People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

Hassiba Benbouali University of Chlef

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Department of Common Core in Science and Technology

POLYCOPY OF COURSES

Analysis course 1 with detailed examples and entrainment exercises

Intended for students of:

LMD Bachelor's Degree (First Year Common Core)

Subject: Mathematics (Analysis 1)

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Distribution of the Analysis 1 Program

Field: *Science and Technology*	***Program:*** *Analysis 1* ***Code :*** *F1.1.1* ***Teaching units:*** *Fundamental* ***Semester hourly volume:*** $67H30'$ ***Weekly hours:*** $(1H30'$ *courses and* $3H$ *Tutorials*$)$ ***Semester 1 :*** 15 *weeks*	**First** *Year* *common core* ***Coefficient :*** 03 ***Credits:*** 06

Chapter	*The subtitles*	*Number of weeks*
1. Properties of the Set $\mathbb{R}$	*1-1 Upper bounded, lower bounded.* *1-2 Supremum and infimum.* *1-3 Maximum element, minimum element.* *1-4 Absolute value, integer part.*	**03**
2. Real Numerical Sequences and applications	*2-1 Convergent Sequences.* *2-2 Comparison Theorems.* *2-3 Monotone Convergence Theorem.* *2-4 Extracted Sequences.* *2-5 Adjacent Sequences.* *2-6 Special Sequences (arithmetic, geometric, recurrent).*	**03**

3. Real Functions of a Single variable	*3-1 Limits and Continuity of Functions.* *3-2 Derivative and Differential of a Function.* *3-3 Applications to Elementary Functions (Power, Exponential, Hyperbolic, Trigonometric,...).*	03
4. Limited development of functions	*4-1 Introduction and definition of limited development.* *4-2 Taylor's Formula.* *4-3 Limited development of useful functions.* *4-4 Limited Development Applications.*	03
5. Simple Integrals	*5-1 Review of the Riemann integral.* 5.2 calculus of primitives. 5.3 Integration by parts. 5.4 Integration by substitution.	03

Introduction

This document includes a set of introductory and fundamental concepts for building a solid foundation in analysis and deduction for university students. It also develops students' research and discovery abilities through analysis and synthesis, building upon their prior and current knowledge.
The primary objective of the program content presented in this publication is to cultivate students capable of addressing problems scientifically and rigorously, developing their computational skills, and exploring new techniques.

There is no doubt that everything you learned in mathematics during your previous school years, and especially in high school, is of paramount importance. However, higher education aims to teach you how to deepen, develop, and explore this knowledge! The first year lays the foundation, familiarizes you with the tools you will need later on, and allows you to use them effectively.
It is also an opportunity to discover the many applications of mathematics in most fields, including physics, chemistry, economics, and many others. This is where the true beauty and richness of mathematics lies, from the infinitely large (limits) to the infinitely small (derivatives), from measuring lengths (absolute values) to areas and volumes (integration), and countless other creative possibilities.
The lessons are presented very clearly with many examples, allowing students to fully understand the program. In each lesson, we have provided practice exercises.
This document contains six chapters, the first of which is "**Properties of Real Numbers**."
This chapter provides a brief overview of some fundamental mathematical concepts necessary for the study of mathematical analysis. While many of these concepts may be familiar to the reader

(with further explanation), others will be new, such as the lower major, upper minor, and integer part functions.

The second is **Numerical sequences.**
Numerical sequences aim to understand the evolution of sequences of numbers (real, complex, etc.). This allows us to model many everyday phenomena. For example, suppose we invest an amount S_1 D.A at an annual interest rate of 0.07%. After 10 years, we will therefore have $S_{10} = (1.07)^{10} S_1$.
The main topics covered in this chapter are, firstly, a review of the fundamental concepts of numerical sequences, such as their definition, limits, and direction of variation; secondly, the convergence of a sequence and some theorems; and finally, sub-sequences, adjacent sequences, arithmetic sequences, and geometric sequences, illustrated with examples.

The third is **real-valued function.**
The main tool explored in this volume of analysis is functions. You are already familiar with some of them: square root, sine, cosine, logarithm, exponential function... . These functions are used when we are concerned with phenomena that change according to specific parameters. For example, the position of a comet as a function of time, or the change in the volume of a gas as a function of temperature and pressure. This mathematical form is applied in physics, chemistry, biology, and even economics, and helps us solve problems.
In this chapter, we will cover limits, continuity, extension by continuity, the intermediate value theorem and its consequences, differentiation, the finite increment theorem, and its applications.

The forth is **Limited developments.**
In this chapter, for any function, we will find the n^{th}-degree polynomial that best approximates the function. The results are only valid for x around a fixed value (this will often be around 0). This polynomial will be calculated from the successive derivatives at the point considered. Let's take the example of an exponential function. The behavior of the function $f(x) = exp(x)$ in the neighborhood of the point $x = 0$ can be understood using its tangent, whose equation is $y = 1 + x$. We have approximated the graph with a straight line. If we want a more precise approximation, which parabola with the equation $y = c_1 + c_2 x + c_3 x^2$ best approximates the graph of f in the neighborhood of $x = 0$?. It is the parabola with the equation $y = 1 + x + \frac{1}{2} x^2$.

In this chapter, we will deal with limited development at a specific point or at infinity and the position of the curve with respect to its tangent or its asymptote, as well as the calculation of limits.

In the chapter five: "**Application to Prime Functions**", we will deal with trigonometric and hyperbolic functions, their inverses, and some basic properties for using them in solving equations and inequalities.

In the last chapter **Simple Integrals.**
The most important result about integration is the fundamental theorem of calculus, which states that integration and differentiation are inverse operations in an appropriately understood sense. Among other things, this connection enables us to compute many integrals explicitly. Integrability is a less restrictive condition on a function than differentiability. Roughly speaking, integration makes functions smoother, while differentiation makes functions rougher. For example, the indefinite integral of every continuous function exists and is differentiable, whereas the derivative of a continuous function need not exist (and generally doesn't). The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function as well as some not-too-badly discontinuous functions. There are, however, many other types of integrals, the most important of which is the Lebesgue integral. The Lebesgue integral allows one to integrate unbounded or highly discontinuous functions whose Riemann integrals do not exist, and it has better mathematical properties than the Riemann integral. The definition of the Lebesgue integral requires the use of measure theory, which we will not describe here. In any event, the Riemann integral is adequate for many purposes, and even if one needs the Lebesgue integral, it's better to understand the Riemann integral first.

Contents

Chapter 1

Properties of the Set $\mathbb{R}$

1.1 Upper bounded, lower bounded, and bounded part

1.1.1 Upper bounds and lower bounds

Definition 1.1.1 *:*

Let E be a non-empty subset of $\mathbb{R}$.

▷ *The set E is bounded from above, if there exist the real number M in $\mathbb{R}$, called an upper bound of E, Such that $x \leq \mathrm{M}$ for every x in E.*

▷ *The set E is bounded from below, if there exist the real number m in $\mathbb{R}$, called an lower bound of E, Such that $x \geq \mathrm{m}$ for every x in E.*

Bounded sets

Definition 1.1.2 *:*

A set is said to be bounded, if it is both bounded above and bounded below. i.e., if there exist the two real numbers m and M, Such that $m \leq x \leq M$ for all x in E .

Examples 1.1.3 ***1.*** *Let $E = \,]-\infty, 3\,]$.*

This set E is bounded from above, and has an infinite number of above bounds, for example, the numbers 3, $\sqrt{10}$, $\frac{10}{3}$ and 7 are its upper bounds of E. It also has no lower bound, so it is unbounded from below.

Therefore, it is unbounded, even though it is bounded from above.

2. *Let* $E = [\,2, +\infty\,[$.

Then E is bounded below, and has an infinite number of below bounds, for example, the numbers 0, $\sqrt{2}$, $\frac{5}{3}$ *and* 2 *are its lower bounds of E. It also has no upper bound, so it is not bounded from above.*

This imply that, it is unbounded, even though it is bounded from below.

3. *Let* $E = [\,0, \sqrt{5}\,[$.

Then E is bounded from below, and has an infinite number of below bounds, for example, the numbers $-\sqrt{2}$, $\frac{-1}{3}$ *and* 0 *are its lower bounds of E. It also has bounded from above, and has an infinite number of above bounds, for example, the numbers* $\sqrt{5}$ *and* 3 *are its upper bounds of E.*

This imply that, the set E is bounded.

4. *Let* $E = \{-1, 0, 1, 3\} \cup \,]-\frac{1}{2}, \frac{11}{3}\,[$.

The set E is both bounded above and bounded below. Because, has an infinite number of below bounds and an infinite number of above bounds, for example, the numbers -2 *and* -1 *are lower bounds of E and the numbers* $\frac{11}{3}$ *and* 4 *are upper bounds of E. This imply that, the set E is bounded.*

1.1.2 The supremum and infimum

Definition 1.1.4 :

The ***supremum*** *of a set is its* ***least upper bound*** *and the* ***infimum*** *is its* ***greatest upper bound****.*

Definition 1.1.5 :

Suppose that $E \subset \mathbb{R}$ *is a set of real numbers.*

$\star$ *If* $M \in \mathbb{R}$ *is an upper bound of E such that* $M \leq M'$ *for every upper bound* M' *of E, then M is called the supremum of E, denoted* $M = supE$.

$\star$ *If* $M \in \mathbb{R}$ *is a lower bound of E such that* $m \geq m'$ *for every upper bound* m' *of E, then m is called the infimum of E, denoted* $m = infE$.

Remarks 1.1.6 : • *If E is not bounded from above, then we write* $supE = +\infty$, *and if E is not bounded from below, we write* $infE = -\infty$.

• *If* $E = \emptyset$ *is the empty set, then every real number is both an upper and a lower bound of* $\emptyset$, *and we write* $sup\emptyset = +\infty$, $inf\emptyset = -\infty$.

Archimedes property

This property is a fundamental mathematical principle.

Property 1.1.7 *For any two real numbers, x and y, and x is positive, there exists a natural number n such that nx is greater than y.*

Proof : We assume, by contradiction, that: $\exists x \in \mathbb{R}_+^*, y \in \mathbb{R}_+^*, \forall n \in \mathbb{N}; nx < y$. We consider the set B defined by $B = \{nx; n \in \mathbb{N}\}$. B is a non empty subset of $\mathbb{R}$, bounded from above by the real number y. This imply that $supA = M$ exist, hence

$$\begin{aligned} nx < M;\ \forall n \in \mathbb{N} &\implies (n+1)x < M;\ \forall n \in \mathbb{N} \\ &\implies nx < M - x;\ \forall n \in \mathbb{N}. \end{aligned}$$

then $M - x$ is an upper bound of B and $M - x < M$, because $x > 0$. This is a contradiction because M is a smallest upper bounds of B. ∎

Example 1.1.8 *: Determine $sup(A)$ and $inf(A)$ where*

$$A = \left\{ \frac{1}{n+1};\ n \in \mathbb{N} \right\}.$$

Solution: *We can see that $sup(A) = 1$ because $1 \in B$ and it is an upper bound of A (since $x \le 1$ for all $x \in A$). We have that zero is a lower bound since $\frac{1}{n+1} > 0$ for all $n \in \mathbb{N}$. Furthermore, we can see that zero is the greatest possible lower bound since, if we assume that m is the minimum of A with $m > 0$, then this assumption leads us to a contradiction, and that is: by Archimedes property on m and 1 we have : there exist n in $\mathbb{N}$ such that $n.m > 1$. This imply that, there exist $n \in \mathbb{N}$ such that $(n+1)\, m > 1 + m > 1 \Rightarrow m > \frac{1}{n+1} \in A$. This means that m is not a lower bound of A, it follows that contradicts to the above hypothesis, and therefore that $inf(A) = 0$.*

1.1.3 Methods of proving the supremum and infimum

To prove that a given number M (resp. m) is a least upper bound (resp. greatest lower bound) of a set E, we can use one of the methods:

Method 1: Characterization of supremum and infimum

In this first method, we generally follow the following two steps:

Step 1. Prove that M is an upper bound (resp. m is a lower bound) for E i.e. show that $M \ge x$ (resp. $m \le x$) for all $x \in E$.

Step 2. Prove that M (resp. m) is the least upper bound (resp. greatest lower bound) for E. Often this is done by assuming that there is an $\varepsilon > 0$ such that $M - \varepsilon$ (resp. $m + \varepsilon$) is also an upper bound (resp. a lower bound) for E. One then exhibits an element $x \in E$ with $x > M - \varepsilon$ (resp. $x < m + \varepsilon$), showing that $M - \varepsilon$ (resp. $m + \varepsilon$) is not an upper bound (resp. a lower bound) of E. Or, to prove that $supE = M$, we prove the two following properties

$$M = supE \Leftrightarrow \begin{cases} 1). \ M \geq x, \ \forall x \in E \\ 2). \ \forall \varepsilon > 0, \ \exists x \in E; \ x > M - \varepsilon \end{cases}.$$

And, for $infE = m$, we prove the two following properties

$$m = infE \Leftrightarrow \begin{cases} 1). \ m \leq x, \forall x \in E \\ 2). \ \forall \varepsilon > 0, \ \exists x \in E; \ x < m + \varepsilon. \end{cases}$$

Method2: Reasoning by contradiction

In this second method, we can use the reasoning by contradiction as follows:
To prove that $supE = M$, we assume that M is not the least upper bound of E. Therefore, there exists an upper bound $M' \in \mathbb{R}$ of E such that $M' < M$. Applying Archimedean property to the number $M - M'$ and calculating based on the given information, we can arrive at a contradiction with our hypothesis that M' is an upper bound of E.

Example 1.1.9 $\star$***1:*** *Find the least upper bound for the following set A and prove that your answer is correct.* $A = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots\}$.
Solution: *By first method: We note that every element of A is less than 1 since $\frac{n}{n+1} < 1$ for all $n \in \mathbb{N}$. This imply that 1 is an upper bound fo A.*
We claim that the least upper bound of A is 1 i.e $supA = 1$. Assume that 1 is not the least upper bound of A. Then there is an $\varepsilon > 0$ such that $1 - \varepsilon$ is also an upper bound of A. However, we claim that there is a natural number n such that $1 - \varepsilon < \frac{n}{n+1}$. This inequality is equivalent with the following sequence of inequalities

$$1 - \frac{n}{n+1} < \varepsilon \Leftrightarrow 1 - \frac{n+1-1}{n+1} < \varepsilon \Leftrightarrow \frac{1}{n+1} < \varepsilon \Leftrightarrow n+1 > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon} - 1.$$

This imply that: there exist $n > \frac{1}{\varepsilon} - 1$ such that $1 - \varepsilon < \frac{n}{n+1}$. Showing that $1 - \varepsilon$ is not an upper bound for A. This means $\forall \varepsilon > 0, \ \exists n_0 = E\left(\frac{1}{\varepsilon} - 1\right) + 1 \in \mathbb{N}; \forall n \geq n_0, x = \frac{n}{n+1} \in A; \ x > 1 - \varepsilon$. Thus show that $supA = 1$.
By second method: We show that $supA = 1$. we assume that 1 is not the least upper bound of E. Therefore, there exists an upper bound $M' \in \mathbb{R}$ of E such that $M' < 1$. Applying Archimedean property to the number $1 - M'$, that there exist $n \in \mathbb{N}$ such that $n(1 - M') > 1$, this imply that $M' < 1 - \frac{1}{n} < 1 - \frac{1}{n+1} \in A$ which contradicts to the fact that M' is an upper bound of the set A. that is $supA = 1$.

