L^2} + \|\nabla v(s,\cdot)\|^p_{L^2}\big)\, ds \\ &\lesssim \int_0^t (t-s)^{-\gamma} \|v(s,\cdot)\|^p_{H^1}\, ds \\ &\lesssim \sup_{t\in(0,T)} \|v(t,\cdot)\|^p_{H^1} \int_0^t (t-s)^{-\gamma}\, ds \lesssim t^{1-\gamma} \|v\|^p_{L^\infty((0,T),H^1)}. \end{aligned}$$

finally, from this estimate we may conclude that $h \in \mathscr{C}\big([0,T], L^2(\mathbb{R}^n)\big)$. □

Proof of Theorem 3.1. Let us define, for some $T > 0$, the space of solutions

$$X(T) = \mathscr{C}\big([0,T], H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T], L^2(\mathbb{R}^n)\big),$$

occupied with the norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} \big\{ \|u(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \big\}.$$

For any $R > 0$, we denote by $B_R(T)$ for the centered ball in $X(T)$ defined by

$$B_R(T) = \{u \in X(T) \,:\, \|u\|_{X(T)} \leq R\},$$

and consider the mapping N on $X(T)$ defined by $N : v \to N(v) = u$, where $u = u(t,x)$ is the energy solution to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + u_t - \Delta u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma} |v(\tau,\cdot)|^p \, d\tau, & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

Using Duhamel's principle with the fundamental solutions E_0 and E_1 together, one can express explicitly $N(v) = u$ as follows:

$$N(v) = E_0(t,0,x) *_x f + E_1(t,0,x) *_x g + \int_0^t E_1(t,s,x) * h(s, v(s,x)) \, ds, \tag{3.7}$$

To establish that N has a unique fixed point in $X(T)$, it suffices to show the existence of some positives T and R such that

- For all $v \in B_R(T)$ we have $N(v) \in B_R(T)$,
- N is a contracting mapping from $B_R(T)$ to $B_R(T)$, that is

$$\exists k \in (0,1) : \|N(v_1) - N(v_2)\|_{X(T)} \le k \|v_1 - v_2\|_{X(T)} \quad \text{for all } v_1, v_2 \in B_R(T). \tag{3.8}$$

Let $v \in B_R(T)$. By applying the estimates from Lemma 3.1, we obtain the following results:

$$\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \le C I_0 + C \int_0^t \|h(s, v(s,\cdot))\|_{L^2} \, ds, \tag{3.9}$$

where

$$I_0 = \|f\|_{L^2} + \|\nabla f\|_{L^2} + \|g\|_{L^2},$$

and

$$\|u(t,\cdot)\|_{L^2} \le C(1+T) I_0 + C \int_0^t \Big(\int_0^s \|h(\tau, v(\tau,\cdot))\|_{L^2} \, d\tau \Big) \, ds. \tag{3.10}$$

From the estimates (3.9) and (3.10) it follows

$$\|u\|_{X(T)} \le C I_0 + C T^{2-\gamma} R^p,$$

where C is a suitable positive constant and T is supposed to be smaller or equal to 1. If we choose $R = 2CI_0$, then we can choose $T \le 1$ so small such that $\|u\|_{X(T)} \le R$. Consequently, $u = N(v) \in B_R(T)$.

Now we deal with the contraction property. So, let us consider $v_1, v_2 \in B_R(T)$ such that $u_1 = N(v_1)$ and $u_2 = N(v_2)$. Putting $w = u_1 - u_2$ the function $w = w(t,x)$ is the energy solution to the homogeneous Cauchy problem:

$$\begin{cases} w_{tt} - \Delta w + w_t - \Delta w_t = \displaystyle\int_0^t (t-s)^{-\gamma}(|v_1(s,\cdot)|^p - |v_2(s,\cdot)|^p)\, ds, \ \ (t,x) \in (0,T) \times \mathbb{R}^n, \\ w(0,x) = w_t(0,x) = 0, \ \ x \in \mathbb{R}^n. \end{cases}$$

As a result of the estimates provided in Lemma 3.1, it can be concluded that

$$\|\nabla w(t,\cdot)\|_{L^2} + \|w_t(t,\cdot)\|_{L^2} \lesssim \int_0^t \int_0^s (s-\tau)^{-\gamma} \||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2}\, d\tau\, ds, \tag{3.11}$$

and

$$\|w(t,\cdot)\|_{L^2} \lesssim \int_0^t \int_0^s \int_0^\tau (\tau-\rho)^{-\gamma} \||v_1(\rho,\cdot)|^p - |v_2(\rho,\cdot)|^p\|_{L^2}\, d\rho\, d\tau\, ds. \tag{3.12}$$

Therefore, by Hölder's inequality we have

$$\||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2} \lesssim \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|_{L^{2p}} \big(\|v_1(\tau,\cdot)\|_{L^{2p}}^{p-1} + \|v_2(\tau,\cdot)\|_{L^{2p}}^{p-1}\big).$$

Applying the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^{2p}(\mathbb{R}^n)$ for $\frac{p}{p-1} \geq \frac{n}{2}$ we arrive at

$$\||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2} \lesssim \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|_{H^1} \big(\|v_1(\tau,\cdot)\|_{H^1}^{p-1} + \|v_2(\tau,\cdot)\|_{H^1}^{p-1}\big). \tag{3.13}$$

Setting (3.13) into (3.11) and (3.12) implies for $T \leq 1$ the relation

$$\|w(t,\cdot)\|_{L^2} + \|\nabla w(t,\cdot)\|_{L^2} + \|w_t(t,\cdot)\|_{L^2} \leq 2CR^{p-1}T^{2-\gamma}\|v_1 - v_2\|_{X(T)}.$$

Choosing $T \leq 1$ so small that $2CR^{p-1}T^{2-\gamma} \leq 1$ leads to the desired contracting property (3.8) by taking for example $k = 2CR^{p-1}T^{2-\gamma} = \frac{1}{2}$, that is,

$$\|N(v_1) - N(v_2)\|_{X(T)} = \|w\|_{X(T)} \leq \frac{1}{2}\|v_1 - v_2\|_{X(T)}.$$

Applying the Banach fixed point theorem, we can finalize the proof of the local (in time) well-posedness for energy solutions as follows

3.3.1 Concluding by the fixed point principle the existence of solution

Let $T > 0$. Define the recurrent functional sequence $(u^{(n)})_n$ on $[0,T] \times \mathbb{R}^n$ as follows:

$$\begin{cases} u^{(0)}(t,x) := u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \\ u^{(n)}(t,x) = N(u^{(n-1)})(t,x), \ n \geq 1, \ 0 < t \leq T, \ x \in \mathbb{R}^n. \end{cases} \tag{3.14}$$

Then, $(u^{(n)})_n$ satisfies the following propriety:

Proposition 3.1. *Let $(u^{(n)})_n$ be the sequence defined by (3.14), then there exists some $\bar{u} \in \mathscr{C}\big([0,T];\, H^1\big)$ such that $(u^{(n))_n}$ converges to $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^1\big)$ as $n \to \infty$ with respect to the topology of $L^\infty\big([0,T];\, H^1\big)$.*

First, let us proof that $(u^{(n)})_n$ is a Cauchy sequence.
For all $n \in \mathbb{N}$ and $t \in [0,T]$ we have by the contraction propriety of N

$$\begin{aligned} \|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^1} &= \|N(u^{(n)})(t,\cdot) - N(u^{(n-1)})(t,\cdot)\|_{H^1} \\ &\le k\|u^{(n)}(t,\cdot) - u^{(n-1)}(t,\cdot)\|_{H^1}. \end{aligned} \tag{3.15}$$

We iterate inequality (3.15) n times we find

$$\|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^1} \le \;\; k^n\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^1}. \tag{3.16}$$

Next, for all $p \in \mathbb{N}$ we get after using (3.16)

$$\begin{aligned} \|u^{(n+p)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^1} &\le \|u^{(n+p)}(t,\cdot) - u^{(n+p-1)}(t,\cdot)\|_{H^1} + \cdots + \|u^{(n+1)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^1} \\ &\le (k^{n+p-1} + \cdots + k^n)\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^1} \\ &\le \frac{k^n(1-k^p)}{1-k}\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^1} \\ &\le \frac{k^n}{1-k}\|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^1}, \quad \text{since} \quad k<1. \end{aligned} \tag{3.17}$$

The inequality (3.17) holds for all $t \in [0,T]$ therefore, we get from (3.17)

$$\sup_{t\in[0,T]} \|u^{(n+p)}(t,\cdot) - u^{(n)}(t,\cdot)\|_{H^1} \le \frac{k^n}{1-k} \sup_{t\in[0,T]} \|u^{(1)}(t,\cdot) - u^{(0)}\|_{H^1} \tag{3.18}$$

Since the right hand side of (3.18) is independent of p and goes to 0 as $n \to +\infty$ then $(u^{(n)})_n$ is a Cauchy sequence in $\mathscr{C}\big([0,T];\, H^1\big)$.

Finally, since the space $\mathscr{C}\big([0,T];\, H^1\big)$ is a complete (and then closed) subspace of $L^\infty\big([0,T];\, H^1\big)$ then $(u^{(n)})_n$ converges to some $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^1\big)$. The aim, now, is to show that this $\bar{u}$ belongs to $X(T)$ and is solution to Problem (3.1). This is the aim of the following proposition.

Proposition 3.2. *Let $(u^{(n)})_n$ be the sequence defined by (3.14) and let $\bar{u}$ be its limit in $\mathscr{C}\big([0,T];\, H^1\big)$. Then $\bar{u}$ belongs to $X(T)$ and satisfies*

$$N(\bar{u}) = \bar{u}.$$

In particular $\bar{u}$ is solution to Problem (3.1).

Since $u^{(n)} \in B_R(T)$, then $\big(u^{(n)}\big)_n$ and $\big(u_t^{(n)}\big)_n$ has a weak convergent subsequences $\big(u^{(n_k)}\big)_k$ and $(u_t^{(n_k)})_k)$ in $L^\infty\big([0,T];\, H^1\big)$ and $L^\infty\big([0,T];\, L^2\big))$ respectively. By Proposition 3.1, $(u^{(n)})_n$ converges to $\bar{u}$ in $\mathscr{C}\big([0,T];\, H^1\big)$, then, the above subsequences converge weakly to $\bar{u}$ and $\bar{u}_t$ in $L^\infty\big([0,T];\, H^1\big)$ and $L^\infty\big([0,T];\, L^2\big))$ respectively. As a consequence, we show that $\bar{u} \in L^\infty\big([0,T]; H^1\big)$ and $\bar{u}_t \in L^\infty\big([0,T]; L^2\big)$. This means that

$$\bar{u} \in L^\infty\big([0,T]; H^1\big) \cap L^\infty\big([0,T]; L^2\big) \cap B_R(T)\}$$

Consequently,

$$N(\bar{u}) \in L^\infty\big([0,T]; H^1\big) \cap L^\infty\big([0,T]; L^2\big) \cap B_R(T).$$

Next, applying the contraction proriety (3.8) of N, we get

$$\sup_{0\le t\le T} \|N(u^{(n)}) - N(\bar{u})\|_{X(T)} \le k \sup_{0\le t\le T} \|u^{(n)}(t,\cdot) - \bar{u}(t,\cdot)\|_{X(T)}, \quad k \in]0,1[. \tag{3.19}$$

Since the right hand side of (3.19) goes to 0 as $n \to +\infty$, then $N(u^{(n)})$ converges to $N(\bar{u})$ in $\mathscr{C}\big([0,T];\, H^1\big)$. Passing to the limit in $u^{(n)} = N(u^{(n-1)})$ as $n \to +\infty$, we get after using the fact that $u_n^{(n)} \to \bar{u}$ in $\mathscr{C}\big([0,T];\, H^1\big)$

$$N(\bar{u}) = \bar{u} \in B_R(T)$$

Finally, by definition of N, this $\bar{u}$ is apparently, the desired solution of Problem (3.1).

3.3.2 Uniqueness of the solution

The uniqueness of a such solution in $X(T)$ follows immediately from formula (3.8). Assume that Problem (3.1) admits two solutions u_1 and u_2. Then one has

$$N(u_1) = u_1 \quad \text{and} \quad N(u_2) = u_2. \tag{3.20}$$

Again, by using the contraction propriety (3.8) of N, we get

$$\sup_{0\leq t\leq T} \|N(u_1)(t,\cdot) - N(u_2)(t,\cdot)\|_{X(T)} \leq k \sup_{0\leq t\leq T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{X(T)}. \tag{3.21}$$

Thanks to (3.20), we get from (3.21) the following inequality

$$(1-k) \sup_{0\leq t\leq T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{X(T)} \leq 0. \tag{3.22}$$

Using the fact that $0 < k < 1$ we get from (3.22)

$$\sup_{0\leq t\leq T} \|u_1(t,\cdot) - u_2(t,\cdot)\|_{X(T)} = 0. \tag{3.23}$$

This implies that

$$u_1(t,x) - u_2(t,x) = 0, \quad \textit{for all } t > 0 \quad \textit{and } x \in \mathbb{R}^n.$$

that is

$$u_1 = u_2.$$

This achieves the proof of Theorem 4.2. □

3.4 Blow-up of solutions

Among the various methods to achieve blow-up results, the test function method is a key approach introduced by Zhang in [61]. This method is successfully applied for parabolic models (See [18, 62]). Recently, it is discovered that this method could also be utilized for classical damped wave models having "parabolic-like behavior" in terms of decay estimates. However, this method fails to give positive results for classical wave models. The model we consider in this section is parabolic-like then to show blow-up results, we may employ the test function method.