Example 1.1.10 $\star$*2: Find the sup and inf of the following set and prove your answer.*

$$B = \{\frac{-2n+1}{n+1},\ n \in \mathbb{N}\}.$$

Solution: *We write the first five terms of B:*

$$B = \{1,\ \frac{-1}{2},\ -1,\ \frac{-5}{4},\ \frac{-7}{5},\cdots,\ \frac{-2n+1}{n+1},\dots\}.$$

It seems that the largest lower bound of B is -2 and the smallest upper bound of B is 1.

We must first show that -2 is a lower bound of B and 1 is an upper bound of B.

For this we have $\frac{-2n+1}{n+1} = -2 + \frac{3}{n+1}$ for all $n \in \mathbb{N}$, and $-2 < -2 + \frac{3}{n+1} \leq 1$ for all $n \in \mathbb{N}$. This proves that the set B is bounded from below by the number -2 and from above by the number 1.

Next we show that -2 is the greatest lower bound of B i.e $infB = -2$, we use the characterization of the lower bound.

For this, we show that:

$$\text{for every } \varepsilon > 0 \text{ there exist } n \in \mathbb{N} \text{ such that } -2+\varepsilon < \frac{-2n+1}{n+1}. \tag{1.1}$$

Suppose $\varepsilon > 0$, we search for the existence of a natural number $n \in \mathbb{N}$ that satisfies : $-2+\varepsilon < \frac{-2n+1}{n+1}$.

However, this leads to the following:

$$-2+\varepsilon < \frac{-2n+1}{n+1} \Leftrightarrow -2+\varepsilon < -2+\frac{3}{n+1} \Leftrightarrow n < \frac{3}{\varepsilon} - 1.$$

Since there exist $n \in \mathbb{N}$ (we can choose $n = E(\frac{3}{\varepsilon})$) satisfying the inequality (1.1), then $-2+\varepsilon$ it is not a lower bound of B. This means $infB = -2$.

Since , 1 is the upper bound of the set B. Therefore, since B is not empty, then, $sup(B)$ exists and $sup(B) \leq 1$. In fact, $sup(B) = 1$ because 1 is an element in B.

Proposition 1.1.11 *The supremum or infimum of a set E if it exists, is unique.*

Moreover, if both exist, then $infE \leq supE$.

Proof : $\star$ **Uniqueness** Proof by contradiction: Suppose that there exist two distinct suprema (or infima) of the set E. and show that this leads to a contradiction.

. Let M_1 and M_2 be two distinct suprema of the set E. This imply that M_1 and M_2 are upper bound of E. Since M_1 is the supremum of E, this means $M_1 \leq M_2$, and since M_1 is the supremum of E, this means $M_2 \leq M_1$. This imply that $M_1 = M_2$. This contradicts the hypothesis that they are distinct.

⋆ **Relationship** If $infE$ and $supE$ exist, then E is non-empty. Choose $x \in E$, this means $infE \leq x \leq supE$, since $infE$ is a lower bound of E and $supE$ is an upper bound of E. It follows that $infE \leq supE$. ■

Now, we state the following axioms:

⊗ Axiom 1: Every non-empty bounded subset of $\mathbb{R}$ has a least upper bound.

⊗ Axiom 2: Every non-empty bounded subset of $\mathbb{R}$ has a greatest lower bound.

⊗ Axiom 3: Every non-empty bounded subset of $\mathbb{R}$ has a least upper bound and a greatest lower bound.

1.2 Maximum element, minimum element

Definition 1.2.1 *:*

Let $E \subset \mathbb{R}$ non-empty. We will say that:

♢ *E has a **maximum** (denoted $max(E)$) if it has an upper bound that belong to E.*

In the other words:

$$max\,(E) = M \Longleftrightarrow (x \leq M,\ \forall x \in E \text{ and } M \in E)\,.$$

♢ *E has a **minimum** (denoted $min(E)$) if it has a lower bound that belong to E.*

In the other words:

$$min\,(E) = m \Longleftrightarrow (x \geq m,\ \forall x \in E \text{ and } m \in E)\,.$$

Examples 1.2.2 ***a)*** *Let $E = B = \left]-\frac{3}{2}, \frac{\sqrt{7}}{3}\,\right]$.*

The set of upper bounds of B is $U = \left[\,\frac{\sqrt{7}}{3}, +\infty\,\right[$. It follows that the smallest upper bounds of B is $supB = \frac{\sqrt{7}}{3}$, since $\frac{\sqrt{7}}{3} \in B$ then $maxB = \frac{\sqrt{7}}{3}$.

The set of lower bounds of B is $L = \left]-\infty, -\frac{3}{2}\,\right]$. It follows that the greatest lower bounds of B is $infB = -\frac{3}{2}$, since $-\frac{3}{2} \notin B$ then $minB$ does not exist.

b) *Let $E = D = \left\{x \in \mathbb{R},\ x^2 + 4x + 3 < 0\right\}$.*

Since $x^2 + 4x + 3 < 0 \Longleftrightarrow -3 < x < -1$, then $D = \left]-3, -1\,\right[$.

The set of upper bounds of D is $U_1 = \left[\,-1, +\infty\,\right[$. It follows that the smallest upper bounds of D is $supD = -1$, and the set of lower bounds of D is $L_1 = \left]-\infty, -3\,\right]$. It follows that the greatest lower bounds of D is $infD = -3$. Since $-1 \notin D$ and $-3 \notin D$ then, $maxD$ and $infD$ does not exists.

c) *We find the infimum, supremum, minimum and the maximum, if they exist, of the set*

$F = \left\{2 - \frac{(-1)^n}{n},\ n \in \mathbb{N}^*\right\}$.

Answer: *We can identify some elements in this set as follows:* $F = \left\{3, \frac{3}{2}, \frac{7}{3}, \frac{7}{4}, \frac{11}{5}, \frac{11}{6}, \cdots\right\}$.
We observe that the $infF = 2$ *and* $supF = 3$. *but this is not sufficient, we must prove it.*
For all $n \in \mathbb{N}^*$, *we have*

$$-1 \leq (-1)^n \leq 1 \Longleftrightarrow 1 \geq \frac{1}{n} \geq -\frac{(-1)^n}{n} \geq -\frac{1}{n} \geq -1 \Longleftrightarrow 3 \geq 2 - \frac{(-1)^n}{n} \geq 1.$$

This imply that f *is bounded from below by* 1 *and from above by* 3.
We show that $supF = 3$. *Consider* M *any upper bound of* F. *That is* $M \geq x, \forall x \in F$, *since* $3 \in F$ *we have* $M \geq 3$ *for any upper bound of* F. *Thus,* $supF = maxF = 3$.
We show that $infF = 1$. *suppose that* m *is a lower bound of* F. *That is* $m \leq x, \forall x \in F$, *since* $1 \in F$ *we have* $m \leq 1$ *for any lower bound of* F. *Thus,* $infF = minF = 1$.

Theorem 1.2.3 *Let* $E \subset \mathbb{R}$. *Then, if* $max(E)$ *(or* $min(E)$*) exists, it is unique.*

Proof : Let's assume that $E \subset \mathbb{R}$ has two different maximum values, M_1 and M_2. Based on the definition of the maximum value of a set, we have:

$$\begin{cases} M_1 \text{ is a maximum of } E \Longleftrightarrow x \leq M_1,\ \forall x \in E \text{ and } M_1 \in E. \text{ for } x = M_2 \text{ we obtain } M_2 \leq M_1. \\ M_2 \text{ is a maximum of } E \Longleftrightarrow x \leq M_2,\ \forall x \in E \text{ and } M_2 \in E. \text{ for } x = M_1 \text{ we obtain } M_1 \leq M_2. \end{cases}$$

It follows that $M_1 = M_2$.
Similarly for the minimum. ■

1.2.1 Compatibility of sup and inf with algebraic operations

We define the following sets where A, B are non-empty subsets of $\mathbb{R}$ and $\alpha \in \mathbb{R}$

$$\alpha A := \{\alpha.x,\ x \in A\}.$$
$$\alpha + A := \{\alpha + x,\ x \in A\}.$$
$$A + B := \{x + y,\ x \in A \text{ and } x \in B\}.$$

Properties 1.2.4 *Let* A, B *are non-empty subsets of* $\mathbb{R}$ *and* $\alpha \in \mathbb{R}$, *then:*
1$\star$ *If* A, B *are bounded and* $A \subseteq B$ *then,* $infB \leq infA \leq supA \leq supB$.
2$\star$ *If* A, B *are bounded from above, then,* $sup\,(A \cup B) = max\,\{supA, supB\}$.
3$\star$ *If* A, B *are bounded from below, then,* $inf\,(A \cup B) = min\,\{infA, infB\}$.
4$\star$ *If* A, B *are bounded from above, then,* $sup\,(A \cap B) \leq min\,\{supA, supB\}$.

5$\star$ *If* A, B *are bounded from below, then,* $inf\,(A \cup B) \geq max\,\{infA, infB\}$.

6$\star$ *If* A, B *are bounded from above, then,* $sup\,(A + B) = supA + supB$.

7$\star$ *If* A, B *are bounded from below, then,* $inf\,(A + B) = infA + infB$.

8$\star$ *If* $\alpha \in \mathbb{R}_+$ *and* A *is bounded from above, then,* $sup\,(\alpha.A) = \alpha.supA$

and if $\alpha \in \mathbb{R}_+$ *and* A *is bounded from below, then,* $inf\,(\alpha.A) = \alpha.infA$.

9$\star$ *If* $\alpha \in \mathbb{R}_-$ *and* A *is bounded, then,* $sup\,(\alpha.A) = \alpha.infA$ *and* $inf\,(\alpha.A) = \alpha.supA$.

10$\star$ $sup\,(A - B) = sup\,(A + (-1)\,B)$ *and* $inf\,(A - B) = inf\,(A + (-1)\,B)$.

Proof : 1$\star$ We show that: If $A \subseteq B$ then, $infB \leq infA \leq supA \leq supB$.

$\odot$ first, we show that $infA \leq supA$ similarly for $infB \leq supB$.

Since $A \neq \phi$ then by definition of $infA$ and $supA$ we have : $\forall x \in A, infA \leq x \leq supA$. So $infA \leq supA$.

$\odot$ We show that $infB \leq infA$.

Since $A \subseteq B$ then, $infB \leq x,\ \forall x \in A$. This imply that $infB$ is a lower bound of A. As $infA$ is the greatest lower bounds of A, conclude that $infB \leq infA$.

$\odot$ We show that $supA \leq supB$.

Since $A \subseteq B$ then, $supB \geq x,\ \forall x \in A$. This imply that $supB$ is an upper bound of A. As $supA$ is the least lower bounds of A, conclude that $supA \leq supB$.

2$\star$ We show that: $sup\,(A \cup B) = max\,\{supA, supB\}$.

Since $\mathbb{R}$ is totally ordered, then, we have $supA \leq supB$ or $supB \leq supA$.

Let us suppose that $supA \leq supB$, then we have $max\,\{supA, supB\} = supB$.

Since $B \subseteq (A \cup B)$, then, from 1$\star$ we have

$$supB \leq sup\,(A \cup B)\,. \tag{1.2}$$

Since $supB$ is an upper bound of B and of A because $supA \leq supB$, then, is also an upper bound of $(A \cup B)$. Since $sup\,(A \cup B)$ is smallest upper bounds of $(A \cup B)$, then,

$$sup\,(A \cup B) \leq supB. \tag{1.3}$$

From (1.2) and (1.3), we obtain the desired result.

3$\star$ $inf\,(A \cup B) = min\,\{infA, infB\}$.

The proof is similar to what came before.

4$\star$ We show that: $sup\,(A \cap B) \leq min\,\{supA, supB\}$.

We know that $(A \cap B) \subseteq A$ and $(A \cap B) \subseteq B$. From 1$\star$ we have $sup\,(A \cap B) \leq supA$ and

$sup\,(A \cap B) \leq supB$. this means that $sup\,(A \cap B) \leq min\,\{supA, supB\}$.

5$\star$ $inf\,(A \cup B) \geq max\,\{infA, infB\}$.

The proof is similar to the previous one.

6$\star$ We show that: $sup\,(A + B) = supA + supB$.

For this we show $sup\,(A + B) \leq supA + supB$ and $sup\,(A + B) \geq supA + supB$.

$\oplus$ Let $a \in A$ and $b \in B$, then $a \leq supA$ and $b \leq supB$. Thus $a + b \leq supA + supB$. this imply that $supA + supB$ is an upper bound of $A + B$. Since $sup\,(A + B)$ is smallest upper bounds of $A + B$, then

$$sup\,(A + B) \leq supA + supB. \tag{1.4}$$

$\oplus$ Let $a \in A$ and $b \in B$, then $a + b \in (A + B)$, and this implies that, for all $a \in A$ and $b \in B$ $a + b \leq sup\,(A + B) \Longrightarrow a \leq sup\,(A + B) - b$. So, $sup\,(A + B) - b$ is an upper bound of A. Since $supA$ is least upper bounds of A, then, for all $b \in B$, $supA \leq sup\,(A + B) - b$ and this equivalent to $b \leq sup\,(A + B) - supA$. Therefore, we can say that $sup\,(A + B) - supA$ is an upper bound of B. Since $supB$ is least upper bounds of B, then,

$$supA + supB \leq sup\,(A + B)\,. \tag{1.5}$$

From (1.4) and (1.5), we arrive at the desired result.

7$\star$ For $inf\,(A + B) = infA + infB$.

The proof is similar to the previous one.

8$\star$ We show that: If $\alpha \in \mathbb{R}_+$ then, $sup\,(\alpha.A) = \alpha.supA$ and $inf\,(\alpha.A) = \alpha.infA$.

$\odot$ We know that, for all $x \in A$ we have : $x \leq supA$ this imply that, if $\alpha \in \mathbb{R}_+$ then, $\alpha.x \leq \alpha.supA$. It follows that $\alpha.supA$ is an upper bound of $\alpha.A$. Since $sup\,(\alpha.A)$ is least upper bounds of $\alpha.A$, then,

$$sup\,(\alpha.A) \leq \alpha.supA. \tag{1.6}$$

$\odot$ We know that, for all $x \in A$ we have : $\alpha.x \leq sup\,(\alpha.A)$ this imply that, $x \leq \frac{1}{\alpha} sup\,(\alpha.A)\,,\ \forall x \in A$. It follows that $\frac{1}{\alpha} sup\,(\alpha.A)$ is an upper bound of A. Since $sup\,(A)$ is least upper bounds of A, then,

$$\alpha.supA \leq sup\,(\alpha.A)\,. \tag{1.7}$$

From (1.6) and (1.7), we obtain $sup\,(\alpha.A) = \alpha.supA$ with $\alpha \in \mathbb{R}_+$.

$\oplus$ We use the same steps to prove that : $inf\,(\alpha.A) = \alpha.infA$ with $\alpha \in \mathbb{R}_+$.

9$\star$ We show that: If $\alpha \in \mathbb{R}_-$ then, $inf\,(\alpha.A) = \alpha.supA$ and $sup\,(\alpha.A) = \alpha.infA$.

$\odot$ We know that, for all $x \in A$ we have : $x \leq supA$ this imply that, if $\alpha \in \mathbb{R}_-$ then, $\alpha.x \geq \alpha.supA$.

It follows that $\alpha.supA$ is a lower bound of $\alpha.A$. Since $inf\,(\alpha.A)$ is greatest lower bounds of $\alpha.A$, then,

$$inf\,(\alpha.A) \geq \alpha.supA. \tag{1.8}$$

$\odot$ On the other hand, for any $z = \alpha.x \in (\alpha.A)$ we have : $z \geq inf\,(\alpha.A)$ this imply that, $\frac{1}{\alpha}z = x \leq \frac{1}{\alpha}inf\,(\alpha.A)\,,\ \forall x \in A$. It follows that $\frac{1}{\alpha}inf\,(\alpha.A)$ is an upper bound of A. Since $sup\,(A)$ is smallest upper bounds of A, then,

$$\alpha.supA \geq inf\,(\alpha.A)\,. \tag{1.9}$$

From (1.8) and (1.9), we obtain $inf\,(\alpha.A) = \alpha.supA$ with $\alpha \in \mathbb{R}_-$.

$\oplus$ We use the same steps to prove that : $sup\,(\alpha.A) = \alpha.infA$ with $\alpha \in \mathbb{R}_-$. ■

1.3 Absolute value, integer part

1.3.1 Absolute value

definition The **absolute value** or **modulus** of a real number x, denoted $|x|$, is a positive real number defined by:

$$|x| = \begin{cases} x, & if\, x \geq 0 \\ -x, & if\, x < 0. \end{cases} \quad Or \quad |x| = max\,\{x, -x\}\,.$$

Examples 1.3.1 $\sqrt{(-3)^2} = |-3| = 3$, $\left|-\sqrt{2}\right| = \sqrt{2}\,$, $\left|\sqrt{3}-3\right| = 3 - \sqrt{3}$ *and* $\left|x^2\right| = x^2$, $\forall x \in \mathbb{R}$.

Properties of absolute value

Properties 1.3.2 *Let $\alpha \in \mathbb{R}_+$, we have :*

1$\oplus$ $|-x| = |x|\,,\ \forall x \in \mathbb{R}$.

2$\oplus$ $x \leq |x|$ *and* $-x \leq |x|$, *(or* $-|x| \leq x \leq |x|$*)*, $\forall x \in \mathbb{R}$.

3$\oplus$ $|x.y| = |x|\,.\,|y|\,,\ \forall x, y \in \mathbb{R}$.

4$\oplus$ $\left|x^2\right| = |x|^2\,,\ \forall x \in \mathbb{R}$.

5$\oplus$ $\left|\dfrac{1}{y}\right| = \dfrac{1}{|y|},\ \forall y \in \mathbb{R}^*$.

6$\oplus$ $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|},\ \forall x, y \in \mathbb{R}$ *with* $y \neq 0$.

7$\oplus$ $|x| \leq \alpha \Longleftrightarrow -\alpha \leq x \leq \alpha,$ *For all* $x \in \mathbb{R}$ *with* $\alpha \geq 0$.

8$\oplus$ $|x| \geq \alpha \Longleftrightarrow (x \geq \alpha$ or $x \leq -\alpha),$ *For all* $x \in \mathbb{R}$.

9$\oplus$ $|x \pm y| \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$.

10$\oplus$ $||x| - |y|| \leq |x \pm y| \leq |x| + |y|$, $\forall x, y \in \mathbb{R}$.

Proof : 1$\oplus$ From definition of absolute value $|-x| = max\{-(-x), -x\} = max\{x, -x\} = |x|$.

2$\oplus$ We know that $x \leq max\{x, -x\}$ and $-x \leq max\{x, -x\}$, for all $x \in \mathbb{R}$. This imply that $x \leq |x|$ and $-x \leq |x|$), $\forall x \in \mathbb{R}$.

3$\oplus$ For all $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |x| . |y| &= max\{-x, x\} .max\{-y, y\} \\ &= max\{xy, (-x)y, x(-y), (-x)(-y)\} \\ &= max\{-xy, xy\} = |x.y| . \end{aligned}$$

4$\oplus$ Based on property number 3$\star$ mentioned previously, and taking $x = y$, we find $|x^2| = |x|^2$, $\forall x \in \mathbb{R}$.

5$\oplus$ For all $y \in \mathbb{R}^*$ we have $\left|\frac{y}{y}\right| = 1$. This imply that $1 = \left|y.\frac{1}{y}\right| = |y| . \left|\frac{1}{y}\right|, \forall y \in \mathbb{R}^*$. this means the property number 5$\star$ is true.

6$\oplus$ Using properties 3$\star$ and 5$\star$ mentioned previously, we find $\left|\frac{x}{y}\right| = \left|x.\frac{1}{y}\right| = |x| \left|\frac{1}{y}\right| = \frac{|x|}{|y|}$, $\forall x, y \in \mathbb{R}$ with $y \neq 0$.

7$\oplus$ For all $x \in \mathbb{R}$ and $\alpha \geq 0$ we have:

$$\begin{aligned} |x| = max\{-x, x\} \leq \alpha &\iff -x \leq \alpha \text{ and } x \leq \alpha \\ &\iff -\alpha \leq x \leq \alpha. \end{aligned}$$

8$\oplus$ For all $x \in \mathbb{R}$ and $\alpha \geq 0$ we have:

$$\begin{aligned} |x| = max\{-x, x\} \geq \alpha &\iff -x \geq \alpha \text{ or } x \geq \alpha \\ &\iff x \leq -\alpha \text{ or } x \geq \alpha. \end{aligned}$$

9$\oplus$ For all $x, y \in \mathbb{R}$ we have

$$\begin{aligned} -|x| \leq x \leq |x| \text{ and } -|y| \leq y \leq |y| &\implies -(|x| + |y|) \leq x + y \leq |x| + |y| \\ &\implies |x + y| \leq |x| + |y|, \text{ (using the property 7}\star\text{).} \end{aligned}$$

And for $|x - y| = |x + (-y)|$ then using the above property.

10$\oplus$ For all $x, y \in \mathbb{R}$ we have

$$\begin{cases} |x| = |(x \pm y) \mp y| \leq |x \pm y| + |y| \\ |y| = |(y \pm x) \mp x| \leq |x \pm y| + |x| \end{cases} \iff \begin{cases} |x| - |y| \leq |x \pm y| \\ |x| - |y| \geq -|x \pm y| \end{cases}$$

This imply that $-|x \pm y| \leq |x| - |y| \leq |x \pm y|$, then, according to 7$\star$ arrive at the desired result. ■

1.3.2 Integer part

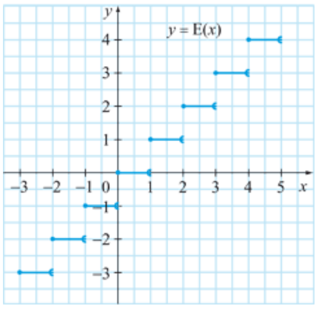
This function appeared in the literature in the early years of the nineteenth century.
For more information on the history of this function, see ([8], p. 3-8). For the development of its fundamental theorem and a discussion of the symbols mentioned above, see Murray's thesis [9].

Definition 1.3.3 *The integer part of a real number, denoted* $[x]$ *or* $E(x)$*, is the unique integer such that* $[x] \leq x < [x] + 1$.
In other words: These symbols $[x]$ *or* $E(x)$ *are also called the largest integer less than or equal to* x*. That is* $[x] = max\{p \in \mathbb{Z}/p \leq x\}$.

Examples 1.3.4 *:*
1. $[3,9999] = 3$. *Because the largest integer less than or equal to* $3,9999$ *is* 3. *Furthermore,* $3 \leq 3,9999 < 4$.
2. $\left[-\sqrt{2}\right] = -2$. *Because the largest integer less than or equal to* $-\sqrt{2} = -1.4142135624...$ *is* -2. *Furthermore,* $-2 \leq -\sqrt{2} < -1$.
3. $[-cos(\pi)] = 1$. *Because* $cos(\pi) = -1$ *and the largest integer less than or equal to* $-cos(\pi)$ *is* 1.

Graph of the integer part function

- *The domain of the integer part function is* $\mathbb{R}$.
- *Range (i.e. values taken by* $[x]$ *) of the integer part function is* $\mathbb{Z}$.
- *The integer part function is increasing in*$\mathbb{R}$*,*
  but it is not strictly increasing.
- *The integer part function is not continuous on all integer points.*



Uniqueness of the integer part

Proposition 1.3.5 *Every real number* x *has a unique integer part.*

Proof : By contradiction, we assume that there exists a real number x that has two distinct integer parts n and m (with $n > m$), that is: they satisfy the definition of the integer part for x as follows:

$$n \leq x < n+1 \quad \text{et} \quad m \leq x < m+1 \Longrightarrow m < n \leq x < m+1 < n+1.$$

This implies that $m - n < 1$ and $nm < 1$, therefore we have $-1 < m - n < 1$, so $m - n = 0$. This contradicts the fact that n and m are distinct. This shows that the hypothesis $n \neq m$ is false. Therefore, the proof is complete. ■

1.3.3 Basic properties

For every $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ we have:

1). $[x] = n$ if and only if $n \leq x < n+1$.

2). $x - 1 < [x] \leq x < [x] + 1$.

3). $[x+n] = [x] + n$.

4). $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right]$ with $n \neq 0$.

5). $[-x] = \begin{cases} -[x], & \textit{if } x \textit{ is an integer} \\ -[x]-1, & \textit{otherwise}. \end{cases}$.

6). If m and n are positive integers, then $\left[\frac{m}{n}\right] - \left[\frac{m-1}{n}\right] = \begin{cases} 1, & \textit{if } n \textit{ divides } m \\ 0, & \textit{otherwise}. \end{cases}$.

7). $\frac{[x]}{2} - \left[\frac{x}{2}\right] = \begin{cases} 0, & \textit{if } 2 \textit{ divides } [x] \\ \frac{1}{2}, & \textit{otherwise}. \end{cases}$.

8). $\left[\frac{x+1}{2}\right] = \begin{cases} \frac{[x]}{2}, & \textit{if } 2 \textit{ divides } [x] \\ \frac{[x]+1}{2}, & \textit{otherwise}. \end{cases}$.

9). $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$, for all x, y in $\mathbb{R}$.

Proof : 1). and 2). By definition.

3). From property 2) we have $[x] + n \leq x + n < [x] + n + 1$. This imply that the integer part of $x+n$ is $[x]+n$.

4). From property 2) we have $\frac{[x]}{n} \leq \frac{x}{n} < \frac{[x]}{n} + \frac{1}{n} < \frac{[x]}{n} + 1$. This imply that $\left[\frac{[x]}{n}\right] \leq \left[\frac{x}{n}\right] \leq \frac{x}{n} \leq \left[\frac{[x]}{n} + 1\right]$. Thus, the integer part of $\frac{x}{n}$ is $\left[\frac{[x]}{n}\right]$.

5). $\oplus$ If x is an integer, then $-x$ is also, and $-x + x = 0$. According to the property 3) we have $[-x+x] = [-x] + [x] = 0$. This imply that, $[-x] = -[x]$.

$\oplus$ If $x \notin \mathbb{Z}$, then $-x \notin \mathbb{Z}$ and $[-x] < -x$. From property 2) we have

$$[x] \leq x < [x] + 1 \Longleftrightarrow -[x] - 1 < -x \leq -[x] \Longleftrightarrow (-[x]-1) < -x \leq (-[x]-1) + 1.$$

This imply that the integer part of $-x$ is $-[x]-1$ for all $x \in \mathbb{R} - \mathbb{Z}$.

6). Firstly, suppose that n and m are two positive integers such that n divides m i.e, there exist q an integer number such that $\frac{m}{n} = q$. Since $\frac{m-1}{n} = \frac{m}{n} - \frac{1}{n}$, then, from property 3) we have $\left[\frac{m-1}{n}\right] = \left[q - \frac{1}{n}\right] = q + \left[-\frac{1}{n}\right] = q - 1$ because $0 < \frac{1}{n} \leq 1$. This imply that, $\left[\frac{m-1}{n}\right] = \left[\frac{m}{n}\right] - 1$, and so the desired result.

Secondly, suppose that n and m are two positive integers such that n does not divides m, i.e, there exist q and r integer numbers such that $m = qn + r$ where $0 < r < n$. then, from property 3) we have $\left[\frac{m-1}{n}\right] = \left[q + \frac{r-1}{n}\right] = q + \left[\frac{r-1}{n}\right] = q$ because $0 < \frac{r-1}{n} \leq 1$. And $\left[\frac{m}{n}\right] = \left[q + \frac{r}{n}\right] = q + \left[\frac{r}{n}\right] = q$ because $0 < \frac{r}{n} \leq 1$. This means that $\left[\frac{m-1}{n}\right] = \left[\frac{m}{n}\right]$ for all m and n positive integer numbers. which completes the proof.

7). From property 2) we have $\frac{[x]-1}{2} < \frac{[x]}{2} \leq \frac{x}{2} < \frac{[x]+1}{2}$.

$\otimes$ If $[x]$ is even integer number, then, $\frac{[x]}{2} \leq \frac{x}{2} < \frac{[x]+1}{2} \Longrightarrow \frac{[x]}{2} \leq \frac{x}{2} < \frac{[x]}{2} + \frac{1}{2} < \frac{[x]}{2} + 1$. By the definition of integer part we conclude that the integer part of $\frac{x}{2}$ is $\frac{[x]}{2}$. That is, $\left[\frac{x}{2}\right] = \frac{[x]}{2}$, if 2 divides $[x]$.

$\otimes$ If $[x]$ is odd integer number, then, $\frac{[x]-1}{2} \leq \frac{[x]}{2} \leq \frac{x}{2} < \frac{[x]-1+2}{2} = \frac{[x]-1}{2} + 1$. By the definition of integer part we conclude that the integer part of $\frac{x}{2}$ is $\frac{[x]-1}{2}$, if 2 does not divides $[x]$. which completes the proof.

8). From property 2) we have $[x] < [x] + 1 \leq x + 1 < [x] + 2$, then we have $\frac{[x]}{2} < \frac{[x]+1}{2} \leq \frac{x+1}{2} < \frac{[x]}{2} + 1$.

$\otimes$ If $[x]$ is even integer number, then, $\frac{[x]}{2} \leq \frac{x+1}{2} < \frac{[x]+1}{2}$. By the definition of integer part we conclude that the integer part of $\frac{x+1}{2}$ is $\frac{[x]}{2}$. That is, $\left[\frac{x+1}{2}\right] = \frac{[x]}{2}$, if 2 divides $[x]$.

$\otimes$ If $[x]$ is odd integer number, then, $\frac{[x]+1}{2} \leq \frac{x+1}{2} < \frac{[x]-1+1}{2} + 1 = \frac{[x]+1}{2} + \frac{1}{2} < \frac{[x]+1}{2} + 1$. By the definition of integer part we conclude that the integer part of $\frac{x+1}{2}$ is $\frac{[x]+1}{2}$, if 2 does not divides $[x]$. which completes the proof. ■

Using the integer part function to solve equations or inequalities

1). Solve in $\mathbb{R}$ the equation $\frac{2}{3}[2x-1] = 4$.

<u>Answer:</u> First, we isolate the bracket of the integer part as fellows: $[2x-1] = 6$, then from the property 3)$\star$ we have $[2x] = 7$. Next, apply the definition of the integer part function, we arrive at $7 \leq 2x < 8$. Therefore, dividing by 2 we obtain $\frac{7}{2} \leq x < 4$. This means that the set of solutions for the proposed equation is $\left[\frac{7}{2}, 4\right[$.

2). Find all $x \in \mathbb{R}$ such that $[3x+2] = 2x+3$.