Before presenting nonexistence results of a global solution for (3.1), let us introduce, first, the notion of weak solution for (3.1). Noting that

$$\int_0^t (t-s)^{-\gamma}|u(s,\cdot)|^p\,ds = \Gamma(1-\gamma) I_{0|t}^{1-\gamma}(|u(t,\cdot)|^p). \tag{3.24}$$

Then one may conclude the definition of weak solution as follows

Definition 3.1. *Let $T > 0$ and $\gamma \in (0,1)$. A weak solution for the Cauchy problem (3.1) on $(0,T)\times\mathbb{R}^n$ with the initial data $f, g \in L^1_{loc}(\mathbb{R}^n)$ is a locally integrable function $u \in L^p\big((0,T), L^p_{loc}(\mathbb{R}^n)\big)$ satisfying the relation*

$$\begin{aligned}
\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)\,dx\,dt &+ \int_{\mathbb{R}^n} g(x)\varphi(0,x)\,dx - \int_{\mathbb{R}^n} f(x)\varphi_t(0,x)\,dx \\
&+ \int_{\mathbb{R}^n} f(x)\varphi(0,x)\,dx - \int_{\mathbb{R}^n} f(x)\Delta\varphi(0,x)\,dx \qquad (3.25)\\
= \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)\,dx\,dt &- \int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)\,dx\,dt \\
- \int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_t(t,x)\,dx\,dt &+ \int_0^T\int_{\mathbb{R}^n} u(t,x)\Delta\varphi_t(t,x)\,dx\,dt, \qquad (3.26)
\end{aligned}$$

for all nonnegative test functions $\varphi \in \mathscr{C}^2([0,T]\times\mathbb{R}^n)$ such that $\varphi(T,\cdot) = \varphi_t(T,\cdot) = 0$ and $\alpha = 1-\gamma$.

The usual method to get the weak formulation is to multiply the equation of (3.1) by φ and making integrations over $[0,T]\times\mathbb{R}^n$. At this stage, we are able to announce the blow-up result for the Cauchy problem (3.1) that reads as follows:

Theorem 3.2. *Let $0<\gamma<1$, $p \in (1,\infty)$ for $n = 1,2$ and $1 < p < \frac{n}{n-2}$ for $n \geq 3$. Assume that $(f,g) \in H^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and satisfy*

$$\int_{\mathbb{R}^n} f(x)dx > 0, \qquad \int_{\mathbb{R}^n} g(x)dx > 0. \tag{3.27}$$

Then,

1. *If $p \leq p_{Fuj}(\gamma,n)$ then the solutions of Cauchy problem (3.1) does not exist globally in time.*
2. *If $p < \frac{1}{\gamma}$ for $n \geq 1$ or $p = \frac{1}{\gamma}$ and $p \leq \frac{n}{n-2}$ for $n \geq 3$ then the solutions of Cauchy problem (3.1) does not exist globally in time.*

W here

$$p_{Fuj}(\gamma,n) = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+}.$$

Proof of Theorem 3.2. The proof of the Theorem 3.2 is achieved by contradiction argument. Assume that u is a non trivial global weak solution for Cauchy problem (3.1) in the sense of (3.1). Since the method depends on the right choice of the test function, we select a test function φ defined for some $T > 0$ as follows:

$$\varphi(t,x) = D^{\alpha}_{t|T}\psi(t,x) = \varphi_1^{\ell}(x)D^{\alpha}_{t|T}\varphi_2(t),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{3.28}$$

where $\ell > 1$ and $D^{\alpha}_{t|T}$ denotes the right fractional derivative operator of order α in the sense of Riemann-Liouville, and the functions φ_1 and φ_2 are given by

$$\varphi_1(x) = \phi\Big(\frac{x^2}{K}\Big), \quad \textit{and} \ \ \varphi_2(t) = \Big(1 - \frac{t}{T}\Big)_+^{\beta}, \tag{3.29}$$

with $\beta > 1$, K is a positive constant that will be appropriately chosen later, and ϕ is a non-increasing cut-off function satisfying the following conditions:

$$\phi(s) = \begin{cases} 1 & \text{if } 0 \le s \le 1, \\ \text{decreasing} & \text{if } 1 \le s \le 2, \\ 0 & \text{if } s \ge 2, \end{cases} \tag{3.30}$$

$0 \le \phi \le 1$ everywhere and satisfies the technique condition

$$|\phi'(s)| \le \frac{C}{1+s}.$$

We denote by Ω_T for the support of φ_1, that is

$$\Omega_T = \text{supp}\varphi_1 = \Big\{x \in \mathbb{R}^N, |x|^2 \le K\Big\}, \tag{3.31}$$

and by Δ_T, for the support of $\Delta\varphi_1$, defined as follows:

$$\Delta_T = \text{supp}\Delta\varphi_1 = \Big\{x \in \mathbb{R}^N, K \le |x|^2 \le 2K\Big\}. \tag{3.32}$$

Treatment of the left-hand side (3.25)

Introducing the test function defined by (3.28), we get by using the integration by parts formula (2.9) and the identity (2.10)

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dtdx &= \int_0^T \int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^p)D^{\alpha}_{t|T}\psi(t,x)dtdx \\ &= \int_0^T \int_{\mathbb{R}^n} D^{\alpha}_{0|T}I^{\alpha}_{0|T}(|u|^p)\psi(t,x)dtdx \\ &= \int_0^T \int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx. \end{aligned} \tag{3.33}$$

For the second term of the left-hand side of equality (3.25), we use Corollary 2.2 with $j = 0$, then we get

$$\begin{aligned}\int_{\mathbb{R}^n} g(x)\varphi(0,x)dx &= \int_{\mathbb{R}^n} g(x)\varphi_1^\ell(x) D^\alpha_{t|T}\varphi_2(t)\Big|_{t=0} dx \\ &= C_0 T^{-\alpha}\int_{\mathbb{R}^n} g(x)\varphi_1^\ell(x)dx, \end{aligned} \tag{3.34}$$

since

$$D^\alpha_{t|T}\varphi_2(t)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha} = C_0 T^{-\alpha}.$$

For the third term, noting that

$$\varphi_t(t,x) = \frac{\partial\varphi}{\partial t}(t,x) = -\varphi_1^\ell(x) D^{\alpha+1}_{t|T}\varphi_2(t),$$

then, after using Corollary 2.2 with $j = 1$ we get the following estimate:

$$\int_{\mathbb{R}^n} f(x)\varphi_t(0,x)dx = C_1 T^{-\alpha-1}\int_{\mathbb{R}^n} f(x)\varphi_1^\ell(x)dx, \tag{3.35}$$

since

$$D^{\alpha+1}_{t|T}\varphi_2(t)_{|t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1} = C_1 T^{-\alpha-1}.$$

In the same way we may estimate the forth term of the left hand-side of the weak formulation(3.25)

$$\int_{\mathbb{R}^n} f(x)\varphi(t,x)\Big|_{t=0} dx = C_0 T^{-\alpha}\int_{\mathbb{R}^n} f(x)\varphi_1^\ell(x)dx, \tag{3.36}$$

since

$$\varphi(t,x)\Big|_{t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}\varphi_1^\ell(x) = C_0 T^{-\alpha}\varphi_1^\ell(x).$$

Finally, after using results form Corollary 2.2 we have the estimate:

$$\int_{\mathbb{R}^n} f(x)\Delta\varphi(0,x)\,dx = CT^{-\alpha}\int_{\mathbb{R}^n} f(x)\Delta\varphi_1^\ell(x)\,dx. \tag{3.37}$$

Treatment of the right-hand side

Now, we deal with the terms of the right handed side of the weak formulation (3.26). First, by using the identity (2.11) we have

$$\varphi_{tt}(t,x) = \varphi_1^\ell(x) d_t^2 D^\alpha_{t|T}\varphi_2(t) = \varphi_1^\ell(x) D^{\alpha+2}_{t|T}\varphi_2(t).$$

Therefore

$$\int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_{tt}(t,x)dtdx = \int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x) D_{t|T}^{\alpha+2}\varphi_2(t)dtdx. \tag{3.38}$$

Using formula (2.11) we have

$$\varphi_t(t,x) = -\varphi_1^\ell(x) d_t D_{t|T}^{\alpha}\varphi_2(t) = -\varphi_1^\ell(x) D_{t|T}^{\alpha+1}\varphi_2(t).$$

Therefore

$$\int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_{t}(t,x)dtdx = -\int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x) D_{t|T}^{\alpha+1}\varphi_2(t)dtdx. \tag{3.39}$$

For the third term the formulation (3.26), after using the following identity

$$\Delta(\varphi_1^\ell) = \ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2, \tag{3.40}$$

we get

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u(t,x)\Delta\varphi(t,x)dtdx \\
&\quad = \int_0^T \int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big) D_{t|T}^{\alpha}\varphi_2(t)dtdx.
\end{aligned} \tag{3.41}$$

Finally, by using (2.11) and the identity (3.40), we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} u(t,x)\Delta\varphi_t(t,x)dtdx \\
&\quad = -\int_0^T \int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big) D_{t|T}^{\alpha+1}\varphi_2(t)dtdx.
\end{aligned} \tag{3.42}$$

Including formulas (3.33), (3.34), (3.35), (3.36), (3.37), (3.38), (3.39), (3.41) and (3.42) in the weak formulation (3.25)-(3.26) we get

$$\begin{aligned}
\Gamma(\alpha)&\int_0^T \int_{\mathbb{R}^n} |u|^p\psi(t,x)dtdx + C_0T^{-\alpha}\int_{\mathbb{R}^n} g(x)\varphi_1^\ell(x)dx + C_1T^{-\alpha-1}\int_{\mathbb{R}^n} f(x)\varphi_1^\ell(x)dx \\
&C_0T^{-\alpha}\int_{\mathbb{R}^n} g(x)\varphi_1^\ell(x)dx - C_0T^{-\alpha}\int_{\mathbb{R}^n} g(x)\Delta\varphi_1^\ell(x)dx \\
&= \int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x) D_{t|T}^{\alpha+2}\varphi_2(t)dtdx + \int_0^T \int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x) D_{t|T}^{\alpha+1}\varphi_2(t)dtdx \\
&- \int_0^T \int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big) D_{t|T}^{\alpha}\varphi_2(t)dtdx \\
&+ \int_0^T \int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big) D_{t|T}^{\alpha+1}\varphi_2(t)dtdx.
\end{aligned} \tag{3.43}$$

The fact that

$$\left|\ell\varphi_1^{\ell-1}\Delta\varphi+\ell(\ell-1)\varphi^{\ell-2}|\nabla\varphi_1|^2\right|\lesssim\varphi^{\ell-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)\tag{3.44}$$

with together $\varphi_1\leq 1$ allow us to have from formula (3.43) the following inequality:

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dtdx+C_0T^{-\alpha}\int_{\mathbb{R}^n}g(x)\varphi_1^\ell(x)dx+C_1T^{-\alpha-1}\int_{\mathbb{R}^n}f(x)\varphi_1^\ell(x)dx\\
&C_0T^{-\alpha}\int_{\mathbb{R}^n}g(x)\varphi_1^\ell(x)dx-C_0T^{-\alpha}\int_{\mathbb{R}^n}g(x)\Delta\varphi_1^\ell(x)dx\\
&\lesssim\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx\\
&\quad-\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi^{\ell-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)|D_{t|T}^{\alpha}\varphi_2(t)|dtdx\\
&\quad+\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi^{\ell-2}(|\Delta\varphi_1|+|\nabla\varphi_1|^2)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx.
\end{aligned}\tag{3.45}$$

Now, applying the following $\varepsilon-$*Young inequality*

$$AB\leq\varepsilon A^p+C(\varepsilon)B^q,\quad \varepsilon>0,\ C(\varepsilon)>0\ \text{ and }\ pq=p+q,$$

to the terms of the right-hand side of inequality (3.45) we get

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx&=\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\psi^{\frac{1}{p}}\psi^{-\frac{1}{p}}\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dtdx\\
&\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\psi dtdx+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2|^{\frac{p}{p-1}}dtdx.
\end{aligned}\tag{3.46}$$

For the second term of the right-hand side in (3.45) we have

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dtdx&\leq\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\psi(t,x)dtdx\\
&+\int_0^T\int_{\mathbb{R}^n}\varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx.
\end{aligned}\tag{3.47}$$

In the same way we estimate the third and forth terms as follows

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}^n}|u|\varphi_1^{\ell-2}\big(|\Delta\varphi_1|+|\nabla\varphi_1|^2\big)|D_{t|T}^{\alpha}\varphi_2|dtdx&\leq\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\psi dtdx\\
+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\nu_q(\varphi_1)\varphi_1^{\ell-2\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2|^{\frac{p}{p-1}}dtdx,
\end{aligned}\tag{3.48}$$

and

$$\int_0^T \int_{\mathbb{R}^n} |u|\varphi_1^{\ell-2}\big(|\Delta\varphi_1| + |\nabla\varphi_1|^2\big)|D_{t|T}^{\alpha+1}\varphi_2|dtdx \leq \varepsilon \int_0^T \int_{\mathbb{R}^n} |u|^p\psi dtdx$$
$$+C(\varepsilon)\int_0^T \int_{\mathbb{R}^n} \sigma_q(\varphi_1)\varphi_1^{\ell-2\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2|^{\frac{p}{p-1}}dtdx, \tag{3.49}$$

where

$$\sigma_q(\varphi_1) = |\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}.$$

Taking into account the fact that (3.27) implies that

$$\int_{\mathbb{R}^n} f(x)\varphi^\ell(x)dx > 0, \quad \int_{\mathbb{R}^n} g(x)\varphi^\ell(x)dx > 0.$$