<u>Answer:</u> We have,

$$[3x+2] = 2x+3 \Longleftrightarrow [3x] = 2x+1.$$

This imply that $2x+1 \in \mathbb{Z}$ and $2x+1 \leq 3x < 2x+2$. So, we can obtain $x = \frac{k-1}{2}$ where $k \in \mathbb{Z}$ and $1 \leq x < 2$. These imply that $3 \leq k < 5$. For $k = 3$ we have $x = 1$ and for $k = 4$ we have $x = \frac{3}{2}$. we conclude that the set of solution of our equation is $S = \left\{1, \frac{3}{2}\right\}$.

3). Solve $[3x+1] = [-x+4]$, where x is a real unknown.

Answer: We have

$$\begin{aligned}
[3x+1] = [-x+4] &\iff [3x]+1 = [-x]+4, \ \textit{by the property } 3)\star \\
&\iff [3x] = [-x]+3 \\
&\iff [-x]+3 \le 3x < [-x]+4, \ \textit{applying the property } 2)\star \ \textit{to } [3x] \\
&\implies -x+2 < 3x < -x+4, \ \textit{applying the property } 2)\star \ \textit{to } [-x] \\
&\iff 2 < 4x < 4 \\
&\iff \tfrac{1}{2} < x < 1.
\end{aligned}$$

Therefore, the set of the solutions to our equation is the subset S of $]\,\frac{1}{2}, 1\,[$.

Case 1: If $x \in \,]\,\frac{1}{2}, \frac{2}{3}\,[$ then $\frac{3}{2} < 3x < 2$ and $-\frac{2}{3} < -x < -\frac{1}{2}$, so $[3x] = 1$ and $[-x] = -1$. Thus, $[3x] \neq [-x]+3$, and consequently $[3x+1] \neq [-x+4]$.

Case 2: If $x \in \left[\frac{2}{3}, 1\,\right[$ then $2 \le 3x < 3$ and $-1 < -x \le -\frac{2}{3}$, so $[3x] = 2$ and $[-x] = -1$. Thus, $[3x] = [-x]+3$, and consequently $[3x+1] = [-x+4]$.

This gives us the set of solutions to our equation which is $S = \left[\frac{2}{3}, 1\,\right[$.

4). Solve the inequality $[x]^2 - [x] - 2 \le 0$.

Answer: We denote $[x] = k$, where $k \in \mathbb{Z}$. Our inequality becomes: $k^2 - k - 2 \le 0$.

Next, we can factor the quadratic polynomial inequality above into $(k+1)(k-2) \le 0$, this imply that: $-1 \le k \le 2$. Since $k \in \mathbb{Z}$, the possible values for k are: $k \in \{-1, 0, 1, 2\}$.

Convert back to x. We now solve the union of the conditions for k: If $[x] = -1$, then $-1 \le x < 0$. If $[x] = 0$, then $0 \le x < 1$. If $[x] = 1$, then $1 \le x < 2$. If $[x] = 2$, then $2 \le x < 3$. Combining these intervals: we obtain the solution set $S = [-1\,, 3\,[$.

5). Solve the inequality $-2[3x+5]^2 + [3x] + 6 \ge 0$.

Answer: We have,

$$-2[3x+5]^2 + [3x] + 6 \ge 0 \iff -2[3x+5]^2 + [3x+5] + 1 \ge 0.$$

We denote $[3x+5] = p$, where $p \in \mathbb{Z}$. Our inequality becomes: $-2p^2 + p + 1 \ge 0$.

Next, we can factor the quadratic polynomial inequality above into $-2\,(p-1)\left(p+\frac{1}{2}\right) \ge 0$, this imply that: $-\frac{1}{2} \le p \le 1$. Since $p \in \mathbb{Z}$, the possible values for p are: $p \in \{0, 1\}$.

Convert back to x. We now solve $[3x+5] = p$ where $p = 0$ or $p = 1$: If $[3x+5] = 1$, then $-\frac{4}{3} \le x < -1$. And if $[3x+5] = 0$, then $-\frac{5}{3} \le x < -\frac{4}{3}$. Combining these intervals: we obtain the solution set $S = \left[-\frac{5}{3}\,, -1\,\right[$.

Chapter 2

Numerical Sequences

2.1 Concepts of Numerical Sequences

2.1.1 Fundamental Concepts of Numerical Sequences

Definition 2.1.1 :

A sequence of real numbers is a real-valued function whose domain is the subset of natural numbers $\mathbb{N}$ or an infinite subset $N_1 \subset \mathbb{N}$ to the real numbers i.e:

$$\begin{array}{rccc} u: & N \subseteq \mathbb{N} & \longrightarrow & \mathbb{R} \\ & n & \longmapsto & u(n) \text{ or } u_n \end{array}.$$

Notations: The sequence u is denoted by $(u_n)_{n\in N\subseteq\mathbb{N}}$.

$u(n)$ or u_n is called the general term or n^{th} term of the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$.

• We can defined the sequence by:

1. Explicit definition

Definition 2.1.2 :

A sequence $(u_{n \in N \subseteq \mathbb{N}})$ *is said to be explicitly defined if we know the function* f *such that:*

$$u_n = f(n); \; n \in N.$$

Example 2.1.3 :

Let $(u_n)_{n\in\mathbb{N}}$ *the sequence defined by :* $u_n = \frac{n}{\sqrt{n+1}}$, $\forall n \in \mathbb{N}$. *Is defined explicitly by the function* f *where* $f(x) = \frac{x}{\sqrt{x+1}}$ *with* $x \in \mathbb{R}_+$.

2. Property definition

Definition 2.1.4 :

We can define a sequence by giving the properties or characterization of the sequence's terms .

Example 2.1.5 :

1) Let $(u_n)_{n\in\mathbb{N}}$ *the sequence of strictly negative even numbers.*

2) Let $(u_n)_{n\in\mathbb{N}}$ *the sequence defined as* $u_n = n$ *if* n *is a multiple of* 3 *and* $u_n = n - 1$ *otherwise.*

3. By induction (recurrence) definition

Definition 2.1.6 :

A sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is defined recursively if there exists a function* f *such that:* $n \in N$; $u_{n+1} = f(u_n)$ *and by defining an initial value.*

Example 2.1.7 :

Let $(u_n)_{n\in\mathbb{N}}$ *the sequence defined by :* $u_{n+1} = (u_n)^2 - 3u_n$, $\forall n \in \mathbb{N}$ *and* $u_0 = 2$. *Is defined recursively as* $u_{n+1} = f(u_n)$, $\forall n \in \mathbb{N}$ *where* $f(x) = x^2 - 3x$ *with* $x \in \mathbb{R}$.

2.1.2 Bounded sequences

Definition 2.1.8 :

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ *be a real sequence.*

$\otimes$ *A sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is bounded from above if and only if:*

$$\exists M \in \mathbb{R}, \forall n \in N; \; u_n \leq M.$$

⊗ *A sequence* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *is bounded from below if and only if:*

$$\exists m \in \mathbb{R}, \forall n \in N;\ u_n \geq m.$$

⊗ *A sequence* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *is bounded if and only if: it is bounded from above and bounded from below which means :*

$$\exists C \in \mathbb{R}_+, \forall n \in N;\ |u_n| \leq C.$$

Example 2.1.9 *:*
Let $(u_n)_{n\in\mathbb{N}}$ *the sequence defined by :* $u_n = \frac{3n-1}{2n+1}, \forall n \in \mathbb{N}$. *Then the sequence* (u_n) *is bounded from below by* -1 *and from above by* $\frac{3}{2}$. *Because for all* $n \in \mathbb{N}$, *we have* $u_n = \frac{3n-1}{2n+1} = \frac{3}{2}(1 - \frac{5}{6n+3})$ *and for all* $n \in \mathbb{N}, 6n+3 \geq 3 \Longrightarrow -1 \leq u_n \leq \frac{3}{2}$.

Proposition 2.1.10 *The sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is bounded if and only if the sequence* $(|u_n|)_{n\in N\subseteq\mathbb{N}}$ *is bounded above, that is, for all* $n \in N \subseteq \mathbb{N}$, *there exist* $M \in \mathbb{R}_+$ *such that* $|u_n| \leq M$.

Example 2.1.11 *:*
Let the sequence $(u_n)_{n\in\mathbb{N}}$ *defined by :* $u_n = \frac{1+sin(3n)}{3}, \forall n \in \mathbb{N}$ *is bounded. Because for all* $n \in \mathbb{N}$, *we have* $|sin(3n)| \leq 1$, *this imply that, for all* $n \in \mathbb{N}$, $|u_n| \leq \frac{2}{3}$.

2.1.3 Operations on sequences

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ and $(v_n)_{n\in N\subseteq\mathbb{N}}$ be two numerical sequences and $\lambda \in \mathbb{R}$.

Sum of sequences

The sum of $(u_n)_{n\in N\subseteq\mathbb{N}}$ and $(v_n)_{n\in N\subseteq\mathbb{N}}$ is the numerical sequence whose general term is $u_n + v_n$ i.e $(u_n)_{n\in N\subseteq\mathbb{N}} + (v_n)_{n\in N\subseteq\mathbb{N}} = (u_n + v_n)_{n\in N\subseteq\mathbb{N}}$.

Product of sequences by a scalar

The product of $(u_n)_{n\in N\subseteq\mathbb{N}}$ by λ is the numerical sequence whose general term is $(\lambda u)_n$, that is, $\lambda(u_n)_{n\in N\subseteq\mathbb{N}} = (\lambda u_n)_{n\in N\subseteq\mathbb{N}}$.

Product of sequences

The product of $(u_n)_{n\in N\subseteq\mathbb{N}}$ and $(v_n)_{n\in N\subseteq\mathbb{N}}$ is the numerecal sequence whose general term is $u_n.v_n$ that is, $(u_n)_{n\in N\subseteq\mathbb{N}}.(v_n)_{n\in N\subseteq\mathbb{N}} = (u_n.v_n)_{n\in N\subseteq\mathbb{N}}$.

Inverse of a sequence

If $u_n \neq 0$, for all $n \in N \subseteq \mathbb{N}$, then the inverse of the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is the numerical sequence whose general term is $\frac{1}{u_n}$. i.e., $\frac{1}{(u_n)_{n\in N\subseteq\mathbb{N}}} = (\frac{1}{u_n})_{n\in N\subseteq\mathbb{N}}$.

Quotient of Sequences

If $v_n \neq 0$ for all $n \in N \subseteq \mathbb{N}$, then the division of $(u_n)_{n\in N\subseteq\mathbb{N}}$ by $(v_n)_{n\in N\subseteq\mathbb{N}}$ is the numerical sequence whose general term is $\frac{u_n}{v_n}$ that is, $\frac{(u_n)_{n\in N\subseteq\mathbb{N}}}{(v_n)_{n\in N\subseteq\mathbb{N}}} = \left(\frac{u_n}{v_n}\right)_{n\in N\ \subseteq\mathbb{N}}$.

2.1.4 Increasing and decreasing sequences

Definition 2.1.12 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a numerical sequence.

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is an increasing sequence iff:* $\forall n \in N \subseteq \mathbb{N};\ u_{n+1} \geq u_n$.

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is a strictly increasing sequence iff:* $\forall n \in N \subseteq \mathbb{N};\ u_{n+1} > u_n$.

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is a decreasing sequence iff:* $\forall n \in N \subseteq \mathbb{N};\ u_{n+1} \leq u_n$.

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is a strictly decreasing sequence iff:* $\forall n \in N \subseteq \mathbb{N};\ u_{n+1} < u_n$.

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is monotonic if it is increasing or decreasing.*

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is strictly monotonic if it is strictly increasing or strictly decreasing.*

$*$ $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is a constant sequence iff:* $\forall n \in N \subseteq \mathbb{N};\ u_{n+1} = u_n$.

Proposition 2.1.13 *Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ a numerical sequence given by a regressive expression:*

$u_{n+1} = f(u_n)\,,\ \forall n \in N \subseteq \mathbb{N}$.

If f is increasing, then $(u_n)_{n\in N\subseteq\mathbb{N}}$ is monotonic.

Example 2.1.14 *:*

Let the numerical sequence $(u_n)_{n\in\mathbb{N}}$ defined by : $u_{n+1} = f(u_n)\,,\ \forall n \in \mathbb{N}$ and $u_0 = 1$ where $f(x) = 2x - 3$. We have the function f is increasing. The sequence $(u_n)_{n\in\mathbb{N}}$ is strictly decreasing in $\mathbb{N}$, because $u_{n+1} - u_n = u_n - 3 < 0$, $\forall n \in \mathbb{N}$, and we can prove by recurrence that $u_n \leq 1$, $\forall n \in \mathbb{N}$.

2.2 Convergent sequences

Finite and infinite limit of a numerical sequence

Definition 2.2.1 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a real sequence.

We say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ tends to l as n tends to infinity if and only if:

$$\forall\varepsilon\ >0,\exists n_0\in\mathbb{N};\ \forall n\in N;\ n\geq n_0\Longrightarrow|u_n-l|\leq\varepsilon.$$

*In this case, we say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ **convergent** and converges to l and we note $\lim\limits_{n\to+\infty}u_n=l$.*

Example 2.2.2 *:*

We show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n=\frac{n-1}{2n+1}$ converges to $\frac{1}{2}$. For this, we use the definition of convergence.

Let $\varepsilon>0$ we have:

$$\begin{aligned}\left|u_n-\tfrac{1}{2}\right|<\varepsilon &\Leftrightarrow \left|\tfrac{n-1}{2n+1}-\tfrac{1}{2}\right|<\varepsilon\\ &\Leftrightarrow \left|\tfrac{-3}{4n+2}\right|<\varepsilon\\ &\Leftrightarrow \tfrac{3}{4n+2}<\varepsilon\\ &\Leftrightarrow 4n+2<\tfrac{3}{\varepsilon}\\ &\Leftrightarrow n<\tfrac{3}{4\varepsilon}-\tfrac{1}{2}.\end{aligned}$$

By setting $n_0=E\left(\frac{3}{4\varepsilon}-\frac{1}{2}\right)+1$, we obtain:

$$\forall\varepsilon>0,\exists n_0=E\left(\frac{3}{4\varepsilon}-\frac{1}{2}\right)+1\in\mathbb{N},\forall n\in\mathbb{N};\ n\geq n_0\Longrightarrow\left|u_n-\frac{1}{2}\right|\leq\varepsilon.$$

This imply that $\lim\limits_{n\to+\infty}u_n=\dfrac{1}{2}$.

Definition 2.2.3 *:*

1. *We say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ tends to $+\infty$ as n tends to infinity if and only if:*

$$\forall A\ >0,\exists n_0\in\mathbb{N},\ \forall n\ \in\mathbb{N};\ n\ \geq\ n0\ \Longrightarrow\ u_n\ \geq\ A.$$

*In this case, we say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is **divergent** and diverges to $+\infty$ and we note $\lim\limits_{n\to+\infty}u_n=+\infty$*

2. *We say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ tends to $-\infty$ as n tends to infinity if and only if:*

$$\forall A\ >0,\exists n_0\in\mathbb{N},\ \forall n\ \in\mathbb{N};\ n\ \geq\ n0\ \Longrightarrow\ u_n\ \leq-\ A.$$

*In this case, we say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is **divergent** and diverges to $-\infty$ and we note $\lim\limits_{n\to+\infty}u_n=-\infty$*

Example 2.2.4 *:*
Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ *and* $(v_n)_{n\in N\subseteq \mathbb{N}}$ *be two sequences defined by:* $u_n = \frac{n^2-1}{2n}$ *and* $v_n = \frac{-n^3-1}{n+1} + 3$ *respectively.*

We show that $\lim\limits_{n\to+\infty} u_n = +\infty$ *and* $\lim\limits_{n\to+\infty} v_n = -\infty$.