We get from (3.45), (3.46), (3.47), (3.48) and (3.49), for ε small enough, the estimate

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p\psi(t,x)dtdx$$
$$\leq CT^{-\alpha}\int_{\mathbb{R}^n} f(x)|\Delta\varphi_1^\ell(x)|\,dx + C(I_1+I_2+I_3+I_4), \tag{3.50}$$

for some positive constant C, where

$$I_1 = \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dtdx$$
$$I_2 = \int_0^T \int_{\mathbb{R}^n} \varphi_1^\ell(x)\varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx,$$
$$I_3 = \int_0^T \int_{\mathbb{R}^n} \sigma_q(\varphi_1(x))\varphi_1^{\ell-2\frac{p}{p-1}}(x)\varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha}\varphi_2(t)|^{\frac{p}{p-1}}dtdx,$$
$$I_4 = \int_0^T \int_{\mathbb{R}^n} \sigma_q(\varphi_1(x))\varphi_1^{\ell-2\frac{p}{p-1}}(x)\varphi_2^{-\frac{1}{p-1}}(t)|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dtdx.$$

Case $p \leq p_{Fuj}(\gamma,n)$. We choose in (3.29) $K = T$. In order to estimate the integrals I_i, $i = 1,2,3,4$, we consider the scaled change of variables

$$x = T^{\frac{1}{2}}y \quad \text{and} \quad t = T\tau. \tag{3.51}$$

Noting that the integrals I_i are null outside Ω_T (defined by (3.31)), then, by using Fubini's theorem, we get, for I_1

$$I_1 = \Big(\int_{\Omega_T} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big) = J_{11}J_{12}. \tag{3.52}$$

We have

$$J_{11} = \int_{\Omega_T} \varphi_1^\ell(x)dx = T^{\frac{n}{2}} \int_0^2 \phi^\ell(y^2)dy = CT^{\frac{n}{2}}. \tag{3.53}$$

In the other hand, after using Proposition 2.4, we get

$$\begin{aligned} J_{12} &= \int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt \\ &= T \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}} \big(T^{-(\alpha+2)}(1-\tau)^{\beta-\alpha-2}\big)^{\frac{p}{p-1}} d\tau \\ &= T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{\beta-(\alpha+2)\frac{p}{p-1}} d\tau = CT^{1-(\alpha+2)\frac{p}{p-1}}. \end{aligned} \tag{3.54}$$

Putting (3.53) and (3.54) into (3.52) we find

$$I_1 = CT^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1}. \tag{3.55}$$

Now we deal with I_2. In the same, we have

$$I_2 = \Big(\int_{\Omega_T} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) J_{21}J_{22}. \tag{3.56}$$

Noting that $J_{21} = J_{11}$ then we have immediately

$$J_{21} = CT^{\frac{n}{2}}. \tag{3.57}$$

The estimate of J_{22} is straight forward. We have

$$J_{22} = \int_0^T |\varphi_2(t)|^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dt = CT^{1-(\alpha+1)\frac{p}{p-1}}. \tag{3.58}$$

Hence replace (3.57) and (4.51) in 3.56 we obtain

$$I_2 = CT^{-(\alpha+1)\frac{p}{p-1}+1+\frac{n}{2}}. \tag{3.59}$$

It remains to estimate I_3 and I_4 First we choose ℓ so large such that $\ell - 2\frac{p}{p-1} > 0$. This guarantees that $\varphi_1^{\ell-2\frac{p}{p-1}}$ is bounded since φ_1 is bounded. Next we observe that for all $j = 1, \cdots, n$ we have

$$\partial_{x_j}\varphi_1 = \partial_{x_j}\phi\Big(\frac{|x|^2}{K}\Big) = \frac{2}{K}\phi'\Big(\frac{|x|^2}{K}\Big)x_j, \ \ |\nabla\varphi_1|^{2q} = \Big(\frac{2}{K}\Big)^{2q}\Big|\phi'\Big(\frac{|x|^2}{K}\Big)\Big|^{2q}|x|^{2q}.$$

Then, after using the change of variables (3.51), the facts that $K = T$ and $|\phi'(s)| \leq \frac{C}{1+s}$ we arrive at

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla\varphi_1(x)|^{2q} dx &= \Big(\frac{2}{T}\Big)^{2q} T^{\frac{n}{2}} \int_{|y|^2\leq 2} \big|\phi'(|y|^2)\big|^{2q} \big(T|y|^2\big)^q dy \\ &\leq CT^{-q+\frac{n}{2}} \int_{|y|^2\leq 2} \frac{|y|^{2q}}{(1+|y|^2)^{2q}} dy \leq C_0T^{-q+\frac{n}{2}}. \end{aligned} \tag{3.60}$$

On the other hand, we have for all $j = 1, \cdots, n$ the following relations:

$$\partial^2_{x_j}\varphi_1 = \frac{2}{K}\partial_{x_j}\Big(\phi'\Big(\frac{|x|^2}{K}\Big)x_j\Big) = \frac{4}{K^2}\phi''\Big(\frac{|x|^2}{K}\Big)x_j^2 + \frac{2}{K}\phi'\Big(\frac{|x|^2}{K}\Big),$$
$$|\Delta\varphi_1(x)|^q = \Big|\frac{4}{K^2}\phi''\Big(\frac{|x|^2}{K}\Big)|x|^2 + \frac{2n}{K}\phi'\Big(\frac{|x|^2}{K}\Big)\Big|^q.$$

Then,

$$\int_{\mathbb{R}^n}|\Delta\varphi_1(x)|^q\,dx = \int_{\mathbb{R}^n}\Big|\frac{4}{K^2}\phi''\Big(\frac{|x|^2}{K}\Big)|x|^2 + \frac{2n}{K}\phi'\Big(\frac{|x|^2}{K}\Big)\Big|^q\,dx. \tag{3.61}$$

Next, applying Minkowski's inequality in (3.61) we get, after using the proposed change of variables, the estimate

$$\begin{aligned}
&\Big(\int_{\mathbb{R}^n}|\Delta\varphi_1(x)|^q\,dx\Big)^{\frac{1}{q}}\\
&\quad\le \Big(\int_{\mathbb{R}^n}\Big|\frac{4}{K^2}\phi''\Big(\frac{|x|^2}{K}\Big)|x|^2\Big|^q\,dx\Big)^{\frac{1}{q}} + \Big(\int_{\mathbb{R}^n}\Big|\frac{2n}{K}\phi'\Big(\frac{|x|^2}{K}\Big)\Big|^q\,dx\Big)^{\frac{1}{q}}\\
&\quad\le \Big(4^qT^{\frac{n}{2}-2q}\int_{|y|^2\le 2}\big|\phi''(|y|^2)(T|y|^2)\big|^q\,dy\Big)^{\frac{1}{q}}\\
&\qquad+\Big((2n)^qT^{\frac{n}{2}-q}\int_{|y|^2\le 2}|\phi'(|y|^2)|^q\,dy\Big)^{\frac{1}{q}}\\
&\quad\le \big(C_14^qT^{\frac{n}{2}-q}\big)^{\frac{1}{q}} + \big((2n)^qC_2T^{\frac{n}{2}-q}\big)^{\frac{1}{q}} \le CT^{\frac{n}{2q}-1},
\end{aligned}\tag{3.62}$$

where

$$C_1 = \int_{|y|^2\le 2}\big|\phi''(|y|^2)(|y|^2)\big|^q\,dy, \ \text{ and } \ C_2 = \int_{|y|^2\le 2}|\phi'(|y|^2)|^q\,dy.$$

Hence, taking into account that $q = \frac{p}{p-1}$ we find from (3.62) the estimate

$$\int_{\mathbb{R}^n}|\Delta\varphi_1(x)|^{\frac{p}{p-1}}\,dx \le CT^{\frac{n}{2}-\frac{p}{p-1}}. \tag{3.63}$$

Finally, both estimates (3.63) and (3.60) with the fact that $\ell - 2\frac{p}{p-1} > 0$ give the estimate

$$\int_{\Omega_T}\big(|\Delta(x)\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1(x)|^{2\frac{p}{p-1}}\big)\varphi_1(x)^{\ell-2\frac{p}{p-1}}dx = CT^{-\frac{p}{p-1}+\frac{n}{2}}. \tag{3.64}$$

Now, let us turn to estimate I_3 and I_4. Let us start with I_3. As before, we begin by applying Fubini's theorem we get

$$I_3 = \Big(\int_{\Omega_T}\sigma_q(\varphi_1(x))\varphi_1^{\ell-2q}dx\Big)\Big(\int_0^T\varphi_2^{-\frac{1}{p-1}}(t)|D^\alpha_{t|T}\varphi_2(t)|^qdt\Big) = J_{31}J_{32}. \tag{3.65}$$

Due to the estimate (3.64) we have immediately

$$J_{31} = \int_{\Omega_T} \big(|\Delta(x)\varphi_1|^{\frac{p}{p-1}} + |\nabla\varphi_1(x)|^{2\frac{p}{p-1}}\big)\varphi_1(x)^{\ell-2\frac{p}{p-1}}dx = CT^{-\frac{p}{p-1}+\frac{n}{2}}. \tag{3.66}$$

The estimate of J_{32} is immediate too by using the same argument as we did with J_{12} (see (3.54)). This leads to

$$J_{32} = \int_0^T \varphi_2(t)^{-\frac{1}{p-1}}|D^{\alpha}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}}dt = CT^{-\alpha\frac{p}{p-1}+1}. \tag{3.67}$$

Then including (3.66) and (3.67) into (3.65) we find

$$I_3 = CT^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}, \tag{3.68}$$

for some constant $C > 0$. In order to estimate I_4 we remark that by Fubini's theorem we have

$$I_4 = \Big(\int_{\Omega_T} \sigma_q(\varphi_1(x))\varphi_1^{\ell-2q}dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}}(t)|D^{\alpha+1}_{t|T}\varphi_2(t)|^q dt\Big) = J_{41}J_{42}. \tag{3.69}$$

Since

$$J_{41} = J_{31} \text{ and } J_{42} = J_{22},$$

then, using the estimates (3.66) and (4.51) we derive the following estimate for I_4

$$I_4 = CT^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1}, \tag{3.70}$$

for some constant $C > 0$. Finally, we replace (3.55), (3.59), (3.68) and (3.70) into (3.50) we obtain

$$\begin{aligned}&\int_0^T\int_{\Omega_T} |u|^p\psi(t,x)dtdx - CT^{-\alpha}\\ &\qquad \le C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1} + T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}\big) \le CT^{-\delta},\end{aligned} \tag{3.71}$$

where

$$\delta = (\alpha+1)\frac{p}{p-1} - \frac{n}{2} - 1.$$

At this stage, to prove the first result in Theorem.3.2, we distinguish between two sub-cases.

i) Sub-case $p < p_{Fuj}(\gamma, n)$.

In this case, one can remark that condition $p < p_\gamma(n)$ is equivalent to $\delta < 0$. Then we pass to the limit as $T \to \infty$ in (3.71), we get

$$\lim_{T\to+\infty} \int_0^T \int_{\Omega_T} |u|^p \psi(t,x)dtdx = 0. \tag{3.72}$$

Next, using the dominated convergence theorem of Lebesgue, the continuity of u with respect to t and x and the fact that

$$\lim_{T\to+\infty} \psi(t,x) = 1, \tag{3.73}$$

we find

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dtdx = 0,$$

which implies that

$$u = 0,$$

and this is a contradiction (because we have supposed that the solution u is not trivial).

ii) Subcase $p = p_{Fuj}(\gamma, n)$ First of all, one can remark that the condition $p = p_\gamma(n)$ is equivalent to $\delta = 0$. Next, taking the limit as $T \to \infty$ in (3.71) with the consideration $\delta = 0$ we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p dtdx < +\infty,$$

from which we can deduce that

$$\lim_{T\to\infty} \int_0^{+\infty} \int_{\Delta_T} |u|^p \psi dtdx = 0, \tag{3.74}$$

where Δ_T is defined by (3.32). Fixing arbitrarily R in $(0, T)$ for some $T > 0$ such that when $T \to \infty$ we don't have $R \to \infty$ at the same time and choosing in (3.29)

$$K = TR^{-1}.$$

Using the Hölder's inequality

$$\int_X uvd\mu \leq \Big(\int_X |u|^p d\mu\Big)^{\frac{1}{p}} \Big(\int_X |v|^q d\mu\Big)^{\frac{1}{q}}; \quad u \in L^p(X), \;\; v \in L^q(X), \, p, q > 1, pq = p + q,$$

instead of the $\varepsilon-$Young's one to estimate the integrals I_3 and I_4 on the set $\Delta_{TR^{-1}}$ (where they are not null in)

$$\Omega_{TR^{-1}} := \{x \in \mathbb{R}^n : |x|^2 \leq 2TR^{-1}\} = supp\varphi_1.$$

Noting that $supp\Delta\varphi_1 \subset \Delta_{TR^{-1}} \subset \Omega_{TR^{-1}}$ where

$$\Delta_{TR^{-1}} := \{x \in \mathbb{R}^n : TR^{-1} \leq |x|^2 \leq 2TR^{-1}\}, \tag{3.75}$$

then we get for $j = 0, 1$

$$\begin{aligned}\int_0^T \int_{\Omega_{TR^{-1}}} |u|\varphi_1^{\ell-2}\big(|\Delta\varphi_1^2 + |\nabla\varphi_1|^2\big)|D_{t|T}^{\alpha+j}\varphi_2|dtdx \leq \Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p\psi dtdx\Big)^{\frac{1}{p}} \\ \times\Big(\int_0^T \int_{\Delta_{TR^{-1}}} \varphi_1^{\ell-2q}\big(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\big)\varphi_2^{\frac{-1}{p-1}}|D_{t|T}^{\alpha+j}\varphi_2|^q dtdx\Big)^{\frac{1}{q}}.\end{aligned}$$

In order to estimate the integrals I_i, $i = 1, \cdots 4$ on the set $\Omega_{TR^{-1}}$ in this case, we consider the change of variables

$$x = T^{\frac{1}{2}}R^{-\frac{1}{2}}y, \quad t = T\tau. \tag{3.76}$$

Following the same steps as we did above we get the estimate

$$I_1 + I_2 \leq C\big(T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1} + T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}\big)R^{-\frac{n}{2}}. \tag{3.77}$$

Taking into account the hypothesis $p = p_{Fuj}(\gamma, n)$ (or equivalently $\delta = 0$) we get from (3.77)

$$I_1 + I_2 \leq CR^{-\frac{n}{2}}. \tag{3.78}$$

Now, let us turn to estimate the integrals I_3 and I_4. It is clear that is sufficient to estimate the integrals

$$\tilde{I}_j = \Big(\int_0^T \int_{\Delta_{TR^{-1}}} \varphi_1^{\ell-2q}\big(|\Delta\varphi_1|^q + |\nabla\varphi_1|^{2q}\big)\varphi_2^{\frac{-1}{p-1}}|D_{t|T}^{\alpha+j}\varphi_2|^q dtdx\Big)^{\frac{1}{q}} \quad j = 0, 1. \tag{3.79}$$

Following the same argument as we did to estimate I_3 above we obtain

$$\tilde{I}_j = C_j T^{-(\alpha+j+1)\frac{p}{p-1}+\frac{n}{2}} R^{\frac{p}{p-1}-\frac{n}{2}}. \tag{3.80}$$

Therefore, since $\delta = 0$, we get, for $j = 0$ the estimate for I_3

$$\begin{aligned}I_3 &= C(T^{-(\alpha+1)q+\frac{n}{2}+1}R^{q-\frac{n}{2}})^{\frac{1}{q}}\Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p\psi dtdx\Big)^{\frac{1}{p}} \\ &= CR^{1-\frac{n}{2q}}\Big(\int_0^T \int_{\Delta_{TR^{-1}}} |u|^p\psi dtdx\Big)^{\frac{1}{p}},\end{aligned} \tag{3.81}$$

and for $j = 1$

$$I_4 = C(T^{-(\alpha+2)q+\frac{n}{2}+1}R^{q-\frac{n}{2}})^{\frac{1}{q}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}$$
$$= CT^{-1}R^{1-\frac{n}{2q}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}. \tag{3.82}$$

Hence for T large enough ($T > 1$ is sufficient) we get from (3.81) and (3.82) the estimate

$$I_3 + I_4 \leq CR^{1-\frac{n}{2q}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}. \tag{3.83}$$

Using the estimates (3.83) and (3.78) we get from (3.50) the estimate

$$\int_0^T\int_{\mathbb{R}^n}|u(t,x)|^p\psi(t,x)dtdx - CT^{-\alpha}\int_{\mathbb{R}^n}u_0(x)|\Delta\varphi_1^\ell(x)|\,dx$$
$$\leq CR^{-\frac{n}{2}} + CR^{1-\frac{n}{2q}}\Big(\int_0^T\int_{\Delta_{TR^{-1}}}|u|^p\psi dtdx\Big)^{\frac{1}{p}}. \tag{3.84}$$

Now, taking the limit as $T \to +\infty$ in (3.84) we find after using (3.74) and the fact that $\lim_{T\to+\infty}\psi(t,x) = 1$ the estimate

$$\int_0^\infty\int_{\mathbb{R}^N}|u|^p\,dtdx \leq CR^{-\frac{n}{2}}.$$

which means that necessarily $R \to +\infty$ and this is a contradiction.

Now we deal with the second result in Theorem.3.2.

Case $p \leq \frac{1}{\gamma}$.

In this case we choose $K = R$ in (3.29) where R is any fixed positive real number. The aim is to estimate the integrals I_i, $i = 1, \cdots 4$ on the set

$$\Omega_R := \{x \in \mathbb{R}^N : |x| \leq 2R^{\frac{1}{2}}\} = supp\varphi_1,$$

since they are null outside $supp\varphi_1$. For this reason we consider the scaled change of variables

$$x = R^{\frac{1}{2}}y \quad t = T\tau.$$

We proceed as above, we may estimate the integrals I_i for $i = 1, \cdots 4$ which appear in 3.50).

Let us begin with I_1. By using Fubini's theorem we have

$$I_1 = \Big(\int_{\Omega_R}\varphi_1(x)^\ell dx\Big)\Big(\int_0^T\varphi_2(t)^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big)$$
$$= \Big(R^{\frac{n}{2}}\int_{|y|^2\leq 2}\phi^\ell(y^2)dy\Big) \times \Big(T^{1-(\alpha+2)\frac{p}{p-1}}\int_0^T(1-\tau)^{-\frac{\beta}{p-1}+(\beta-\alpha-2)\frac{p}{p-1}}d\tau\Big)$$
$$= CR^{\frac{n}{2}}T^{1-(\alpha+2)\frac{p}{p-1}}. \tag{3.85}$$

Similarly we find for I_2 the estimate

$$I_2 = CR^{\frac{n}{2}} T^{1-(\alpha+1)\frac{p}{p-1}}. \tag{3.86}$$

In the same way we estimate the integrals I_3 and I_4 on Δ_R where they are not null. Again by Fubini's theorem we have for $j = 0, 1$

$$\begin{aligned} I_j &= \int_0^T \int_{\Delta_R} \sigma_q(\varphi_1)\varphi_1^{\ell-2q} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+j}\varphi_2|^q dtdx \\ &= \Big(\int_{\Delta_R} \sigma_q(\varphi_1)\varphi_1^{\ell-2q} dx \Big) \Big(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+j}\varphi_2|^q dt \Big) \\ &= CR^{\frac{n}{2}-\frac{p}{p-1}} T^{1-(\alpha+j)\frac{p}{p-1}}. \end{aligned} \tag{3.87}$$

Including the estimates (3.87), (3.86) and (3.85) in (3.50) we get

$$\begin{aligned} &\int_0^T \int_{\Omega_R} |u|^p \psi(t,x) dtdx - CT^{-\alpha} \int_{\mathbb{R}^n} u_0(x) |\Delta\varphi_1^\ell(x)|\, dx \\ &= CR^{\frac{n}{2}} \big(T^{1-(\alpha+2)\frac{p}{p-1}} + T^{-(\alpha+1)\frac{p}{p-1}+1} \big) + CR^{\frac{n}{2}-\frac{p}{p-1}} (T^{1-\alpha\frac{p}{p-1}} + T^{1-(\alpha+1)\frac{p}{p-1}}) \\ &\leq CR^{\frac{n}{2}} T^{1-(\alpha+1)\frac{p}{p-1}} + CR^{\frac{n}{2}-\frac{p}{p-1}} T^{1-\alpha\frac{p}{p-1}}. \end{aligned} \tag{3.88}$$

Then, we distinguish between two cases.

i) Subcase $p < \frac{1}{\gamma}$.

First, noting that $p < \frac{1}{\gamma}$ implies $1 - \alpha\frac{p}{p-1} < 0$. So, the facts that

$$(\alpha+1)\frac{p}{p-1} > \alpha\frac{p}{p-1}$$

and

$$\lim_{T\to+\infty} \psi(t,x) = \varphi_1^\ell(x), \tag{3.89}$$

allow us to get from (3.88) by taking the limit as $T \to +\infty$

$$\int_0^{+\infty} \int_{\Omega_R} |u|^p \varphi_1^\ell(x)\, dtdx = 0. \tag{3.90}$$

Next, we take the limit in (3.90) as $R \to +\infty$ and use the fact that $\lim_{R\to+\infty} \varphi_1^\ell(x) = 1$ we get

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u|^p dtdx = 0.$$

This implies that $u = 0$ and this is a contradiction.

ii) Subcase $p = \frac{1}{\gamma}$.

In this case, the following assumption is useful

$$p < \frac{n}{n-2} \quad \text{if} \quad n \geq 3. \tag{3.91}$$

Noting that (3.91) implies

$$\frac{n}{2} - \frac{p}{p-1} < 0. \tag{3.92}$$

Since $p = \frac{1}{\gamma}$ and $\alpha = 1 - \gamma$ then

$$1 - \alpha\frac{p}{p-1} = 0, \quad \text{and} \quad 1 - (\alpha+1)\frac{p}{p-1} = -\frac{p}{p-1}. \tag{3.93}$$

Hence, taking the limit as $T \to \infty$ in (3.88) we get after using (3.93) the estimate

$$\int_0^\infty \int_{\Omega_R} |u|^p \varphi_1^\ell(x) dtdx = CR^{\frac{n}{2} - \frac{p}{p-1}}. \tag{3.94}$$

Next, by noting that if $n = 1, 2$ then $\frac{n}{2} - \frac{p}{p-1} < 0$ for all $p > 1$, then by taking the limit as $R \to \infty$ in (3.94) and using (3.92) and the fact that $\lim_{R \to +\infty} \varphi_1^\ell(x) = 1$ we derive

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p dtdx = 0. \tag{3.95}$$

This implies that $u = 0$ and this is a contradiction.

Remark 3.1. *One can note that the Gagliardo-Nirenberg condition (3.91) is needed only in the study of the critical case $p = \frac{1}{\gamma}$ and not otherwise. Note also that is equivalent to*

$$\frac{n-2}{n} < \gamma < 1, \quad n \geq 3,$$

in models containing a singular potential term like $\int_0^t (t-s)^{-\gamma}|u(s,\cdot)|^p ds$ which is known as a memory term in the PDE analysis.

Remark 3.2. *Noting that*

$$\lim_{\gamma \to 1} \int_0^t (t-s)^{-\gamma}|u(s,\cdot)|^p ds = \Gamma(1-\gamma)|u(t,\cdot)|^p,$$

in distributional sense. In the other hand

$$\lim_{\gamma \to 1} p_{Fuj}(\gamma, n) = 1 + \frac{2}{n} := p_{Fuj}(n).$$

*This interpret to us why does $p_{Fuj}(\gamma, n)$ coincide, when $\gamma \to 1$, with the critical exponent founded in [**?**] for the Cauchy problem corresponding to*

$$u_{tt} - \Delta u + u_t - \Delta u_t = |u|^p, \tag{3.96}$$

without additional regularity (that is when $m = 1$).

□

3.5 Global existence of small data energy solution

In this section, we present, some results of global existence of small data solution for Cauchy problem (4). The authors of the paper [14] present some interesting results of global small data well posedness for Cauchy problem (4) for various regularities. In order to close the cycle, it seems that is better to show what happens if $p > p_{Fuj}(\gamma, n)$? This what we will discover in this paragraph. In fact, If $p > p_{Fuj}(\gamma, n)$ then Cauchy problem (4) has global solution. Since we were interesting in the energy solutions, we will select from the paper [14] only the results concerning energy solution without proof. For this reason, we announce the related results in two theorems, the first one concerns with $p > p_{Fuj}(\gamma, n)$ while the second one is interested in the case $p > \frac{1}{\gamma}$. Before that, let us, First, fix for any real $\sigma > 0$ the data space

$$X_\sigma = X_\sigma(\mathbb{R}^n) := (H^\sigma(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \times (L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)),$$

occupied with the norm $\|\cdot\|$

$$\|(u, v)\|_{X_\sigma} = \|u\|_{H^\sigma} + \|u\|_{L^1} + \|v\|_{L^2} + \|v\|_{L^1}.$$

Then one has the following theorem of global existence of small data solution. Unfortunately, this result is valid in space dimensional smaller then 4.

Theorem 3.3. *Let $\sigma \in (0, 2)$ be a real number. Assume that $n < 2(2 - \sigma)$, the data $(u_0, u_1) \in X_\sigma$ and the exponent p satisfies the conditions:*

$$p > p_{Fuj}(\gamma, n). \tag{3.97}$$

and

$$\begin{cases} 2 \le p & \text{if} \quad \sigma \ge \frac{n}{2}, \\ 2 \le p \le \frac{n}{n-2\sigma} & \text{if} \quad \sigma < \frac{n}{2}. \end{cases} \tag{3.98}$$

Then, there exists a small ε such that if

$$\|(u_0, u_1)\|_{X_\sigma} \le \varepsilon, \tag{3.99}$$

then there exists a uniquely determined global (in time) energy solution to (3.1)

$$u \in \mathscr{C}([0,\infty), H^\sigma) \cap \mathscr{C}^1([0,\infty), L^2).$$

Moreover, u and some of its derivatives satisfies the following Matsumura type estimates

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4}+(1-\gamma)}\|(u_0,u_1)\|_{X_\sigma}, \\ \||D|^\sigma u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4}-\frac{\sigma}{2}+(1-\gamma)}\|(u_0,u_1)\|_{X_\sigma}, \\ \|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4}-1+(2-\gamma)}\|(u_0,u_1)\|_{X_\sigma}. \end{aligned}$$

The following theorem shows global existence of small data solution for $p > \frac{1}{\gamma}$. It is valid in space dimensional larger then 4.