1. *Let* $A > 0$ *we have:*

$$u_n \geq A \Leftrightarrow \frac{n^2-1}{2n} \geq A \Leftrightarrow \frac{n}{2} - \frac{1}{2n} \geq A \Leftrightarrow \frac{n}{2} > \frac{n}{2} - \frac{1}{2n} \geq A \Longrightarrow \frac{n}{2} > A \Longrightarrow n > 2A.$$

Let's put $n_0 = E(2A) + 1 > 2A$ *this imply that*

$$\forall A > 0, \exists n_0 = E(2A) + 1 \in \mathbb{N},\ \forall n \in \mathbb{N};\ n \geq n_0 \Longrightarrow u_n \geq A.$$

2. *Let* $A > 0$ *we have:*

$$\begin{aligned} v_n \leq -A \Leftrightarrow \tfrac{-n^3-1}{n+1} + 3 \leq -A \quad \Leftrightarrow -\tfrac{(n+1)(n^2-n+1)}{n+1} \leq -A - 3 \quad &\Leftrightarrow n^2 - n + 1 \geq A + 3 \\ &\Longrightarrow n^2 + 1 \geq n^2 - n + 1 \geq A + 3 \\ &\Longrightarrow n^2 \geq A + 2 \Longrightarrow n \geq \sqrt{A+2}. \end{aligned}$$

Let's put $n_0 = E\left(\sqrt{A+2}\right) + 1$ *we obtain*

$$\forall A > 0, \exists n_0 = E\left(\sqrt{A+2}\right) + 1 \in \mathbb{N},\ \forall n \in \mathbb{N};\ n \geq n_0 \Longrightarrow v_n \leq -A.$$

Divergent sequences

Definition 2.2.5 *:*
Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ *be a sequence of real numbers. We say that the sequence* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *is divergent if it is not convergent, i.e* $\forall l \in \mathbb{R},\ \exists \varepsilon > 0,\ \forall n_0 \in \mathbb{N};\ \exists n \in N;\ n \geq n_0 \wedge |u_n - l| > \varepsilon$.

Remark 2.2.6 *Here are two types of divergence*
1* *Divergence of infinite type: in this case the sequence converges to* $+\infty$ *or* $-\infty$*. For example the sequence with general term* $u_n = -2n^2 + 3$ *is converges to* $-\infty$.
2* *Divergence of type limit does not exist: in this case the sequence does not have a finite or infinite limit.*

Example 2.2.7 *:*
The sequence with general term $u_n = (-1)^n + \frac{1}{n+1}$ *is divergent. Because* $\lim\limits_{n\to+\infty} u_{2n} = 1$ *and* $\lim\limits_{n\to+\infty} u_{2n+1} = -1$*. That is the limit of this sequence does not exist.*
Or *We will show that the sequence* $(u_n)_{n\in\mathbb{N}}$ *does not have a finite or infinite limit.*

1). By contradiction, suppose that: $\lim_{n\to+\infty} u_n = l,\ l \in \mathbb{R}$. *According to the convergence definition with* $\varepsilon = \frac{1}{4}$ *we get:* $\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}$ *such that:*

$$\begin{aligned} n \geq n_0 &\Longrightarrow u_n \in \left[l - \tfrac{1}{4}, l + \tfrac{1}{4}\right] \\ &\Longrightarrow (-1)^n + \tfrac{1}{n+1} \in \left[l - \tfrac{1}{4}, l + \tfrac{1}{4}\right] \\ &\Longrightarrow l - \tfrac{1}{4} \leq -1 + \tfrac{1}{2n+2} \leq l + \tfrac{1}{4} \text{ and } l - \tfrac{1}{4} \leq 1 + \tfrac{1}{2n+1} \leq l + \tfrac{1}{4} \\ &\Longrightarrow -l - \tfrac{1}{4} \leq 1 - \tfrac{1}{2n+2} \leq -l + \tfrac{1}{4} \text{ and } l - \tfrac{1}{4} \leq 1 + \tfrac{1}{2n+1} \leq l + \tfrac{1}{4} \\ &\Longrightarrow -\tfrac{1}{2} \leq 2 + \tfrac{1}{(2n+2)(2n+1)} \leq \tfrac{1}{2} \\ &\Longrightarrow -\tfrac{5}{2} \leq \tfrac{1}{(2n+2)(2n+1)} \leq -\tfrac{3}{2}. \end{aligned}$$

It's a contradiction.

This means that the sequence $(u_n)_{n\in\mathbb{N}}$ *has no finite limit.*

2). By contradiction, suppose that: $\lim_{n\to+\infty} u_n = +\infty$.

According to the limit definition with A = 4 we have: $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}$ *such that:*

$$\begin{aligned} n \geq n0 &\Longrightarrow u_n \geq 4 \\ &\Longrightarrow (-1)^n + \tfrac{1}{n+1} \geq 4 \\ &\Longrightarrow \tfrac{1}{n+1} \geq 4 - (-1)^n \geq 3 \\ &\Longrightarrow n \leq -\tfrac{2}{3}. \end{aligned}$$

It's a contradiction.

3). We use the same method for the case: $\lim_{n\to+\infty} u_n = -\infty$ *.*

According to the limit definition with A = 4 we have: $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}$ *such that:*

$$\begin{aligned} n \geq n_0 &\Longrightarrow u_n \leq -4 \\ &\Longrightarrow -1 + \tfrac{1}{2n+2} \leq -4 \text{ and } 1 + \tfrac{1}{2n+1} \leq -4 \\ &\Longrightarrow \tfrac{1}{2n+2} \leq -3 \text{ and } \tfrac{1}{2n+1} \leq -5. \end{aligned}$$

It's a contradiction.

From 2) and 3) we imply that the sequence $(u_n)_{n\in\mathbb{N}}$ *does not have an infinite limit.*

Proposition 2.2.8 *If a sequence of real numbers* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *has a limit, then this limit is unique.*

Proof : By contradiction. Suppose that: $\lim_{n\to+\infty} u_n = l,\ l \in \mathbb{R}$ and $\lim_{n\to+\infty} u_n = m,\ m \in \mathbb{R}$, where $l \neq m$. Taking $\varepsilon = \frac{l-m}{4}$ which implies :

$\exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N};\ n \geq n_0 \Longrightarrow |u_n - l| \leq \varepsilon$, and $\exists n_1 \in \mathbb{N},\ \forall n \in \mathbb{N};\ n \geq n_1 \Longrightarrow |u_n - m| \leq \varepsilon$.

Putting $n_2 = max(n_0, n_1)$, and this imply that :

$$\forall n \in \mathbb{N};\ n \geq n_2 \Longrightarrow |u_n - l| + |u_n - m| \leq 2\varepsilon.$$

With $n \geq n_2$ we have $|l-m| = |l-u_n+u_n-m| \leq |u_n-l|+|u_n-m| \leq 2\varepsilon \Longrightarrow \varepsilon \leq \frac{\varepsilon}{2}$. it's a contradiction. ∎

Proposition 2.2.9 *If $(u_n)_{n\in N\subseteq\mathbb{N}}$ is a convergent sequence, then is a bounded sequence.*

Proof : We'll show the following implication:

$$(u_n)_{n\in N\subseteq\mathbb{N}} \text{ is a convergent sequence } \Longrightarrow (u_n)_{n\in N\subseteq\mathbb{N}} \text{ is bounded.}$$

Suppose that $(u_n)_{n\in N\subseteq\mathbb{N}}$ is convergent, then for $\varepsilon = \frac{1}{2}$ we have:

$$\begin{aligned}\exists n_1 \in \mathbb{N}, \forall n \in \mathbb{N};\ n \geq n_1 \ &\Longrightarrow |u_n - l| \leq \varepsilon = \tfrac{1}{2}\\ &\Longrightarrow m = l - \tfrac{1}{2} \leq u_n \leq l + \tfrac{1}{2} = M, \forall n \geq n_1 \in \mathbb{N}.\end{aligned}$$

So the set $\{u_{n_1},\ u_{n_1+1},\ \ldots\}$ is bounded. On the other hand the set $\{u_{n_0},\ u_{n_0+1},\ \ldots,\ u_{n_1-1}\}$ (where n_0 is the first number of N) is bounded. (Because it contains a finite number of terms). Then the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is bounded. ∎

Remark 2.2.10 *In general, the converse is false. Indeed, there exist bounded sequences that don't converge.*

Example 2.2.11 *:*

The sequence defined by general term $w_n = sin\left(\frac{n\pi}{4}\right)$, $n \in \mathbb{N}$ is bounded, because $\left|sin\left(\frac{n\pi}{4}\right)\right| \leq 1, \forall n \in \mathbb{N}$, but it is not convergent because the two sub-sequences (w_{8n+1}) and (w_{8n+5}) tends to $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$ respectively. This imply that (w_n) has no limit.

limit's operations

Theorem 2.2.12 *Let l_1, l_2 and λ are real numbers, $(u_n)_{n\in N\subseteq\mathbb{N}}$ and $(v_n)_{n\in N\subseteq\mathbb{N}}$ two convergent sequences with: $\lim\limits_{n\to+\infty} u_n = l_1$ and $\lim\limits_{n\to+\infty} v_n = l_2$, then:*

1. $\lim\limits_{n\to+\infty} \lambda u_n = \lambda l_1$ with $\lambda \in \mathbb{R}$.

2. $\lim\limits_{n\to+\infty} (u_n + v_n) = l_1 + l_2$.

3. $\lim\limits_{n\to+\infty} u_n.v_n = l_1.l_2$.

4. If $u_n \neq 0$ for all $n \in N$ and $l_1 \neq 0$ then $\lim\limits_{n\to+\infty} \dfrac{1}{u_n} = \dfrac{1}{l_1}$.

5. If $v_n \neq 0$ for all $n \in N$ and $l_2 \neq 0$ then $\lim\limits_{n\to+\infty} \dfrac{u_n}{v_n} = \dfrac{l_1}{l_2}$.

Infinite limit's operations

In the case where l_1 and l_2 are infinite, the above theorem remains true if the results of the operations are not indeterminate.

2.3 Comparison Theorems

Theorem 2.3.1 ***Limits and inequalities***

Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ *and* $(v_n)_{n\in N\subseteq \mathbb{N}}$ *be two convergent sequences, then:*

a). *If, there exist* $n_0 \in N$ *such that* $\forall n \geq n_0$; $u_n \leq v_n$ *this implies* $\lim\limits_{n\to+\infty} u_n \leq \lim\limits_{n\to+\infty} v_n$.

b). *If, we have* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *and* $(v_n)_{n\in N\subseteq \mathbb{N}}$ *be two sequences, satisfies :* $\exists n_0 \in N$, $\forall n \geq n_0$; $u_n \leq v_n$ *and* $\lim\limits_{n\to+\infty} u_n = +\infty$ *this implies* $\lim\limits_{n\to+\infty} v_n = +\infty$.

c). *If, we have* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *and* $(v_n)_{n\in N\subseteq \mathbb{N}}$ *be two sequences, satisfies :* $\exists n_0 \in N$, $\forall n \geq n_0; u_n \leq v_n$ *and* $\lim\limits_{n\to+\infty} v_n = -\infty$ *this implies* $\lim\limits_{n\to+\infty} u_n = -\infty$.

d). Squeeze Theorem *(gendarme theorem): If* $(u_n)_{n\in N\subseteq \mathbb{N}}$, $(v_n)_{n\in N\subseteq \mathbb{N}}$ *and* $(w_n)_{n\in N\subseteq \mathbb{N}}$ *three sequences satisfying:* $\exists n_0 \in N, \forall n \geq n_0; u_n \leq v_n \leq w_n$ *and* $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty} w_n = l$ *then* $\lim\limits_{n\to+\infty} v_n = l$.

2.3.1 Monotone Convergence Theorem

Theorem 2.3.2 ***Convergence of monotonic sequences***

$\star$ *If a sequence of real numbers* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *is increasing and bounded from above, then it converges to* $\sup\limits_{n\in N} u_n$.

$\star$ *If a sequence of real numbers* $(u_n)_{n\in N\subseteq \mathbb{N}}$ *is decreasing and bounded from below, then it converges to* $\inf\limits_{n\in N} u_n$.

Examples 2.3.3 *:*

Let $(u_n)_{n\in \mathbb{N}}$ *be a numerical sequence defined by:* $u_0 = 1$ *and* $4u_{n+1} = {u_n}^2 + 4$.

a. *We prove that:* $\forall n \in \mathbb{N}; u_n \leq 2$.

b. *We deduce that the sequence* $(u_n)_{n\in \mathbb{N}}$ *is convergent.*

Answer: a. *Using proof by induction: we denote* $P_n : u_n \leq 2$.

For $n = 0$, *we have* $u_0 = 1 \leq 2$, *so* P_0 *is true.*

Let's assume that the proposition P_n *is true for an* $n \in \mathbb{N}$ *and we'll show that* $u_{n+1} \leq 2$.

According to the assumption we have: $u_n \leq 2 \Longrightarrow {u_n}^2 + 4 \leq 8 \Longrightarrow u_{n+1} \leq 2$. *So, assertion the proposition* P_n *is true for all* $n \in \mathbb{N}$.

b. *We have* $\forall n \in \mathbb{N}; u_{n+1} - u_n = \frac{{u_n}^2+4}{4} - u_n = \left(\frac{u_n - 2}{2}\right)^2 \geq 0$. *Since* $(u_n)_{n\in \mathbb{N}}$ *is increasing in* $\mathbb{N}$ *and bounded from above by* 2 *this imply that the sequence* $(u_n)_{n\in \mathbb{N}}$ *is convergent.*

2.4 Adjacent sequences

Definition 2.4.1 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ *and* $(v_n)_{n\in N'\subseteq\mathbb{N}}$ *be two real sequences.*
We say that (u_n) *and* (v_n) *are* ***adjacent*** *iff:* (u_n) *is increasing in* N *and* (v_n) *is decreasing in* N' *and* $\lim\limits_{n\to+\infty} u_n - v_n = 0$.

Example 2.4.2 *:*

Let $(u_n)_{n\in\mathbb{N}^*}$ *and* $(v_n)_{n\in\mathbb{N}^*}$ *be two real sequences defined by:* $u_n = \sum\limits_{k=1}^{n}\dfrac{1}{k!}$ *and* $v_n = u_n + \frac{1}{n!}$*. This sequences are adjacent, because :*

1. $u_{n+1} - u_n = \sum\limits_{k=1}^{n+1}\dfrac{1}{k!} - \sum\limits_{k=1}^{n}\dfrac{1}{k!} = \dfrac{1}{(n+1)!} \geq 0,\ \forall n\in\mathbb{N}^*$*, this imply that* $(u_n)_{n\in\mathbb{N}^*}$ *is increasing in* $\mathbb{N}^*$.

2. $v_{n+1} - v_n = u_{n+1} - u_n + \frac{1}{(n+1)!} - \frac{1}{n!} = \frac{2}{(n+1)n!} - \frac{1}{n!} \leq 0,\ \forall n\in\mathbb{N}^*$*, because: we have*
$\frac{2}{(n+1)} \leq 1,\ \forall n\in\mathbb{N}^* \Longrightarrow \frac{2}{(n+1)n!} \leq \frac{1}{n!},\ \forall n\in\mathbb{N}^*$*. So* $(v_n)_{n\in\mathbb{N}^*}$ *is decreasing in* $\mathbb{N}^*$.

3. $\lim\limits_{n\to+\infty} u_n - v_n = \lim\limits_{n\to+\infty}\dfrac{1}{n!} = 0$.

Theorem 2.4.3 *If the sequences* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *and* $(v_n)_{n\in N'\subseteq\mathbb{N}}$ *are adjacent then they converge to the same limit.*

Training exercise: $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ be two real sequences defined by: $u_n = \sum\limits_{k=1}^{n}\dfrac{1}{k(k+1)}$ and $v_n = u_n + \frac{1}{n}$.
Show that the sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

Cauchy sequence

Definition 2.4.4 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ *be a numerical sequence.The sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is said to be a Cauchy sequence in* $\mathbb{R}$ *if and only if :* $\forall\varepsilon>0, \exists n_0\in N, \forall p,q\in\ \mathbb{N};\ p,q\geq\ n_0 \Longrightarrow |u_p - u_q| \leq \varepsilon$.

Example 2.4.5 *:*

The sequence $(u_n)_{n\in\mathbb{N}^*}$ *defined by* $u_n = \sum_{k=1}^{n}\frac{1}{k^2}$ *is a Cauchy sequence. Because: Let* $p,q\in\mathbb{N}^*$ *with* $p\leq\ q$ *then we have:*

$$\begin{cases} u_p &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{p^2} \\ u_q &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{p^2} + \frac{1}{(p+1)^2} + \cdots + \frac{1}{q^2} \end{cases}.$$

And

$$\left|u_p - u_q\right| = \frac{1}{(p+1)^2} + \cdots + \frac{1}{q^2}. \tag{2.1}$$

As

$$\begin{cases} \frac{1}{(p+1)^2} & \leq \frac{1}{p(p+1)} = \frac{1}{p} - \frac{1}{p+1} \\ \frac{1}{(p+2)^2} & \leq \frac{1}{(p+1)(p+2)} = \frac{1}{p+1} - \frac{1}{p+2} \\ \vdots & \\ \frac{1}{q^2} & \leq \frac{1}{(q-1)q} = \frac{1}{q-1} - \frac{1}{q} \end{cases}. \tag{2.2}$$

Then from (2.1) *and* (2.2) *we arrive at:*

$$\left|u_p - u_q\right| = \left|\frac{1}{p} - \frac{1}{q}\right| \leq \frac{1}{p} + \frac{1}{q} \leq \frac{2}{p}.$$

Let $\varepsilon > 0$, *we can choose* $n_0 = E\left(\frac{2}{\varepsilon}\right) + 1$. *So,*

$$\forall \varepsilon > 0, \exists n_0 = E\left(\frac{2}{\varepsilon}\right) + 1 \in \mathbb{N}^*, \forall p, q \in \mathbb{N}^*;\ p, q \geq n_0 \Longrightarrow \left|u_p - u_q\right| \leq \varepsilon.$$

Remark 2.4.6 *If* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is Cauchy sequence, then* $\lim\limits_{n\to+\infty} u_{n+1} - u_n = 0$. *The converse, however, does not hold.*

Example 2.4.7 *If* $u_n = \sqrt{2n}$, *then* $\lim\limits_{n\to+\infty} (u_{n+1} - u_n) = \lim\limits_{n\to+\infty} \dfrac{\sqrt{2}}{\sqrt{n+1} + \sqrt{n}} = 0$, *but* (u_n) *is not Cauchy. Indeed, for* $\varepsilon = 1$, *and any* $n_1 \in \mathbb{N}$, *we have* $\left|u_{4n_1} - u_{n_1}\right| = 2\sqrt{2n_1} - \sqrt{2n_1} = \sqrt{2n_1} \geq 1$.

2.4.1 Limit inferior and limit superior

Definition 2.4.8 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ *be a sequence of real numbers. Denoting by* $S = \{u_n;\ n \in N \subseteq \mathbb{N}\}$ *i.e the set of cluster points of the sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$. *We define the limit superior (resp. inferior) of* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *as* $\limsup u_n = \sup S$ *(resp.* $\liminf u_n = \inf S$*).*

Theorem 2.4.9 *If* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is a Cauchy sequence, then it is bounded.*

Proof : Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be Cauchy sequence, taking $\varepsilon = 1$. Then there exists $N_1 \in N$ such that for all $p,\ q \geq N_1$, we have $\left|u_p - u_q\right| < 1$. Take $q = N_1$. Then for $n \geq N_1$, $\left|u_n - u_{N_1}\right| < 1$. It follows that $|u_n| \leq \left|u_n - u_{N_1}\right| + \left|u_{N_1}\right| < 1 + \left|u_{N_1}\right|$ for all $n \geq N_1$.
If we let $M = max\left\{|u_1|, |u_2|, \cdots, \left|u_{N_1} - 1\right|, 1 + \left|u_{N_1}\right|\right\}$, then $|u_n| \leq M$ for all $n \in N$. This means that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is a bounded. ■

Theorem 2.4.10 *Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ be a real sequence then: $(u_n)_{n\in N\subseteq \mathbb{N}}$ is a Cauchy sequence if and only if $(u_n)_{n\in N\subseteq \mathbb{N}}$ is convergent.*

Proof : $\star$ We show that if (u_n) is convergent then (u_n) is Cauchy sequence.
Suppose that $\lim\limits_{n\to+\infty} u_n = l$. Note that

$$|u_p - u_q| = |u_p - l + l - u_q| \leq |u_p - l| + |u_q - l|.$$

Thus, given any $\varepsilon > 0$ there is an $N_1 \in \mathbb{N}$ so that if $k > N_1$ then $|u_k - l| \leq \frac{\varepsilon}{2}$. Thus, if $p, q > N_1$ we have $|u_p - u_q| \leq |u_p - l| + |u_q - l| \leq \varepsilon$. This means, (u_n) is a Cauchy sequence.
$\star$ We show that if (u_n) is a Cauchy sequence then (u_n) is convergent.
Let (u_n) be a Cauchy sequence. By Theorem (2.4.9), (u_n) is bounded. Bolzano Weierstrass Theorem implies the existence of a converging sub-sequence (u_{n_k}). Let $u_{n_k} \to l$. So, we have a candidate for the limit of (u_n). We claim that $x_n \to x$. Let $\varepsilon > 0$.

$$\textit{Since } x_{n_k} \to x, \textit{ there exists } N_1 \in \mathbb{N} \textit{ such that } |x_{n_k} - x| < \frac{\varepsilon}{2}, \forall k \geq N_1. \tag{2.3}$$

For the same ε, since (u_n) is Cauchy, there exists $N_2 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2}, \textit{ for all } n, m \geq N_2. \tag{2.4}$$

Let $N = max\{N_1, N_2\}$. Fix an $k \geq N$, and let $m = n_k$. It follows from Equation (2.3) that $|x_m - x| \leq \frac{\varepsilon}{2}$, because $m = n_k \geq k \geq N \geq N1$ (Check this). On the other hand, from Equation (2.4) it follows that $|x_n - x_m| < \frac{\varepsilon}{2}$, whenever $n \geq N$ (Why $m \geq N2$). Everything is ready now For $n \geq N$, we have $|x_n - x| \leq |x_n - x_m| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This completes the proof. ∎

2.5 Extracted Sequences

Definition 2.5.1 *:*
Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ be a numerical sequence.
We say that the sequence $(v_n)_{n\in N\subseteq \mathbb{N}}$ is a sub-sequence of $(u_n)_{n\in N\subseteq \mathbb{N}}$ if there exists a strictly increasing function $\varphi : N \longrightarrow N$ such that $v_n = u_{\varphi(n)}$.

Examples 2.5.2 *:*
1. *Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ the sequence with general term $u_n = (-1)^n + \frac{1}{n+1}$, then the two sequences defined by $v_n = 1 + \frac{1}{2n+1}$ and $w_n = -1 + \frac{1}{2n+2}$ are sub-sequences of $(u_n)_{n\in N\subseteq \mathbb{N}}$ where $v_n = u_{2n}$ and $w_n = u_{2n+1}$.*
2. *Let $(u_n)_{n\in N\subseteq \mathbb{N}}$ the sequence with general term $u_n = sin\left(\frac{n\pi}{2}\right)$, then the sequence defined by $v_n = u_{2n+1} = sin\left(n\pi + \frac{\pi}{2}\right) = cos\,(n\pi) = (-1)^n$ is a sub-sequence of $(u_n)_{n\in N\subseteq \mathbb{N}}$.*

Proposition 2.5.3 *Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a sequence of real numbers:*

$\oplus$ *Every bounded sequence has a convergent sub-sequence.(Bolzano-Weierstrass Property).*

$\oplus$ *The sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ converges to l if and only if all sub-sequences of $(u_n)_{n\in N\subseteq\mathbb{N}}$ converge to the limit l.*

$\oplus$ *If the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ has a divergent sub-sequence, then $(u_n)_{n\in N\subseteq\mathbb{N}}$ is divergent.*

$\oplus$ *If the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ has two sub-sequences that converge to distinct limits, then $(u_n)_{n\in N\subseteq\mathbb{N}}$ is divergent.*

2.6 Special Sequences

2.6.1 Arithmetic sequences

Definition 2.6.1 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a numerical sequence and r a real number.
We say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is arithmetic sequence if and only if:

$$u_{n+1} = u_n + r; \quad \text{for all } n \in N.$$

The real number r is then called the common difference of the sequence.

Examples 2.6.2 *:*

$\triangleright$ *The sequence defined as: $\frac{1}{2}, \frac{7}{2}, \frac{13}{2}, \frac{19}{2}, \cdots, \frac{1}{2}+3n, \ldots$ is arithmetic sequence with a common difference* 3.

$\triangleright$ *The sequence defined in $\mathbb{N}$ by general term $u_n = 3-2n$ is arithmetic sequence with common difference* -2 .

$\triangleright$ *The sequence defined in $\mathbb{N}$ by general term $w_{n+1} = w_n + 5$ is arithmetic sequence with common difference* 5.

Theorem 2.6.3 *Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be an arithmetic sequence with common difference r.*

$\odot$ *If $r > 0$, then the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is increasing in N.*

$\odot$ *If $r < 0$, then the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is decreasing in N.*

$\odot$ *If $r = 0$, then the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ is constant in N.*

Expression in terms of n

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be an arithmetic sequence whose first term is u_{n_0} and common difference r.
Its general term is given by the following relation:

$$u_n = u_{n_0} + (n - n_0)\, r,\ \forall n \geq n_0.$$

Examples 2.6.4 *:*
$\oslash$ *The general term of the arithmetic sequence with a common difference* $r = 2$ *and* $u_0 = -3$ *is* $u_n = -3 + 2n;\ \forall n \in \mathbb{N}$. $\oslash$ *The general term of the arithmetic sequence with a common difference* $r = \frac{2}{3}$ *and* $v_1 = \frac{1}{3}$ *is*

$$v_n = \frac{1}{3} + \frac{2}{3}(n-1)\,;\ \forall n \in \mathbb{N}^*.$$

Sum of consecutive terms of an arithmetic sequence

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be an arithmetic sequence with common difference r. We denote by S_n the sum of the first n terms of the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$, that is:

$$\begin{aligned} S_n &= u_{n_0} + \cdots + u_{n_0+n-1} \\ &= (Number\ of\ terms) \times \tfrac{(\text{first term}+\text{last term}\ )}{2} \\ &= \tfrac{n}{2}\left(u_{n_0} + u_{n_0+n-1}\right). \end{aligned}$$

Examples 2.6.5 *:*
1$*$ *The sum of the first fifteen terms of the arithmetic sequence* $(u_n)_{n\in\mathbb{N}^*}$ *defined by its general term is* $u_n = 3 - 2n; \forall n \in \mathbb{N}^*$, *is*

$$\begin{aligned} S_{15} &= u_1 + dots + u_{15} \\ &= (15) \times \tfrac{(u_1+u_{15})}{2} \\ &= \tfrac{15}{2}(1 + (-27)) \\ &= 195. \end{aligned}$$

2$*$ *The sum of the twenty consecutive terms starting from the tenth term of an arithmetic sequence defined by its first term* $u_0 = 2$ *and its common difference* 3 *is*

$$\begin{aligned} \mathrm{S}_{20} &= u_9 + \cdots + u_{28} \\ &= (20) \times \tfrac{(u_9+u_{28})}{2} \\ &= \tfrac{20}{2}(29 + 86) \\ &= 1150. \end{aligned}$$

Because the general term of this sequence is $u_n = u_0 + nr = 2 + 3n;\ n \in \mathbb{N}$.

2.6.2 Geometric Sequences

Definition 2.6.6 *:*

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ *be a numerical and* q *a real number no Zero.*

We say that the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is geometric sequence if and only if:*

$$u_{n+1} = u_n \times r;\ \text{for all } n \in N.$$

The real number q *is called the common ratio of the sequence.*

Examples 2.6.7 *:*

1∘ The sequence defined in $\mathbb{N}$ *as:* $2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\cdots,2\left(\frac{1}{2}\right)^n,\ \ldots$ *is geometric sequence with a common ratio* $q = \frac{1}{2}$ *and the first term* $u_0 = 2$.

2∘ The sequence defined in $\mathbb{N}^*$ *by general term* $v_n = 3(-2)^{n-1}$ *is geometric sequence with common ratio* -2 *and the first term* $v_1 = 3$.

3∘ The sequence defined in $\mathbb{N}$ *as* $w_{n+1} = \frac{2}{3}w_n$ *and* $w_0 = 1$ *is geometric with common ratio* $\frac{2}{3}$.

Theorem 2.6.8 *Let* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *be a geometric sequence with common ratio* $q \neq 0$.

∗ *If* $q > 1$*, then the sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is increasing in* N.

∗ *If* $0 < q < 1$*, then the sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is decreasing in* N.

∗ *If* $q = 1$*, then the sequence* $(u_n)_{n\in N\subseteq\mathbb{N}}$ *is constant in* N.

Expression in terms of n

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a geometric sequence whose first term is u_{n_0} and common ratio $q \neq 0$.

Its general term is given by the following relation:

$$u_n = u_{n_0} \times q^{(n-\ n_0)},\ \forall n \geq n_0.$$

Examples 2.6.9 *:*

a. *The general term of the geometric sequence with a common ratio* $q = 3$ *and* u_0= 2 *is* $u_n = 2\ \times 3^n;\ \forall n \in \mathbb{N}$.

b. *The general term of the geometric sequence with a common ratio* $q = \frac{2}{3}$ *and* $v_1 = -3$ *is* $v_n = -3 \times \left(\frac{2}{3}\right)^{n-1};\ \forall n \in \mathbb{N}^*$.