Theorem 3.4. *Let $n > 4$ and $\sigma \ge 2$ be a real number. Assume that the data $(u_0, u_1) \in X_\sigma$, Suppose that the exponent p satisfies both conditions*

$$p > \frac{1}{\gamma}, \tag{3.100}$$

and

$$\begin{cases} 2 \le p & \text{if} \quad \sigma \ge \frac{n}{2}, \\ 2 \le p \le \frac{n}{n-2\sigma} & \text{if} \quad \sigma < \frac{n}{2}. \end{cases} \tag{3.101}$$

Then, there exists a small ε such that if

$$\|(u_0, u_1)\|_{X_\sigma} \le \varepsilon, \tag{3.102}$$

then there exists a uniquely determined global (in time) energy solution to (3.1)

$$u \in \mathscr{C}([0,\infty), H^\sigma) \cap \mathscr{C}^1([0,\infty), L^2).$$

Moreover, the solution and some of its derivatives verify the estimates

$$\|u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}+\xi_1}\|(u_0,u_1)\|_{X_\sigma},$$
$$\||D|^\sigma u(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-\frac{\sigma}{2}+\xi_2}\|(u_0,u_1)\|_{X_\sigma},$$
$$\|u_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{4}-1+\xi_3}\|(u_0,u_1)\|_{X_\sigma},$$

where $\xi_1 \geq \frac{n}{4}-\sigma$, $\xi_2 = \xi_1+\frac{\sigma}{2}$ *and* $\xi_3 = \xi_1+1$.

Chapter

4 Influence of the effective damping on the critical exponent and blow-up results

4.1 Introduction

In this chapter, we will focus on the following Cauchy problem for effectively viscoelastic damped wave models with nonlinear memory on the right-hand side for any effective damping $b(t)$.

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^p d\tau, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $\gamma \in (0,1)$, $p > 1$ and $b = b(t)$ is a non negative effective function in the sense of Definition 1.1. The model (4.1) is *parabolic-like* from the point of view of energy decay estimates. For this reason, parabolic like tools and tricks are appropriate to derive existence or blow-up results. The Cauchy problem (4.1) brings a time dependent coefficient that turns the situation more delicate. For this reason, we consider, first, some positive decreasing $b(t)$.

Concrete examples of effective $b(t)$ arising in our model

In this section we set, without proof, some typical examples of effective dissipation in the sense of Definition 1.1 from [11]. For the proof we refer the reader to check [11]. The simplest example is

Example 4.1. $b(t) = (1+t)^{-r}, \quad r \in (0,1)$ *Here, we have* $b'(t) = -r(1+t)^{-r-1}$. *Since* $r \in (0,1)$ *then* $b'(t) < 0$ *for all* $t > 0$. *This means that* $b(t)$ *is effective decreasing.*

Example 4.2. *As a second example we propose the function*

$$b(t) = \mu \frac{(\log(e+t))^\nu}{(e+t)^r}, \;\; \mu > 0, \;\; r \in (0,1), \;\; \nu \in (0,r].$$

Due to [11], the function $b(t)$ is effective in the sense of Definition 1.1. In the other hand, for all $t \geq 0$, it holds

$$b'(t) = \mu r \frac{(\log(e+t))^\nu}{(1+t)^{r+1}} \Big(-1 + \frac{\nu}{r \log(e+t)} \Big).$$

Since

$$\frac{1}{\log(e+t)} < 1 \textit{ for all } \; t > 0,$$

then according to the conditions of the parameters μ, r and η one has $b'(t) < 0$. Consequently, $b(t)$ is effective decreasing.

The aim of this chapter is to study the influence of the effective damping term on the critical exponent in the sense of Fujita and to generalize the results of reference [2]. In the following section, we establish the local existence for suitable data of model 4.1 using the Banach fixed-point theorem. The main result of this chapter is the study of the blow-up phenomenon and the determination of the critical exponent in the Fujita sense. The model we consider is parabolic-like in terms of energy decay estimates for the corresponding linear Cauchy problem. Therefore, we use the modified test function method to prove the results defined in Chapter 2.

4.2 Local existence

The main goal of this section is the proof of a local (in time) well-posedness result for the Cauchy problem (4.1) assuming that the data (u_0, u_1) belong to the energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$.

We proceed as we did in Section 3.3. So, first let us turn for some $T > 0$ to the linear Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t) u_t - \Delta u_t = F(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{4.2}$$

Among the achieved results about Cauchy problem (4.2), replying the requirement of local existence of solution for Cauchy problem (4.1), we select the following:

Lemma 4.1. *Let* $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *and* $F \in \mathscr{C}\big([0,T], L^2(\mathbb{R}^n)\big)$*, where* $T > 0$ *is given. Then there exists a unique energy solution* $u \in \mathscr{C}\big([0,T], H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T], L^2(\mathbb{R}^n)\big)$ *to (4.2). Moreover, the energy solution* $u = u(t,x)$ *satisfies the following estimates for all* $t \in (0,T)$*:*

$$\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \leq \|(u_1, \nabla u_0)\|_{L^2\times L^2} + \int_0^t \|F(s,\cdot)\|_{L^2}\, ds, \tag{4.3}$$

$$\|u(t,\cdot)\|_{L^2} \leq \|u_0\|_{L^2} + t\big(\|(u_1,\nabla u_0)\|_{L^2\times L^2}\big) + \int_0^t \Big(\int_0^s \|F(\tau,\cdot)\|_{L^2}\, d\tau\Big)\, ds. \tag{4.4}$$

Proof. Multiplying the equation $u_{tt} - \Delta u + b(t)u_t - \Delta u_t = F(t,x)$ by u_t and integrating over $\mathbb{R}^n$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \big(|u_t|^2 + |\nabla u|^2\big)dx - \int_{\mathbb{R}^n} \nabla.(u_t \nabla u)dx - \int_{\mathbb{R}^n} \nabla.(u_t\nabla u_t)dx \\ + \int_{\mathbb{R}^n} b(t)u_t^2 dx + \int_{\mathbb{R}^n} |\nabla u_t|^2 dx = \int_{\mathbb{R}^n} F(t,x)u_t dx.\end{aligned} \tag{4.5}$$

Applying the divergence theorem with using the fact that $(u_t\nabla u, u_t\nabla u_t) \in (L^1(\mathbb{R}^n))^2$ and Green's formula we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \big(|u_t|^2 + |\nabla u|^2\big)dx + \int_{\mathbb{R}^n} b(t)u_t^2 dx + \int_{\mathbb{R}^n} |\nabla u_t|^2 dx &\leq \int_{\mathbb{R}^n} F(t,x)u_t(t,x)dx \\ &\leq \|F(t,\cdot)\|_{L^2}\|u_t(t,\cdot)\|_{L^2},\end{aligned}$$

where we have used Cauchy-Schwartz inequality to get the last inequality.
Next, integrating over $[0,t]$ we get

$$\begin{aligned}\frac{1}{2}\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2}\big) + \int_0^t \int_{\mathbb{R}^n} b(t)u_t^2 dx + \int_0^t \int_{\mathbb{R}^n} |\nabla u_s(s,x)|^2 dxds \\ \leq \frac{1}{2}\big(\|u_1\|_{L^2} + \|\nabla u_0\|_{L^2}\big)^2 + \int_0^t \|F(s,\cdot)\|_{L^2}\|u_s(s,\cdot)\|_{L^2} ds.\end{aligned}$$

Taking into account the fact that $b(t) > 0$ we may conclude the following estimate:

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} |\nabla u_s(s,x)|^2 dxds &\leq \frac{1}{2}\|(u_1,\nabla u_0)\|^2_{L^2\times L^2} + \int_0^t \|F(s,\cdot)\|_{L^2}\|u_s(s,\cdot)\|_{L^2} ds, \\ \frac{1}{2}\big(\|u_t(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2}\big) &\leq \frac{1}{2}\|(u_1,\nabla u_0)\|^2_{L^2\times L^2} + \int_0^t \|F(s,\cdot)\|_{L^2}\big(\|u_s(s,\cdot)\|_{L^2} + \|\nabla u(s,\cdot)\|_{L^2}\big)^{\frac{1}{2}} ds\end{aligned}$$

Applying Gronwall type Lemmas (Eg. Lemma 9.12 in [56]) to the above estimate, we find

$$\|u_t(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} \le \|(u_1, \nabla u_0)\|_{L^2\times L^2} + \int_0^t \|F(s,\cdot)\|_{L^2} ds. \tag{4.6}$$

The estimate of (4.4) follows immediately from the formula

$$u(t,\cdot) = u_0 + \int_0^t u_s(s,\cdot) ds, \tag{4.7}$$

and the estimate (4.6). We achieve our aim. This completes the proof. □

Now let us turn to the Cauchy problem (4.1).

Theorem 4.1. *Let $b(t)$ be a decreasing function, $\gamma \in (0,1)$ Assume that $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. If $p > 1$ for $n = 1, 2$ or $1 \le p \le \frac{n}{n-2}$ for $n \ge 3$, then the Cauchy problem (4.1) has a unique local (in time) energy solution*

$$u \in \mathscr{C}\big([0, T_{max}), H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0, T_{max}), L^2(\mathbb{R}^n)\big),$$

where $T_{max} < \infty$.

Proof. Let us introduce for $T > 0$ the space of solutions

$$X(T) = \mathscr{C}\big([0,T], H^1(\mathbb{R}^n)\big) \cap \mathscr{C}^1\big([0,T], L^2(\mathbb{R}^n)\big),$$

with the norm

$$\|u\|_{X(T)} = \max_{t\in[0,T]} \big\{\|u(t,\cdot)\|_{L^2} + \|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2}\big\}.$$

The method we base on to show the existence and uniqueness of solution for Cauchy problem (4.1) is the fixed point theorem (Theorem 2.5). For this reason we need to suitable space and a contracting mapping. Then, let us denoting, for $R > 0$, by $B_T(R)$ for the centred ball in $X(T)$ defined by

$$B_T(R) = \{u \in X(T) \,:\, \|u\|_{X(T)} \le R\},$$

and defining a mapping Φ on $X(T)$ by $\Phi : v \to \Phi(v) = u$, where $u = u(t,x)$ is the energy solution to the Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \displaystyle\int_0^t (t-\tau)^{-\gamma} |v(\tau,\cdot)|^p \, d\tau, & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

Noting that, by Duhamel's principle and with the fundamental solutions E_0 and E_1 we may express $u = \Phi(v)$ by

$$\Phi(v) = E_0(t,0,x) *_x u_0 + E_1(t,0,x) *_x u_1 + \int_0^t E_1(t,s,x) * h(s, v(s,x))\, ds, \tag{4.8}$$

where

$$h(s, v(s,x)) = \int_0^s (s-\tau)^{-\gamma} |v(\tau,x)|^p \, d\tau.$$

In order to conclude that Φ admits a unique fixed point in $X(T)$, it is sufficient to show that there exist positive T and R such that

- for all $v \in B_T(R)$ we have $\Phi(v) \in B_T(R)$,
- Φ is a contracting mapping from $B_T(R)$ to $B_T(R)$ satisfying

$$\|\Phi(v_1) - \Phi(v_2)\|_{X(T)} \le \frac{1}{2} \|v_1 - v_2\|_{X(T)} \quad \text{for all } v_1, v_2 \in B_T(R). \tag{4.9}$$

Let $v \in B_T(R)$. Thanks to Lemma 3.1, we may using the estimates from Lemma 4.1, then, we get the estimates:

$$\|\nabla u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2} \le C I_0 + C \int_0^t \|h(s, v(s,\cdot))\|_{L^2}\, ds, \tag{4.10}$$

where

$$I_0 = \|u_0\|_{L^2} + \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2},$$

and

$$\|u(t,\cdot)\|_{L^2} \le C(1+T) I_0 + C \int_0^t \Big(\int_0^s \|h(\tau, v(\tau,\cdot))\|_{L^2}\, d\tau \Big)\, ds. \tag{4.11}$$

From the estimates (4.10) and (4.11) it follows

$$\|u\|_{X(T)} \le C I_0 + C T^{2-\gamma} R^p,$$

where C is a suitable positive constant and T is supposed to be smaller or equal to 1. If we choose $R = 2CI_0$, then we can choose $T \le 1$ so small that $\|u\|_{X(T)} \le R$. Consequently, $u = \Phi(v) \in B_T(R)$.