Sum of consecutive terms of a geometric sequence

Let $(u_n)_{n\in N\subseteq\mathbb{N}}$ be a geometric sequence with common ratio $q \neq 1$.
We denote by S_n the sum of the first n terms of the sequence $(u_n)_{n\in N\subseteq\mathbb{N}}$, that is:

$$\begin{array}{rcl} S_n & = & u_{n_0} + \cdots + u_{n_0+n-1} \\ & = & (\textit{First term }) \times \frac{1-q^{(\textit{Number of terms})}}{1-q} \\ & = & u_{n_0} \times \frac{1-q^n}{1-q}. \end{array}$$

Examples 2.6.10 *:*

$\oslash$ *The sum of the first ten terms of the geometric sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by its general term $u_n = 3 \times 2^n; \forall n \in \mathbb{N}^*$, is*

$$\begin{array}{rcl} S_{10} & = & u_1 + \cdots + u_{10} \\ & = & u_1 \times \frac{1-q^{10}}{1-q} \\ & = & 6 \times \frac{1-2^{10}}{1-2} \\ & = & 1138 \end{array}.$$

$\oslash$ *The sum of the five consecutive terms starting from the forth term of a geometric sequence defined by its first term $v_0 = 3$ and its common ratio $\frac{3}{2}$ is*

$$\begin{array}{rcl} S_5 & = & v_3 + \cdots + u_7 \\ & = & v_3 \times \frac{1-q^5}{1-q} \\ & = & \frac{81}{8} \times \frac{1-\left(\frac{3}{2}\right)^5}{1-\frac{3}{2}} \\ & = & 133.5234375 \end{array}.$$

Because the general term of this sequence is $v_n = v_0 \times q^n = 3 \times \left(\frac{3}{2}\right)^n$; $n \in \mathbb{N}$.

Chapter 3

Real functions of on real variable

3.1 Numerical functions, definitions and properties

3.1.1 Notion of numerical functions

Definition 3.1.1 *Let E and F be two sets and f be a relation from the set E to the set F. We say that f is a function, if every element of E is associated with at most one element of F, and we write:*

$$\begin{array}{rccc} f: & E & \longrightarrow & F \\ & x & \longmapsto & y = f(x) \end{array}$$

Definition 3.1.2 *We say that f is a **real functions of one real variable** if and only if f is a function from the set $E \subseteq \mathbb{R}$ to the set $F \subseteq \mathbb{R}$, and we write:*

$$\begin{array}{rccc} f: & E \subseteq \mathbb{R} & \longrightarrow & F \subseteq \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

Examples 3.1.3 *:*

$$\begin{array}{rccc} u: & \mathbb{N} & \longrightarrow & \mathbb{R} \\ & n & \longmapsto & u(n) = \frac{(-1)^n}{n+1} + \sqrt{n}. \end{array}$$

u is a real function of a natural variable.

$$\begin{array}{rccc} g: & \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & g(x) = \frac{1}{x} sin(\frac{1}{x}) \end{array}$$

g is a real function of a real variable.

$$\begin{array}{rcl} h: \mathbb{C} & \longrightarrow & \mathbb{R} \\ z & \longmapsto & h(z) = \frac{z \times \overline{z}}{Re(z)-1} \end{array}$$

h is a real function of a complex variable.

Domain of definition of a function

Definition 3.1.4 :

Let f be a real-valued function of a real variable.

The domain of the function f, denoted by D_f, is the set of real numbers that have images by the function f: if $f : I \subseteq \mathbb{R} \longrightarrow I' \subseteq \mathbb{R}$, then, we have,

$$D_f = \left\{ x \in I \text{ such that } f(x) \in I' \right\}.$$

Examples 3.1.5 :

$$\begin{array}{rcl} v: \mathbb{N} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & v(n) = \frac{1}{\sqrt{n-2}} sin(\frac{1}{n-\pi}), \end{array}$$

$$\begin{aligned} D_v &= \left\{ n \in \mathbb{N} \text{ such that } \frac{1}{\sqrt{n-2}} sin(\frac{1}{n-\pi}) \in \mathbb{R} \right\} \\ &= \{ n \in \mathbb{N} \text{ such that } n > 2 \wedge n \neq \pi \} \\ &= \mathbb{N}^* - \{1;2\}. \end{aligned}$$

$$\begin{array}{rcl} w: \mathbb{Z} & \longrightarrow & \mathbb{R} \\ n & \longmapsto & w(n) = \frac{ln(-n)}{\sqrt{16-n^2}}, \end{array}$$

$$\begin{aligned} D_w &= \left\{ n \in \mathbb{Z} \text{ such that } \frac{ln(-n)}{\sqrt{16-n^2}} \in \mathbb{R} \right\} \\ &= \left\{ n \in \mathbb{Z} \text{ such that } 16 - n^2 > 0 \wedge -n > 0 \right\} \\ &= \{ -3, \ -2, \ -1 \}. \end{aligned}$$

$$\begin{array}{rcl} f: \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & f(x) = \frac{1}{\sqrt{x-2}} sin(\frac{1}{x-\pi}), \end{array}$$

$$\begin{aligned} D_f &= \left\{ x \in \mathbb{R} \text{ such that } \frac{1}{\sqrt{x-2}} sin(\frac{1}{x-\pi}) \in \mathbb{R} \right\} \\ &= \{ x \in \mathbb{R} \text{ such that } x > 2 \wedge x \neq \pi \} \\ &=]2, \pi[\bigcup]\pi, +\infty[. \end{aligned}$$

Graphical representation of a function

Definition 3.1.6 :

The graph (or representative curve) of a function $f : E \subseteq \mathbb{R} \longrightarrow F \subseteq \mathbb{R}$, is the subset G_f (resp. or C_f) of $E \times F$ such that:

$$G_f = \left\{ M(x, f(x)) \ / \ x \in D_f \wedge f(x) \in F \right\}.$$

Operations on functions

Definition 3.1.7 :

Let f and g be two functions defined respectively on D_f and D_g. The domain of definition of $f+g$, $f-g$, and $f\times g$ is the set $D_f \cap D_g$, that is:

$$D_{f+g} = D_{f-g} = D_{f\times g} = D_f \cap D_g.$$

For all $x \in D_f \cap D_g$, we have

$$(f+g)(x) = f(x) + g(x), (f-g)(x) = f(x) - g(x) \text{ and } (f\times g)(x) = f(x)\times g(x).$$

The domain of definition of $\dfrac{f}{g}$ is the set

$$D_{\frac{f}{g}} = \left(D_f \cap D_g\right) - \{x \in D_g \text{ such that } g(x) = 0\},$$

and for all $x \in D_{\frac{f}{g}}$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Comparison of two functions

⊗ Two functions f and g are equal ($f = g$), iff : $D_f = D_g = D$ and $\forall x \in D$; $f(x) = g(x)$.

⊗ Let f and g be two numerical functions defined on the same domain $D \subseteq D_f \cap D_g$.

We say that f is less than (resp. greater than) or equal to g, denoted as $f \leq g$ (resp. $f \geq g$): if

$$\forall x \in D,\ f(x) \leq g(x)\ (\text{resp. } f(x) \geq g(x)).$$

Definition 3.1.8 • *The function f is said to be positive (or strictly positive) on $D \subseteq D_f$ if:*

$$\forall x \in D;\ f(x) \geq 0 \quad (f(x) > 0).$$

- *The function f is said to be negative (or strictly negative) on $D \subseteq D_f$ if:*

$$\forall x \in D;\ f(x) \leq 0 \quad (f(x) < 0).$$

Remark 3.1.9 • *If the function f is positive, its graph lies above the X-axis.*

- *If the function f is negative, its graph lies below the X-axis.*

3.1.2 Operations and Monotonicity of real functions

Operations on real functions

Let $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ and $g : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be two numerical functions .
The sum of the functions f and g is the function $f + g :\rightarrow \mathbb{R}$ defined as follows:

$$\forall x \in U; (f+g)(x) = f(x) + g(x).$$

The product of the functions f and g is the function $f \times g : U \rightarrow \mathbb{R}$ defined as follows:

$$\forall x \in U; \ (f \times g)(x) = f(x) \times g(x).$$

The product by scalar λ in $\mathbb{R}$ and the function f is the function $\lambda \times f : U \longrightarrow \mathbb{R}$ defined as follows:

$$\forall x \in U, (\lambda \times f)(x) \ = \lambda \times f(x).$$

Remark 3.1.10 *If the function f is greater than or equal to g, then the graph of the function f lies above the graph of the function g.*

Monotonicity of a function

Definition 3.1.11 *:*
Let f be a real function of a real variable defined on an interval $I \subseteq \mathbb{R}$. We say that :
• *f is **increasing** on I if :*

$$\forall (x_1, \ x_2) \in I^2; \ x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2).$$

• *f is **strictly increasing** on I if:*

$$\forall (x_1, \ x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) < f(x_2).$$

• *f is **decreasing** on I if:*

$$\forall (x_1, x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

• *f is **strictly decreasing** on I if:*

$$\forall (x_1, x_2) \in I^2 : x_1 < x_2 \Longrightarrow f(x_1) > f(x_2).$$

• *f is **constant** on I if:*

$$\forall (x_1, x_2) \in I^2;\ f(x_1) = f(x_2).$$

Or,

$$\forall x \in I;\ f(x) = C, \text{ where } C \text{ is a real constant.}$$

• *f is **monotone** (resp. **strictly monotone**) on I if f is either increasing on I or decreasing on I. (resp. f is strictly increasing or strictly decreasing on I).*

Examples 3.1.12 :

• *Let n be a positive integer. The function f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = ln(x^{2n+1})$, is strictly increasing on $\mathbb{R}_+^*$. Therefore, it is strictly monotonic on $\mathbb{R}_+^*$.*

• *The function g from $\mathbb{R}$ to $\mathbb{R}$ defined by $g(x) = 2E(x)$ is increasing on $\mathbb{R}$. However, it is not strictly increasing on $\mathbb{R}$ because it is constant on each interval $[k, k+1]$ with $k \in \mathbb{Z}$.*

Direction of variation of $f + g$ and $f - g$

Let $I \subseteq D_f \cap D_g$ be an interval:

• If f and g are **increasing** (resp decreasing) on I, then $f + g$ is **increasing** (resp decreasing) on I.

• If f and g do not have the same direction of variation on I, we cannot say anything about the direction of variation of $f + g$ on I.

• The direction of variation of $f - g$ on I, is the direction of variation of $f + (-g)$ on I.

3.1.3 Definition and properties of Composite functions

Let $f : D \longrightarrow \mathbb{R}$ and $g : D' \longrightarrow \mathbb{R}$ be two functions such that $f(D) \subseteq D'$. The function defined by f and g, denoted $g \circ f$, is called the composition of f and g. For every $x \in D_{g\circ f}$; $(g \circ f)(x) = g(f(x))$ where $D_{g\circ f} = \{x \in D \text{ such that } x \in D_f \text{ and } f(x) \in D_g\}$.

Example 3.1.13 :

Let f and g be two functions from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = \sqrt{x}$ and $g(x) = lnx$. Let's determine the *domains of definition $D_{g\circ f}$ and $D_{f\circ g}$ and the functions $g\circ f$ and $f\circ g$.*

$$\begin{aligned}\bullet D_{g\circ f} &= \{x\in\mathbb{R} \;/\; x\in D_f \text{ and } f(x)\in D_g\}\\ &= \{x\in\mathbb{R} \;/\; x\in\mathbb{R}_+ \text{ and } \sqrt{x}\in\mathbb{R}_+^*\}\\ &= \mathbb{R}_+^*.\end{aligned}$$

$\bullet \quad \forall x\in D_{g\circ f}=\mathbb{R}_+^*,$ *we have :*

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= ln\left(\sqrt{x}\right).\end{aligned}$$

$$\begin{aligned}\bullet D_{f\circ g} &= \{x\in\mathbb{R} \;/\; x\in D_g \text{ and } g(x)\in D_f\}\\ &= \{x\in\mathbb{R} \;/\; x\in\mathbb{R}_+^* \text{ and } lnx\geq 0\}\\ &= [\,1,+\infty\,[\,.\end{aligned}$$

$\bullet \quad \forall x\in D_{f\circ g}=[\,1,+\infty\,[\,,$ *we have :*

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= \sqrt{lnx}.\end{aligned}$$

Proposition 3.1.14 *:*

Let f and g be two real functions defined on D and D' respectively with $f(D)\subseteq D^{'}$.

1• *The composite of two functions, decreasing or increasing, is an increasing function.*

2• *The composite of two functions, one decreasing and the other increasing, is a decreasing function.*

Proof :

Suppose f and g are increasing on D and D' respectively, and let's show that $g\circ f$ is increasing. For all x_1 and x_2 in D with $x_1<x_2$, we have: $f(x_1)\leq f(x_2)$ because the function f is increasing on D. Since g is increasing on D' and $f(D)\subseteq D^{'}$, then $g(f(x_1))\leq g(f(x_2))$. Therefore, $(g\circ f)(x_1)=(g\circ f)(x_2)$. This shows that $g\circ f$ is increasing. Similarly, for the other cases. ■

3.1.4 Parity and periodicity of real functions

Let f be a real function of a real variable with domain of definition D_f.

Definition 3.1.15 *:*

∗ *We say that f is an **even** (or **odd**) function on D_f if and only if*

$$\forall x\in D_f;\ (-x)\in D_f \ \wedge\ f(-x)=f(x)\ \ (\textbf{\textit{resp.}}f(-x)=-f(x)).$$

∗ *We say that f is a periodic function on D_f with period $p\in\mathbb{R}^*$ if and only if*

$$\forall x\in D_f;\ \left((x+p)\in D_f\ \wedge\ f(x+p)=f(x)\right)\ \wedge\ \left((x-p)\in D_f\ \wedge\ f(x-p)=f(x)\right).$$

Graphic interpretation

Let f be a real function of a real variable.

• f is even if and only if its graph is symmetric with respect to the y-axis $(Y'Y)$. • f is odd if and only if its graph is symmetric with respect to the origin. • f is periodic with period p if and only if its graph on an interval of length $|p|$ is invariant under translations by vectors $kp\overrightarrow{i}$ where $k \in \mathbb{Z}$ and $\overrightarrow{i}$ is the unit vector with respect to the x-axis $(X'X)$.

Examples 3.1.16 *:*

• *The functions defined by* $x \longmapsto x^{2n}$ *with* $n \in \mathbb{Z}$ *and* $x \longmapsto cos(\frac{1}{x})$ *are even on* $\mathbb{R}^*$. • *The functions defined by* $x \longmapsto x^{2n+1}$ *with* $n \in \mathbb{Z}$ *and* $x \longmapsto sin(\frac{1}{x})$ *are odd on* $\mathbb{R}^*$. • *The functions defined on* $\mathbb{R}$ *by* $x \longmapsto sin(x)$, $x \longmapsto cos(x)$, *and* $x \longmapsto sin(x) + cos(x)$ *are periodic and have period* 2π. *The function defined on* $\mathbb{R}$ *by* $x \longmapsto x - E(x)$ *is periodic and has period* 1.

Proposition 3.1.17 *:*

Let f *be a real function of a real variable defined on* $\mathbb{R}$.

If f *periodic function of period* $p \in \mathbb{R}^*$, *then,*

$$\forall k \in \mathbb{Z},\ \forall x \in \mathbb{R};\ f(x+kp) = f(x).$$

Proof :

Let f be a real-periodic function of a real variable with period $p \in \mathbb{R}^*$

• By induction on $\mathbb{N}$,

Let $\mathcal{P}(n)$ be the periodicity property: $\forall x \in \mathbb{R};\ f(x+np) = f(x)$.

It is clear that $\mathcal{P}(0)$ and $\mathcal{P}(1)$ are true.

Assume that $\mathcal{P}(n)$, for $n \in \mathbb{N}$ fixed, is true, and we show that $\mathcal{P}(n+1)$ is also true.

For all $x \in \mathbb{R}$ we have:

$$f(x+(n+1)\,p) = f((x+np)+p) = f(x+np) = f(x)\ (\text{by hypothesis}).$$

Then, the property $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$. • For all $k \in \mathbb{Z}_-$, there exists $n \in \mathbb{N}$ such that $k = -n$.