Now let us prove the contraction property. Let $v_1, v_2 \in B_T(R)$ such that $u_1 = \Phi(v_1)$ and

$u_2 = \Phi(v_2)$. Putting $w = u_1 - u_2$ the function $w = w(t,x)$ is the energy solution to the Cauchy problem

$$\begin{cases} w_{tt} - \Delta w + b(t)w_t - \Delta w_t = \int_0^t (t-s)^{-\gamma}(|v_1(s,\cdot)|^p - |v_2(s,\cdot)|^p)\,ds, & (t,x) \in (0,T)\times\mathbb{R}^n, \\ w(0,x) = w_t(0,x) = 0, & x \in \mathbb{R}^n. \end{cases}$$

From the estimates of Lemma 4.1 it follows

$$\|\nabla w(t,\cdot)\|_{L^2} + \|w_t(t,\cdot)\|_{L^2} \lesssim \int_0^t \int_0^s (s-\tau)^{-\gamma} \||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2}\,d\tau\,ds, \tag{4.12}$$

and

$$\|w(t,\cdot)\|_{L^2} \lesssim \int_0^t \int_0^s \int_0^\tau (\tau-\rho)^{-\gamma} \||v_1(\rho,\cdot)|^p - |v_2(\rho,\cdot)|^p\|_{L^2}\,d\rho\,d\tau\,ds. \tag{4.13}$$

By Hölder's inequality we have

$$\||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2} \lesssim \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|_{L^{2p}} \big(\|v_1(\tau,\cdot)\|_{L^{2p}}^{p-1} + \|v_2(\tau,\cdot)\|_{L^{2p}}^{p-1}\big).$$

Applying the embedding $H^1(\mathbb{R}^n) \hookrightarrow L^{2p}(\mathbb{R}^n)$ for $\frac{p}{p-1} \geq \frac{n}{2}$ we arrive at

$$\||v_1(\tau,\cdot)|^p - |v_2(\tau,\cdot)|^p\|_{L^2} \lesssim \|v_1(\tau,\cdot) - v_2(\tau,\cdot)\|_{H^1} \big(\|v_1(\tau,\cdot)\|_{H^1}^{p-1} + \|v_2(\tau,\cdot)\|_{H^1}^{p-1}\big). \tag{4.14}$$

Setting (4.14) into (4.12) and (4.13) implies for $T \leq 1$ the relation

$$\|w(t,\cdot)\|_{L^2} + \|\nabla w(t,\cdot)\|_{L^2} + \|w_t(t,\cdot)\|_{L^2} \leq C\|v_1 - v_2\|_{X(T)} 2R^{p-1}T^{2-\gamma}.$$

Choosing $T \leq 1$ so small that $2CR^{p-1}T^{2-\gamma} = \frac{1}{2}$ leads to the desired contracting property (3.8), that is,

$$\|u_1 - u_2\|_{X(T)} = \|w\|_{X(T)} \leq \frac{1}{2}\|v_1 - v_2\|_{X(T)}.$$

By Banach's fixed point theorem we may complete the proof of the local (in time) well-posedness result for energy solutions. □

4.3 Blow-up of solutions

This section is devoted to prove blow-up results for (4.1) and find the Fujita critical exponent as well. The presence of the effective coefficient $b(t)$ suggests to us to follow the modified test function method instead of the classical (or standard) test function method since the last one does not give optimal results. The principle of modified test function method requires the introduction of an axillary function. For this reason, we consider the following initial value problem

$$\begin{cases} g'(t) - b(t)g(t) = -1, \quad t > 0, \\ g(0) = B_0, \end{cases} \tag{4.15}$$

where

$$B_0 = \int_0^\infty \exp\big(- \int_0^t b(s)ds\big) dt.$$

In the following lemma we select among all the results concerning Cauchy problem (4.15) those we need in the sequel

Lemma 4.2. *([26, 3]) Let g be the classical solution of Cauchy problem (4.15). Then g is well defined and satisfies the following properties:*

1) There exist constants $T_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that it holds for any $t \geq T_0$

$$\frac{C_1}{b(t)} \leq g(t) \leq \frac{C_2}{b(t)}. \tag{4.16}$$

2) For every $t \geq T_0$

$$|g'(t)| \leq \frac{1 + b_0}{1 - b_0} \tag{4.17}$$

where

$$b_0 = \limsup_{t \to +\infty} \frac{|b'(s)|}{b(s)^2} < 1. \tag{4.18}$$

Taking into account formula (3.24) and the fact that g is solution for Cauchy problem (4.15), we get

$$(gu)_{tt} + (-\Delta)(gu) - \big((g' - 1)u\big)_t + (-\Delta)(gu_t) = g(t)\Gamma(\alpha) I^\alpha_{0|t}(|u|^p). \tag{4.19}$$

Based on formula (4.19) we introduce the so-called $g-$solution or weak solution for Cauchy problem (4.1).

Definition 4.1. *Let $T > 0$ and $\gamma \in (0,1)$. A weak solution (or $g-$solution) for the Cauchy problem (4.1) on $(0,T) \times \mathbb{R}^n$ with the initial data $u_0, u_1 \in L^1_{loc}(\mathbb{R}^n)$ is a locally integrable function $u \in L^p_{loc}\big((0,T), L^p_{loc}(\mathbb{R}^n)\big)$ satisfying the relation*

$$\begin{aligned}
\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} & g(t) I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)\,dx\,dt + g(0)\int_{\mathbb{R}^n} u_1(x)\varphi(0,x)\,dx + g(0)\int_{\mathbb{R}^n} u_0(x)\varphi_t(0,x)\,dx \\
& +(2g^{'}(0)-1)\int_{\mathbb{R}^n} u_0(x)\varphi(0,x)\,dx - g(0)\int_{\mathbb{R}^n} u_0(x)\Delta\varphi(0,x)\,dx \qquad (4.20)\\
&= \int_0^T \int_{\mathbb{R}^n} g(t)u(t,x)\varphi_{tt}(t,x)\,dx\,dt - \int_0^T \int_{\mathbb{R}^n} g(t)u(t,x)\Delta\varphi(t,x)\,dx\,dt \\
& - \int_0^T \int_{\mathbb{R}^n} (g^{'}(t)-1)u(t,x)\varphi_t(t,x)\,dx\,dt + \int_0^T \int_{\mathbb{R}^n} g^{'}(t)u(t,x)\Delta\varphi(t,x)\,dx\,dt \\
& + \int_0^T \int_{\mathbb{R}^n} g(t)u(t,x)\Delta\varphi_t(t,x)\,dx\,dt, \qquad (4.21)
\end{aligned}$$

for all non-negative test functions $\varphi \in \mathscr{C}^2([0,T]\times\mathbb{R}^n)$ such that $\varphi(T,\cdot) = \varphi_t(T,\cdot) = 0$ and $\alpha = 1-\gamma$, where $g(t)$ is the solution to the Cauchy problem 4.15.

Remark 4.1. *Assume $g(t)$ is the solution to 4.15. By multiplying the first equation in 4.1 by $g(t)$ and conducting a direct calculation, one obtains the following result:*

Now, we may announce the main result of blow-up for Cauchy problem (4.1). So, put for all $n \geq 1$ and $\gamma \in (0,1)$

$$\tilde{p}_\gamma(n) := 1 + \frac{2-2\gamma}{n+2\gamma} \quad \text{and} \quad p^*_\gamma(n) = \max\Big\{\tilde{p}_\gamma(n),\ \frac{1}{\gamma}\Big\}. \qquad (4.22)$$

Theorem 4.2. *Let $0 < \gamma < 1$, $p \in (1,\infty)$ for $n = 1,2$ and $1 < p < \frac{n}{n-2}$ for $n \geq 3$. Assume that $(u_0,u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and satisfy*

$$\int_{\mathbb{R}^n} \big(u_1(x) + (b(0) - B_0^{-1} - T^{-1})u_0(x) - \Delta u_0(x)\big)dx > 0. \qquad (4.23)$$

Then,
If $p \leq p^*_\gamma(n)$, then weak solutions, in the sense of Definition 4.1, of Cauchy problem (4.1) do not exist globally in time.

Proof of Theorem 4.2. The proof of Theorem 4.2 is based on a contradiction argument. The model (4.1) is parabolic like as explained above. For this reason we use the modified test function method to prove our results. Sins the principal of the method is based on the right choice of the test function then, we select a test function φ in the form:

$$\varphi(t,x)=\frac{1}{g(t)}D^{\alpha}_{t|T}\psi(t,x)=\varphi_1^{\ell}(x)\frac{1}{g(t)}D^{\alpha}_{t|T}\varphi_2(t),\ (t,x)\in(0,\infty)\times\mathbb{R}^n, \tag{4.24}$$

where $\ell>1$ and $D^{\alpha}_{t|T}$ is the right fractional derivative operator of order α in the Riemann-Liouville sense, the functions φ_1 and φ_2 are as in (3.29) and $g(t)$ is the solution of Cauchy problem (4.15).

4.3.1 Treatment of the weak formulation (4.20)-(4.21)

Treatment of the left-hand side (4.20)

Introducing the test function defined by (4.24), we have by Proposition 2.2 and the identity (2.10)

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} g(t)I^{\alpha}_{0|t}(|u|^p)\varphi(t,x)dxdt &= \int_0^T\int_{\mathbb{R}^n} I^{\alpha}_{0|t}(|u|^pD^{\alpha}_{t|T}\psi(t,x)dxdt\\ &= \int_0^T\int_{\mathbb{R}^n} D^{\alpha}_{0|T}I^{\alpha}_{0|T}(|u|^p)\psi(t,x)dxdt\\ &= \int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dxdt. \end{aligned}\tag{4.25}$$

For the second term of the left-hand side of (4.20), we use Corollary 2.2 with $j=0$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}u_1(x)\varphi(0,x)dx \qquad (4.26)\\ &=\int_{\mathbb{R}^n}u_1(x)\varphi_1^{\ell}(x)\frac{1}{g(0)}D^{\alpha}_{t|T}\varphi_2(t)\Big|_{t=0}dx=C_0\frac{T^{-\alpha}}{g(0)}\int_{\mathbb{R}^n}u_1(x)\varphi_1^{\ell}(x)dx,\end{aligned}$$

since

$$D^{\alpha}_{t|T}\varphi_2(t)\Big|_{t=0}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}T^{-\alpha}=C_0T^{-\alpha}. \tag{4.27}$$

For the third term, noting, first, that by the identity (2.11) from Proposition 2.3 one has

$$\varphi_t(t,x)=\frac{\partial\varphi}{\partial t}(t,x)=-\varphi_1^{\ell}(x)\frac{g'(t)}{g^2(t)}D^{\alpha}_{t|T}\varphi_2(t)-\varphi_1^{\ell}(x)\frac{1}{g(t)}D^{\alpha+1}_{t|T}\varphi_2(t).$$

Then, after using Corollary 2.2 with $j=1$ we get the following estimate:

$$\int_{\mathbb{R}^n}u_0(x)\varphi_t(0,x)dx=C_1\frac{T^{-\alpha-1}}{g(0)}\int_{\mathbb{R}^n}u_0(x)\varphi_1^{\ell}(x)dx, \tag{4.28}$$

since

$$D_{t|T}^{\alpha+1}\varphi_2(t)_{|t=0} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)}T^{-\alpha-1} = C_1 T^{-\alpha-1}.$$

Concerning the fourth term, again by Corollary 2.2, we have

$$\int_{\mathbb{R}^n} u_0(x)\varphi(t,x)\Big|_{t=0}dx = C_0\frac{T^{-\alpha}}{g(0)}\int_{\mathbb{R}^n} u_0(x)\varphi_1^\ell(x)dx. \tag{4.29}$$

Finally, after using results form Corollary 2.2 we get the estimate:

$$\int_{\mathbb{R}^n} u_0(x)\Delta\varphi(0,x)\,dx = C_0\frac{T^{-\alpha}}{g(0)}\int_{\mathbb{R}^n} u_0(x)\Delta\varphi_1^\ell(x)\,dx. \tag{4.30}$$

Treatment of the right-hand side (4.21)

First, by using the identity (2.11) from Proposition 2.3, we have

$$\varphi_{tt}(t,x) = \varphi_1^\ell(x)\Big(-\frac{g''g-2(g')^2}{g^3}D_{t|T}^{\alpha}\varphi_2(t) + \frac{2g'}{g^2}D_{t|T}^{\alpha+1}\varphi_2(t) + \frac{1}{g}D_{t|T}^{\alpha+2}\varphi_2(t)\Big).$$

Therefore we have

$$\int_0^T\int_{\mathbb{R}^n} g(t)u(t,x)\varphi_{tt}(t,x)dxdt =$$
$$\int_0^T\int_{\mathbb{R}^n} u(t,x)\varphi_1^\ell(x)\Big(-\frac{g''g-2(g')^2}{g^2}D_{t|T}^{\alpha}\varphi_2(t) + \frac{2g'}{g}D_{t|T}^{\alpha+1}\varphi_2(t) + D_{t|T}^{\alpha+2}\varphi_2(t)\Big)dxdt. \tag{4.31}$$

For the second term of the right-hand side (4.21), after using the identity (3.40), we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} g(t)u(t,x)\Delta\varphi(t,x)dxdt\\ &\quad=\int_0^T\int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1+\ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big)D_{t|T}^{\alpha}\varphi_2(t)dxdt.\end{aligned} \tag{4.32}$$

For the third term of the formulation (4.21), using the identity (2.11) with $n=1$ we find

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}(g'(t)-1)u(t,x)\varphi_t(t,x)dxdt\\ &\quad=-\int_0^T\int_{\mathbb{R}^n}(g'(t)-1)u(t,x)\varphi_1^\ell(x)\left(\frac{g'}{g^2}D_{t|T}^{\alpha}\varphi_2(t)+\frac{1}{g}D_{t|T}^{\alpha+1}\varphi_2(t)\right)dxdt.\end{aligned} \tag{4.33}$$

Next, we have directly

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n} g'(t)u(t,x)\Delta\varphi(t,x)dxdt\\ &\quad=\int_0^T\int_{\mathbb{R}^n} g'(t)(g(t))^{-1}u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1+\ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big)D_{t|T}^{\alpha}\varphi_2(t)dxdt\end{aligned} \tag{4.34}$$

Finally, for the last term we use (2.11) and the identity (3.40) again, we get

$$\int_0^T\int_{\mathbb{R}^n} g(t)u(t,x)\Delta\varphi_t(t,x)dxdt = -\int_0^T\int_{\mathbb{R}^n} u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1 + \ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big)$$
$$\times\big(g'(t)(g(t))^{-1}D^{\alpha}_{t|T}\varphi_2(t) + D^{\alpha+1}_{t|T}\varphi_2(t)\big)dxdt. \tag{4.35}$$

Plugging the estimates (4.25), (4.26), (4.28), (4.29), (4.30), (4.31), (4.32), (4.33), (4.34) and (4.35) in the weak formulation (4.20)-(4.21) we get

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u|^p\psi(t,x)dxdt + T^{-\alpha}\int_{\mathbb{R}^n}u_1(x)\varphi_1^{\ell}(x)dx\\
&\quad+(b(0)-B_0^{-1})T^{-\alpha}\int_{\mathbb{R}^n}u_0(x)\varphi_1^{\ell}(x)dx - T^{-\alpha}\int_{\mathbb{R}^n}u_0(x)\Delta\left(\varphi_1^{\ell}(x)\right)dx\\
&\quad=\int_0^T\int_{\mathbb{R}^n}u(t,x)\varphi_1^{\ell}(x)D^{\alpha+2}_{t|T}\varphi_2(t)dxdt + 2\int_0^T\int_{\mathbb{R}^n}u(t,x)\varphi_1^{\ell}(x)\frac{g'(t)}{g(t)}D^{\alpha+1}_{t|T}\varphi_2(t)dxdt\\
&\qquad-\int_0^T\int_{\mathbb{R}^n}\frac{g''g-2(g')^2}{g^2}u(t,x)\varphi_1^{\ell}(x)D^{\alpha}_{t|T}\varphi_2(t)dxdt\\
&\qquad-\int_0^T\int_{\mathbb{R}^n}u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1+\ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big)D^{\alpha}_{t|T}\varphi_2(t)dxdt\\
&\qquad+\int_0^T\int_{\mathbb{R}^n}\frac{(g'(t)-1)g'(t)}{g^2(t)}u(t,x)\varphi_1^{\ell}(x)D^{\alpha}_{t|T}\varphi_2(t)dxdt\\
&\qquad+\int_0^T\int_{\mathbb{R}^n}\frac{(g'(t)-1)}{g(t)}u(t,x)\varphi_1^{\ell}(x)D^{\alpha+1}_{t|T}\varphi_2(t)dxdt\\
&\qquad-\int_0^T\int_{\mathbb{R}^n}u(t,x)\big(\ell\varphi_1^{\ell-1}\Delta\varphi_1+\ell(\ell-1)\varphi_1^{\ell-2}|\nabla\varphi_1|^2\big)D^{\alpha+1}_{t|T}\varphi_2(t)dxdt.
\end{aligned}\tag{4.36}$$

Taking into consideration the assumption (4.23), the estimate (3.44) and the facts that $0\le\varphi_1\le 1$, $\Gamma(\alpha)>0$ and $T>0$ we realize the estimate:

$$I_R \lesssim I_1+I_2+I_3+I_4+I_5+I_6+I_7, \tag{4.37}$$

where

$$\begin{aligned}
I_R = I_R(T) &= \int_0^T \int_{\mathbb{R}^n} |u|^p \psi(t,x)dxdt \\
I_1 = I_1(T) &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dxdt, \\
I_2 = I_2(T) &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^\ell(x)\frac{|g'(t)|}{g(t)}|D_{t|T}^{\alpha+1}\varphi_2(t)|dxdt, \\
I_3 = I_3(T) &= \int_0^T \int_{\mathbb{R}^n} \frac{|g''g-2(g')^2|}{g^2}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha}\varphi_2(t)|dxdt, \\
I_4 = I_4(T) &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^{\ell-2}\left(|\Delta\varphi_1|+|\nabla\varphi_1|^2\right)|D_{t|T}^{\alpha}\varphi_2(t)|dxdt, \\
I_5 = I_5(T) &= \int_0^T \int_{\mathbb{R}^n} \frac{|g'(t)-1||g'(t)|}{g^2(t)}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha}\varphi_2(t)|dxdt, \\
I_6 = I_R(T) &= \int_0^T \int_{\mathbb{R}^n} \frac{|g'(t)-1|}{g(t)}|u(t,x)|\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|dxdt, \\
I_7 = I_7(T) &= \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^{\ell-2}\left(|\Delta\varphi_1|+|\nabla\varphi_1|^2\right)|D_{t|T}^{\alpha+1}\varphi_2(t)|dxdt.
\end{aligned}$$

The aim, now, is to reformulate and estimate the terms of the right-hand side of (4.37). Employing Hölder's inequality with $\frac{1}{p} = 1 - \frac{1}{q}$ we can proceed as follows:

$$I_1 = \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\psi^{\frac{1}{p}}\psi^{-\frac{1}{p}}\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|dxdt \le I_R^{\frac{1}{p}} I_{1,R}^{\frac{1}{q}}$$

where

$$I_{1,R} = \int_0^T \int_{\mathbb{R}^n} \varphi_2^{-\frac{1}{p-1}}\varphi_1^\ell(x)|D_{t|T}^{\alpha+2}\varphi_2(t)|^q dxdt,$$

To estimate I_2 we note initially by Lemma 4.2 that the term $|g'(t)|$ is bounded. The behavior $g(t) \sim b(t)^{-1}$, as stated in the first assertion from Lemma 4.2, implies that

$$I_2 = \int_0^T \int_{\mathbb{R}^n} |u(t,x)|\varphi_1^\ell(x)b(t)|D_{t|T}^{\alpha+1}\varphi_2(t)|dxdt \le I_R^{\frac{1}{p}} I_{2,R}^{\frac{1}{q}}$$

where

$$I_{2,R} = \int_0^T \int_{\mathbb{R}^n} b(t)^{\frac{p}{p-1}}\varphi_2^{-\frac{1}{p-1}}\varphi_1^\ell(x)|D_{t|T}^{\alpha+1}\varphi_2(t)|^q dxdt.$$

Now, let us turn to estimate I_3. First, noting that $g''(t) = b'(t)g(t) + b(t)g'(t)$ implies

$$\frac{|g''g-2(g')^2|}{g^2} \le \frac{|g''|}{g} + 2\frac{(g')^2}{g^2} \lesssim |b'(t)| + \frac{b(t)}{g(t)} + \frac{1}{g^2(t)}.$$

Then by using Lemma 4.2 again we may conclude the estimate From assumption 1) of Lemma 4.2 and the properties of function $b(t)$ from definition 1.1, we obtain

$$\frac{|g''g - 2(g')^2|}{g^2} \lesssim b(t) + b^2(t).$$

Consequently, we get the estimate

$$I_3 \leq I_R^{\frac{1}{p}} I_{3,R}^{\frac{1}{q}}$$

where

$$I_{3,R} = \int_0^T \int_{\mathbb{R}^n} \left(b(t) + b^2(t)\right)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} \varphi_1^\ell(x) |D_{t|T}^\alpha \varphi_2(t)|^q dxdt.$$

Next, we estimate the fourth term as we did previously for the first term in the following form

$$I_4 \leq I_R^{\frac{1}{p}} I_{4,R}^{\frac{1}{q}}, \tag{4.38}$$

where

$$I_{4,R} = \int_0^T \int_{\mathbb{R}^n} \varphi_2^{-\frac{1}{p-1}} \varphi_1^{-\frac{\ell}{p-1}}(x) \left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)^q |D_{t|T}^\alpha \varphi_2(t)|^q dxdt.$$

Now, we proceed to estimate the term I_5. Thanks to Lemma 4.2, the terms $|g'(t)|$ is bounded. Since g is solution to Cauchy problem (4.15) then

$$|g^{'}(t) - 1| \sim b(t)g(t).$$

Consequently, these considerations allow to get the estimate

$$I_5 \leq I_R^{\frac{1}{p}} I_{5,R}^{\frac{1}{q}}, \tag{4.39}$$

where

$$I_{5,R} = \int_0^T \int_{\mathbb{R}^n} b(t)^{2q} \varphi_2^{-\frac{1}{p-1}} \varphi_1^\ell(x) |D_{t|T}^\alpha \varphi_2(t)|^q dxdt.$$

In an analogous way, we may conclude the following estimate for I_6 :

$$I_6 \leq I_R^{\frac{1}{p}} I_{6,R}^{\frac{1}{q}}, \tag{4.40}$$

where

$$I_{6,R} = \int_0^T \int_{\mathbb{R}^n} b(t)^q \varphi_2^{-\frac{1}{p-1}} \varphi_1^\ell(x) |D_{t|T}^{\alpha+1} \varphi_2(t)|^q dxdt.$$

And finally, we estimate the seventh term as before in the following form:

$$I_7 \leq I_R^{\frac{1}{p}} I_{7,R}^{\frac{1}{q}}, \tag{4.41}$$

where

$$I_{7,R} = \int_0^T \int_{\mathbb{R}^n} \varphi_2^{-\frac{1}{p-1}} \varphi_1^{-\frac{\ell}{p-1}}(x) \left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)^q |D_{t|T}^{\alpha+1}\varphi_2(t)|^q dxdt.$$

At this stage, we distinguish between two main cases as follows:

Case $p \leq \tilde{p}_\gamma(n)$**:**

Even this case is divided into two sub-cases.

i) Sub-case $p < \tilde{p}_\gamma(n)$**:** In this case, we choose $K = T$ and β suitably large in the form of the test function (3.29). Then, to estimate the integrals $I_{i,R}, i = 1, \cdots, 7$ we suggest the scaled change of variables

$$x = T^{\frac{1}{2}} y \quad \text{and} \quad t = T\tau. \tag{4.42}$$

Noting that the integrals $I_{i,R}$ are trivial outside Ω_T (defined by (3.31)), then, by using Fubini's theorem, we get, for $I_{1,R}$

$$I_{1,R} = \Big(\int_{\Omega_R} \varphi_1^\ell(x)dx\Big)\Big(\int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) = I_{11}I_{12}. \tag{4.43}$$

We have

$$I_{11} = \int_{\Omega_R} \varphi_1^\ell(x)dx = T^{\frac{n}{2}} \int_{|y|\leq 2} \phi^\ell(y^2)dy = CT^{\frac{n}{2}}, \tag{4.44}$$

where

$$C = \int_{|y|\leq 2} \phi^\ell(y^2)dy > 0.$$

In the other hand, after using Proposition 2.4, we get

$$\begin{aligned} I_{12} &= \int_0^T \varphi_2(t)^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+2}\varphi_2(t)|^{\frac{p}{p-1}} dt \\ &= T \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}} \left(T^{-(\alpha+2)}(1-\tau)^{\beta-\alpha-2}\right)^{\frac{p}{p-1}} d\tau \\ &= T^{1-(\alpha+2)\frac{p}{p-1}} \int_0^1 (1-\tau)^{\beta-(\alpha+2)\frac{p}{p-1}} d\tau = CT^{1-(\alpha+2)\frac{p}{p-1}}. \end{aligned} \tag{4.45}$$

Putting (4.44) and (4.45) into (4.43) we find

$$I_{1,R} \lesssim T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1}. \tag{4.46}$$

Continuing, we make estimations for terms $I_{2,R}, I_{3,R}, I_{5,R}$ and $I_{6,R}$. Again, by fubini's theorem we have

$$I_{2,R} = \Big(\int_{\Omega_R} \varphi_1^\ell(x)\Big)\Big(\int_0^T b(t)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) = I_{21}I_{22}, \tag{4.47}$$

$$I_{3,R} = \Big(\int_{\Omega_R} \varphi_1^\ell(x)\Big)\Big(\int_0^T \big(b(t)+b^2(t)\big)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D^\alpha_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dxdt\Big) = I_{31}I_{32}, \tag{4.48}$$

$$I_{5,R} = \Big(\int_{\Omega_R} \varphi_1^\ell(x)\Big)\Big(\int_0^T b(t)^{2\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D^\alpha_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dxdt\Big) = I_{51}I_{52}, \tag{4.49}$$

and

$$I_{6,R} = \Big(\int_{\Omega_R} \varphi_1^\ell(x)\Big)\Big(\int_0^T b(t)^{\frac{p}{p-1}} \varphi_2^{-\frac{1}{p-1}} |D^{\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dxdt\Big) = I_{61}I_{62}. \tag{4.50}$$

As one may remark that the estimates for I_{21}, I_{31}, I_{51} and I_{61} coincide with the estimate for I_{11}, that is, with (4.44). Hence it remains to estimate the integrals I_{22}, I_{32}, I_{52} and I_{62}. Let's begin. To estimate the integral I_{22}, we use Proposition (2.4) again with $j = 1$ and the fact that b is a positive decreasing function. In this way, we get

$$\begin{aligned}
I_{22} &= \int_0^T \varphi_2(t)^{-\frac{1}{p-1}} b(t)^{\frac{p}{p-1}} |D^{\alpha+1}_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dt \\
&= T^{1-(\alpha+1)\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\frac{\beta}{p-1}} b(T\tau)^{\frac{p}{p-1}} (1-\tau)^{(\beta-\alpha-1)\frac{p}{p-1}}\, d\tau \\
&\le T^{1-(\alpha+1)\frac{p}{p-1}} \sup_{0\leqslant\tau\leqslant 1} b(T\tau)^{\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\beta-(\alpha+1)\frac{p}{p-1}}\, d\tau \\
&\le T^{1-(\alpha+1)\frac{p}{p-1}} b(0)^{\frac{p}{p-1}} \int_0^1 (1-\tau)^{-\beta-(\alpha+1)\frac{p}{p-1}}\, d\tau \\
&\lesssim T^{1-(\alpha+1)\frac{p}{p-1}}.
\end{aligned} \tag{4.51}$$

Putting (4.44) and (4.51) into (4.47) we find

$$I_{2,R} \lesssim T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}. \tag{4.52}$$

In the same way, we have

$$I_{3,R} \lesssim T^{-\alpha\frac{p}{p-1}+\frac{n}{2}+1}, \tag{4.53}$$

$$I_{5,R} \lesssim T^{-\alpha\frac{p}{p-1}+\frac{n}{2}+1}, \tag{4.54}$$

and

$$I_{6,R} \lesssim T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}. \tag{4.55}$$

Now we deal with $I_{4,R}$. Again, by Fubini's theorem we have

$$\begin{aligned}
I_{4,R} &= \Big(\int_{\Omega_R} \varphi_1^{-\frac{\ell}{p-1}}(x)\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)^q dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}} |D^\alpha_{t|T}\varphi_2(t)|^{\frac{p}{p-1}} dt\Big) \\
&:= I_{41}I_{42}.
\end{aligned} \tag{4.56}$$

In order to estimate the integral I_{41} we proceed as we did to get the estimates (3.60) and (3.61) (pages 48 and 49). Then we get the estimates

$$I_{41} \lesssim T^{\frac{n}{2}-\frac{p}{p-1}} \quad \text{and} \quad I_{42} \lesssim T^{1-\alpha\frac{p}{p-1}}. \tag{4.57}$$

Counting on the estimates (4.57) yields

$$I_{4,R} \lesssim T^{1-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}}. \tag{4.58}$$

Finally, we estimate the term $I_{7,R}$ in a similar way, we obtain

$$\begin{aligned} I_{7,R} &= \Big(\int_{\Omega_R} \varphi_1^{-\frac{\ell}{p-1}}(x)\left(|\Delta\varphi_1| + |\nabla\varphi_1|^2\right)^q dx\Big)\Big(\int_0^T \varphi_2^{-\frac{1}{p-1}}|D_{t|T}^{\alpha+1}\varphi_2(t)|^{\frac{p}{p-1}}dt\Big) \\ &= I_{71}I_{72}. \end{aligned} \tag{4.59}$$

Since $I_{71} = I_{41}$, it is sufficient to estimate I_{72}. For this purpose, we use Proposition (2.4) with $j = 1$, we obtain:

$$I_{72} \lesssim T^{1-(\alpha+1)\frac{p}{p-1}}. \tag{4.60}$$

Therefore, putting (4.57) and (4.60) into (4.59) we find

$$I_{7,R} \lesssim T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}. \tag{4.61}$$

Finally, we replace the estimates (4.46), (4.52), (4.53), (4.54),(4.55), (4.58) and (4.61) into (4.37) we may conclude

$$\begin{aligned} \int_0^T \int_{\Omega_T} |u(t,x)|^p \psi(t,x)dxdt \lesssim \big(&T^{-\alpha\frac{p}{p-1}+\frac{n}{2}+1} + T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1} \\ &+T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1}\big). \end{aligned} \tag{4.62}$$

In order to guarantee that all the exponents of T in the right-hand side of (4.62) are negatives, we require that

$$-\alpha\frac{p}{p-1} + \frac{n}{2} + 1 \leq 0.$$

In this way, we have

$$\int_0^T \int_{\Omega_T} |u(t,x)|^p \psi(x,t)dxdt \lesssim T^{-\nu}, \tag{4.63}$$

where

$$\nu = \nu(n,p,\alpha) = \alpha\frac{p}{p-1} - \frac{n}{2} - 1. \tag{4.64}$$

Noting that $\nu > 0$ is equivalent to $p < \tilde{p}_\gamma(n)$. Then, taking the limit as $T \to \infty$ in (4.63), we obtain

$$\lim_{T\to+\infty} \int_0^T \int_{\Omega_T} |u(t,x)|^p \psi(x,t)dxdt = 0.$$

Using the dominated convergence theorem of Lebesgue and the facts that

$$\lim_{T\to+\infty} \psi(t,x) = 1 \quad \text{for all} \quad (t,x) \in (0,T) \times \mathbb{R}^n, \tag{4.65}$$

and

$$\lim_{T\to\infty} \chi_{\Omega_T} = \chi_{\Omega_{\mathbb{R}^n}} = 1,$$

we find

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p dxdt = 0.$$

It follows, immediately, that $u \equiv 0$ and this is a contradiction to (4.23).

Case $p = \tilde{p}_\gamma(n)$

Firstly, Noting that the condition $p = \tilde{p}_\gamma(n)$ is equivalent to $\nu = 0$. Moreover, the fact that $p = \tilde{p}_\gamma(n)$ shows, after using the estimate (4.62), the existence of some positive constant C independent of T satisfying

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p dxdt \leq C.$$

So, it implies that

$$\lim_{T\to+\infty} \int_0^T \int_{\Delta_T} |u(t,x)|^p \psi(x,t)dxdt = 0. \tag{4.66}$$

where Δ_T is defined by 3.32. Then, fixing arbitrarily R in $(0;T)$ for some $T > 0$ such that when $T \to \infty$ we don't have $R \to \infty$ simultaneously and choosing $K = R^{-1}T$ in 3.29. In order to estimate the integrals $I_{j,R}$, $j = 1,..,7$ on the set Ω_T we compute essentially on the change of variables $x = T^{\frac{1}{2}}R^{-\frac{1}{2}}y$. We proceed as we did in the first case we may conclude the estimate

$$I_{1,R}+I_{2,R}+I_{3,R}+I_{5,R}+I_{6,R} \lesssim \left(T^{-\alpha\frac{p}{p-1}+\frac{n}{2}+1}+T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1}+T^{-(\alpha+2)\frac{p}{p-1}+\frac{n}{2}+1}\right)R^{-\frac{n}{2}} \tag{4.67}$$

Taking into account the hypothesis $p = \tilde{p}_\gamma(n)$, we get from 4.67

$$I_{1,R} + I_{2,R} + I_{3,R} + I_{5,R} + I_{6,R} \lesssim \left(1 + T^{-\frac{p}{p-1}} + T^{-2\frac{p}{p-1}}\right)R^{-\frac{n}{2}} \tag{4.68}$$

and

$$\begin{aligned} I_{4,R} + I_{7,R} &\lesssim T^{-(\alpha+1)\frac{p}{p-1}+\frac{n}{2}+1} R^{-\frac{n}{2}+\frac{p}{p-1}} \\ &\lesssim T^{-\frac{p}{p-1}} R^{-\frac{n}{2}+\frac{p}{p-1}}. \end{aligned} \tag{4.69}$$

taking into account the estimates (4.68) and (4.69), we deduce from (4.37) the estimate

$$\int_0^T \int_{\Delta_T} |u(t,x)|^p \psi(t,x) dx dt \lesssim \big(T^{-\frac{p}{p-1}} + 1\big) R^{-\frac{n}{2}}.$$

Passing to the limit as $T \to \infty$, we find after using 4.66

$$\int_0^T \int_{\mathbb{R}^n} |u(t,x)|^p dx dt \lesssim R^{-\frac{n}{2}}.$$

which means that necessarily $R \to \infty$ and this is a contradiction.

The second main result in Theorem 4.2 is :

Case : $p < \frac{1}{\gamma}$.

In this case, we choose $K = R$ in 3.29, where R is any fixed positive real number. Our aim is to estimate the integrals $I_{i,R}, i = 1, \cdots, 7$ on the set

$$\Omega_R := \text{supp}\varphi_1 = \big\{x \in \mathbb{R}^n, |x|^2 \le 2R\big\}, \tag{4.70}$$

since they are null outside suppφ_1. For this reason, we consider the scaled change of variable

$$x = R^{-\frac{1}{2}} y \quad \text{and} \quad t = \tau T.$$

Following the same steps as we did above we get the estimate

$$I_{1,R} + I_{2,R} + I_{3,R} + I_{5,R} + I_{6,R} \lesssim \big(T^{-\alpha\frac{p}{p-1}+1} + T^{-(\alpha+1)\frac{p}{p-1}+1} + T^{-(\alpha+2)\frac{p}{p-1}+1}\big) R^{-\frac{n}{2}} \tag{4.71}$$

and

$$I_{4,R} + I_{7,R} \lesssim T^{-\alpha\frac{p}{p-1}+1} R^{-\frac{n}{2}-\frac{p}{p-1}}. \tag{4.72}$$

Including the estimates (4.71), and (4.72) in (4.37) we get

$$\int_0^T \int_{\Omega_R} |u(t,x)|^p \psi(t,x) dx dt \lesssim R^{-\frac{n}{2}} T^{-\alpha\frac{p}{p-1}+1} + R^{-\frac{n}{2}-\frac{p}{p-1}} T^{-\alpha\frac{p}{p-1}+1}. \tag{4.73}$$

Noting that $p < \frac{1}{\gamma}$ implies that $1 - \alpha\frac{p}{p-1} < 0$. Furthermore, we have

$$\lim_{T \to +\infty} \psi(t,x) = \varphi_1^{\ell}(x). \tag{4.74}$$

Thus it follows, immediately, by taking the limit as $T \longrightarrow +\infty$ in (4.73) that

$$\int_0^{+\infty} \int_{\Omega_R} |u(t,x)|^p \varphi_1^\ell(x) dx dt = 0, \tag{4.75}$$

Next, we take the limit in (4.75) as $R \to +\infty$ and noting that $\lim_{R\to+\infty} \varphi_1^\ell(x) = 1$, we conclude that

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u(t,x)|^p dx dt = 0.$$

This implies that $u \equiv 0$, which is a contradiction to our assumption (4.23).

Case: $p = \frac{1}{\gamma}$. In this case, Noting that the condition $p = \frac{1}{\gamma}$ is equivalent to $1 - \alpha \frac{p}{p-1} = 0$. Hence, by taking the limit as $T \to \infty$ in (4.73) and using the condition $p = \frac{1}{\gamma}$, we obtain the following estimate:

$$\int_0^{\infty} \int_{\Omega_R} |u|^p \varphi_1^\ell(x) dt dx \lesssim R^{-\frac{n}{2}}. \tag{4.76}$$

Next, by taking the limit as $R \to \infty$ in (4.76) and the fact that $\lim_{R\to+\infty} \varphi_1^\ell(x) = 1$ we derive

$$\int_0^{\infty} \int_{\mathbb{R}^N} |u|^p dt dx = 0. \tag{4.77}$$

This implies that $u \equiv 0$ and this is a contradiction.

□

Chapter 5 Conclusion and concluded remarks

In this chapter we interprets some of results we obtained and explain the connection with the old results in this direction, we try also to explore some open projects based on what done and understand from the study of Cauchy problems for wave equations with double damping . Let us first explain the connection between semi-linear and non linear memory terms, by using fractional analysis techniques.

5.1 Comments on Chapter 3

We succeeded to generalize recent results obtained in [6] and [17]. In fact, the author prove in [17] that critical exponent for damped wave equation with nonlinear memory is the same as the critical exponent of the heat equation with nonlinear memory considered in [6].

For us, we add a third model that has the same critical exponent with the heat equation and damped wave equation with non linear memory that is the three models have the same critical exponent. One can interpret this results by saying that the viscoelastic term does not influence neither the critical exponent nor the blow-up results.

As a second accomplished achievement, is issued from the combining of the results of Section 3.4 and Section 3.5. Summarizing,

- If $p \leq p_{Fuj}(\gamma, n)$ then all solutions of Cauchy problem (4) blow-up in finite time.
- If $p > p_{Fuj}(\gamma, n)$ then the solutions of Cauchy problem (4) exist globally in time.

These notes show that the parameter $p_{Fuj}(\gamma, n)$ is critical and sharp.

5.2 Benefits of Chapter 4

In [2] and [14], the authors showed that the critical exponent for the following Cauchy problem of visco-elastic damped wave equation with non linear memory

$$u_{tt} - \Delta u + b(t)u_t - \Delta u_t = \int_0^t (t-\tau)^{-\gamma}|u(\tau,\cdot)|^p d\tau, \quad (t,x) \in (0,\infty)\times\mathbb{R}^n \tag{5.1}$$

in the case $b(t) = 1$ is

$$p_{Fuj}(\gamma,n) = 1 + \frac{2(2-\gamma)}{(n-2+2\gamma)_+}. \tag{5.2}$$

In this paper we studied the influence of the effective dissipation on the critical exponent by studying Cauchy problem (4.1). That is, problem (5.1) with $b(t) \neq 1$. We proved that the critical exponent for Cauchy problem for the model (5.1) is

$$\tilde{p}_\gamma(n) = 1 + \frac{2(1-\gamma)}{n+2\gamma}. \tag{5.3}$$

In term of comparison, one can show easily that $\tilde{p}_\gamma(n) < p_{Fuj}(\gamma,n)$ for all $n \geq 1$ and $\gamma \in (0,1)$. This means that the range of blow-up is contracted since

$$[1,\ \tilde{p}_\gamma(n)] \subset [1,\ p_{Fuj}(\gamma,n)].$$

Summarizing, the effective dissipative term contracts the range of blow-up and systematically it dilates the range of global existence.

5.3 Concluding remarks

In this section, we present further comments on the results we obtain and some open problems. First, one can note that for Cauchy problem (4.1), we find a threshold for the critical exponent of fujita which is given by (4.22). Indeed, since the global (in time) existence is not established yet we can not expect that the parameter $\tilde{p}_\gamma(n)$ is sharp.

5.3.1 Open problems

Base on the previous comments, we may conclude the following:

1. the study of global (in time) existence of small data solutions for Cauchy problem (4.1) is *still open research project.*

2. In the other hand, counting on the results of Section 1.3 one may predict that the critical exponent of Fujita of Cauchy problem for effectively semi-linear damped wave equation
$$u_{tt} - \Delta u + b(t)u_t - \Delta u_t = |u|^p$$
is $p^* = 1$. One may interpret this result by the fact that all energy solutions (without additional regularity) are global in time.

3. Since the additional regularity L^m $(m \in [1, 2))$ improves the range (for p) of global existence we predict that even by taking into account the additional regularity, the solutions of Cauchy problem (4.1) remain exist globally in time.

4. If some think to get some blow-up results for Cauchy problem (4.1), he has to think about hight regular solution by assuming high regular data.

We finish this part by our wish to solve the above still open problems and to generalize our results for $\sigma-$evolution problems as well.

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