For all $x \in \mathbb{R}$, we have:

$$f(x+kp) = f(x-np) = f((x-np)+np)\ \ (\text{from the preceding result}).$$

Therefore,

$$\forall x \in \mathbb{R}, \forall k \in \mathbb{Z}, f(x+kp) = f(x).$$

This completes the proof. ∎

Proposition 3.1.18 *:*

Let f *and* g *be two real functions defined on* D *and* D' *respectively, with* $f(D) \subset D'$.

a • *If f is even on D, then $g \circ f$ is also even.*

b • *If f is odd on D and g is even on D', then $g \circ f$ is even on D.*

c • *If f is odd on D and g is odd on D', then $g \circ f$ is odd on D.*

d • *If f is periodic with period p and function g such that $H(f) \subseteq D(g)$ then a composition $h(x) = (g \circ f)(x)$ is periodic with the same period p.*

e • *If f is periodic with period p and $\alpha \in \mathbb{R}^*$, then function $g(x) = f(\alpha x)$ is periodic with period $\frac{p}{\alpha}$.*

Proof :

a • Let f be an even function on D, then for all $x \in D$, we have:

$$(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x),$$

therefore, $g \circ f$ is even.

Similarly for properties **b** • and **c** •.

d • Let f be a periodic function of the period p on D, then for all $x \in D$, we have:

$$(g \circ f)(x+p) = g(f(x+p)) = g(f(x)) = (g \circ f)(x)$$

, Therefore, $g \circ f$ is p periodic on D. ■

3.1.5 Upper, Lower bound of real functions

Upper bound, lower bound of a subset of $\mathbb{R}$

Definition 3.1.19 *:*

The upper bound (resp. lower bound) of a subset E of $\mathbb{R}$, if it exists, is the smallest upper bound (resp. greatest lower bound) of E, and is denoted $supE$ (resp. $infE$).

Example 3.1.20 *:*

• *For $I =]-2, 1]$, we have $supI = 1$ and $infI = -2$.*

Upper bound, lower bound of a function

Definition 3.1.21 *Let f be a real function of a real variable defined on its domain D_f.*

1. *We say that f is bounded above if and only if there exists a real number M such that:*

$$\forall x \in D_f; f(x) \leq M.$$

2. *We say that f is bounded below if and only if there exists a real number m such that:*

$$\forall x \in D_f; f(x) \geq m.$$

3. *We say that f is bounded if and only if there exist two real numbers m and M such that:*

$$\forall x \in D_f; m \leq f(x) \leq M.$$

Or f is bounded on D_f if f is both bounded above and bounded below on I.

Examples 3.1.22 *:*

• *The functions defined on $\mathbb{R}$ by $x \longmapsto sin(x)$, $x \longmapsto cos(x)$ and $x \longmapsto sin(x) \times cos(x)$, are bounded above by 1 and below by -1.*

• *The function defined by $x \longmapsto \frac{x}{x^2+1}$ on $\mathbb{R}$ is a bounded function on $\mathbb{R}$.*

Indeed, for all $x \in \mathbb{R}$, we have:

$$\begin{aligned} (|x|-1)^2 \geq 0 \quad &\Longleftrightarrow \quad |x| \leq \tfrac{1}{2}(x^2+1) \\ &\Longleftrightarrow \quad |f(x)| \leq \tfrac{1}{2}. \end{aligned}$$

Maximum and minimum values of a function

Definition 3.1.23 *Let f be a numerical function defined on the set D_f, and let $x_0 \in I \subset D_f$.*

1) *We say that the number $f(x_0)$ is the absolute maximum value of the function f if:*

$$\forall x \in D_f;\ f(x) \leq f(x_0).$$

We write $\sup\limits_{x \in D_f} f(x) = f(x_0)$.

2) *We say that the number $f(x_0)$ is a relative maximum value of the function f at the point x_0 on the domain I if:*

$$\forall x \in I;\ f(x) \leq f(x_0).$$

We write $\sup\limits_{x \in I} f(x) = f(x_0)$.

3) *We say that the number $f(x_0)$ is the absolute minimum value of the function f at the point x_0 if:*

$$\forall x \in D_f;\ f(x) \geq f(x_0).$$

We write $\min\limits_{x \in D_f} f(x) = f(x_0)$.

4) *We say that the number $f(x_0)$ is a relative minimum value of the function f at the point x_0 on the domain I if:*

$$\forall x \in I;\ f(x) \geq f(x_0).$$

We write $\min\limits_{x \in I} f(x) = f(x_0)$.

Examples 3.1.24 :

• *Let* $f : x \longmapsto 2e^x - 1$, *then we have,* $\inf_{x\in\mathbb{R}} (2e^x - 1) = -1$. *Because* $f(\mathbb{R}) =]-1, +\infty[$.

• *Let* $g : x \longmapsto \ln x - x$, *then we have,* $\sup_{x\in\mathbb{R}_+^*} (\ln x - x) = -1$. *Because* $g(\mathbb{R}_+^*) =]-\infty, -1]$.

3.1.6 Limits of real functions with a real variable

Limit of a function at a point x_0

Definition 3.1.25 :

Let f be a real function defined on an bounded interval I of $\mathbb{R}$, and $x_0 \in \mathbb{R}$ a point of I or a on the boundary of I.

• *We say that f tends to $\ell \in \mathbb{R}$ at x_0 (or $f(x)$ tends to ℓ when x tends to x_0) if and only if:*

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \forall x \in I, |x - x_0| \leq \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

We then note $\lim_{x\longrightarrow x_0} f(x) = \ell$.

• *We say that f tends to $+\infty$ at x_0 if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists \delta > 0;\ \forall x \in I, |x - x_0| \leq \delta \Longrightarrow f(x) \geq A.$$

We then note $\lim_{x\longrightarrow x_0} f(x) = +\infty$.

• *We say that f tends to $-\infty$ at x_0 if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists \delta > 0;\ \forall x \in I, |x - x_0| \leq \delta \Longrightarrow f(x) \leq -A.$$

We then note $\lim_{x\longrightarrow x_0} f(x) = -\infty$.

Examples 3.1.26 :

1)• *Let* $f : x \in]-1, 2[\longmapsto \dfrac{2x+2}{2x+3}$.

Let's show that $\lim_{x\longrightarrow 1} f(x) = \dfrac{4}{5}$.

For $x \in]-1, 2[$ *we have :*

$$1 < 2x+3 < 7 \Longrightarrow |2x+3| < 7 \Longrightarrow \frac{1}{|2x+3|} < \frac{1}{7} \Longrightarrow \left| f(x) - \frac{4}{5} \right| \leq \frac{2}{35}|x-1| < 2|x-1|.$$

Therefore, for all $\varepsilon > 0$, we can find $\delta > 0$ (for example, $\delta = \dfrac{\varepsilon}{2}$) such that

$$\forall x \in I, |x-1| \leq \delta = \frac{\varepsilon}{2} \Longrightarrow \left| f(x) - \frac{4}{5} \right| \leq \varepsilon.$$

2)• *Let* $g : x \in]-1, +\infty[\longmapsto \dfrac{2}{|x+3|}$.
Let's show that $\lim\limits_{x \longrightarrow -3} g(x) = +\infty$.
For all $A > 0$, *we can find* $\delta > 0$ *(for example,* $\delta = \dfrac{2}{A}$*) such that*

$$\forall x \in I, |x - (-3)| \leq \delta = \frac{2}{A} \Longrightarrow |g(x)| \geq A.$$

Limit of a real function at infinity

Definition 3.1.27 *:*

$\otimes$ *Let* f *be a real-valued function defined on an interval of the form* $I =]a, +\infty[\subseteq \mathbb{R}$.

• *We say that* f *tends to* $\ell \in \mathbb{R}$ *at* $+\infty$ *if and only if:*

$$\forall \varepsilon > 0,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I, x \geq B \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

We then note $\lim\limits_{x \longrightarrow +\infty} f(x) = \ell$.
• *We say that* f *tends to* $+\infty$ *at* $+\infty$ *if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I, x \geq B \Longrightarrow f(x) \geq A.$$

We then note $\lim\limits_{x \longrightarrow +\infty} f(x) = +\infty$.
• *We say that* f *tends to* $-\infty$ *at* $+\infty$ *if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I, x \geq B \Longrightarrow f(x) \leq -A.$$

We then note $\lim\limits_{x \longrightarrow +\infty} f(x) = -\infty$.
$\otimes$ *Let* f *be a real-valued function defined on an interval of the form* $I =]-\infty, a[\subseteq \mathbb{R}$.

• *We say that* f *tends to* $\ell \in \mathbb{R}$ *en* $-\infty$ *if and only if:*

$$\forall \varepsilon > 0,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I', x \leq -B \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

We then note $\lim\limits_{x \longrightarrow -\infty} f(x) = \ell$.
• *We say that* f *tends to* $+\infty$ *at* $-\infty$ *if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I', x \leq -B \Longrightarrow f(x) \geq A.$$

We then note $\lim\limits_{x \longrightarrow -\infty} f(x) = +\infty$.
• *We say that* f *tends to* $-\infty$ *at* $-\infty$ *if and only if:*

$$\forall A \in \mathbb{R}_+^*,\ \exists B \in \mathbb{R}_+^*;\ \forall x \in I', x \leq -B \Longrightarrow f(x) \leq -A.$$

We then note $\lim\limits_{x \longrightarrow -\infty} f(x) = -\infty$.

Example 3.1.28 :

• *Let* $h : x \in \,]-\infty, 2[\, \longmapsto \dfrac{2x}{x-3}$.

Let's show that $\lim\limits_{x \longrightarrow -\infty} f(x) = 2$.

For all $B > 0$ *we can find* $B > 0$ *(for example* $B = \dfrac{6}{\varepsilon} - 3$ *) such that*

$$\forall x \in I, x \leq -B = -\frac{6}{\varepsilon} + 3 \Longrightarrow |h(x) - 2| \leq \varepsilon.$$

Right-hand limit, left-hand limit at a point

Let f be a real function defined on D, $\alpha > 0$ and $]\, x_0 - \alpha, x_0 \,[\,\cup\,]\, x_0, x_0 + \alpha \,[\,\subset D$.

$*$ The right-hand limit of f at x_0 is defined as the restriction of f to $]\, x_0, x_0 + \alpha \,[$ which has a limit ℓ (finite or infinite) at x_0.

We then note: $\lim\limits_{\substack{x \longrightarrow x_0 \\ x > x_0}} f(x)$ (or $\lim\limits_{\substack{x \longrightarrow x_0 \\ >}} f(x)$). Precisely

$$\lim_{\substack{x \longrightarrow x_0 \\ >}} f(x) = \ell \Longleftrightarrow \forall \varepsilon > 0, \exists \delta > 0; \forall x \in D, x_0 < x < x_0 + \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

$*$ We call left limit of f at x_0 if the restriction of f to $]\, x_0 - \alpha, x_0 \,[$ admits a limit ℓ' (finite or infinite) in x_0.

We then note: $\lim\limits_{\substack{x \longrightarrow x_0 \\ x < x_0}} f(x)$ (or $\lim\limits_{\substack{x \longrightarrow x_0 \\ <}} f(x)$).

$$\lim_{\substack{x \longrightarrow x_0 \\ <}} f(x) = \ell \Longleftrightarrow \forall \varepsilon > 0, \exists \delta > 0; \forall x \in D, x_0 - \delta < x < x_0 \Longrightarrow |f(x) - \ell| < \varepsilon.$$

Remark 3.1.29 :

If f *has* ℓ *as a left-hand and right-hand limit at* x_0 *and* $f(x_0) = \ell$ *then* ℓ *is the limit of* f *at* x_0.

Examples 3.1.30 :

1)• *The function* $E : x \in \mathbb{R} \longmapsto E(x)$ *(the integer part function), does not admit a limit in* $x_0 = k$ *where* $k \in \mathbb{Z}$. *Because for all* $x \in \,]k, k+1[$ *we have* $\lim\limits_{\substack{x \longrightarrow k \\ >}} E(x) = k$ *and for all* $x \in \,]k-1, k[$ *we have* $\lim\limits_{\substack{x \longrightarrow k \\ <}} E(x) = k - 1$.

2)• *The function* $f : x \in \mathbb{R}^* \longmapsto \dfrac{1}{|x|}$
has a left limit and a right limit at 0 *and* $\lim\limits_{\substack{x \longrightarrow 0 \\ >}} f(x) = \lim\limits_{\substack{x \longrightarrow 0 \\ <}} f(x) = +\infty$.

3)• *The function* $g : x \in \mathbb{R} \longmapsto \begin{cases} x^2+1 & if\, x > 1 \\ -2 & if\, x = 1 \\ 3-x^3 & if\, x < 1. \end{cases}$

has no limit at the point 1 . *Because* $\lim\limits_{\substack{x \longrightarrow 1 \\ >}} g(x) = \lim\limits_{\substack{x \longrightarrow 1 \\ >}} (x^2+1) = \lim\limits_{\substack{x \longrightarrow 1 \\ <}} g(x) = \lim\limits_{\substack{x \longrightarrow 1 \\ >}} (3-x^3) = 2$ *and* $g(1) = -2$.

Proposition 3.1.31 :

The limit of a function at a point, if it exists, is unique.

Proof :

Suppose that ℓ and ℓ' are two distinct limits of f at x_0. By the definition of a limit, we have:

$$\forall \varepsilon > 0,\ \exists \delta_1 > 0;\ \forall x \in D_f, |x - x_0| \leq \delta_1 \Longrightarrow |f(x) - \ell| \leq \varepsilon,$$

and

$$\forall \varepsilon > 0,\ \exists \delta_2 > 0;\ \forall x \in D_f, |x - x_0| \leq \delta_2 \Longrightarrow \left|f(x) - \ell'\right| \leq \varepsilon.$$

We choose $\varepsilon < \dfrac{|\ell - \ell'|}{2}$ and take $\delta_\varepsilon = min\,\{\delta_1, \delta_2\}$, then for all $0 < \varepsilon < \dfrac{|\ell - \ell'|}{2}$

$$\forall x \in D_f, x \in \left]x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon\right[\Longrightarrow f(x) \in \left]\ell - \varepsilon, \ell + \varepsilon\right[\cap \left]\ell' - \varepsilon, \ell' + \varepsilon\right[.$$

This is absurd, because:

in the case $\ell < \ell'$ we have $\varepsilon < \frac{\ell' - \ell}{2} \Longrightarrow \ell + \varepsilon < \ell' - \varepsilon$,

and in the case $\ell' < \ell$ we have $\varepsilon < \frac{\ell - \ell'}{2} \Longrightarrow \ell' + \varepsilon < \ell - \varepsilon$.

We deduce that the limit is unique. ■

3.1.7 Convergence and divergence of a function

Definition 3.1.32 :

We say that f *converges at* x_0 *if there exists a finite real number* ℓ *such that* $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$. *Otherwise, we say that* f *diverges at* x_0.

Theorem 3.1.33 :

If f *converges at* x_0 *then* f *is bounded in the neighborhood of* x_0.

Proof :

Suppose $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$. For $\varepsilon < 1$, there exists $\delta > 0$; $\forall x \in \left]x_0 - \delta, x_0 + \delta\right[$; $|f(x) - \ell + \ell| \leq$

$|f(x) - \ell| + |\ell| < 1 + |\ell|$.

Thus f is bounded in the neighborhood of x_0. ■

Attention: The converse is false, the function $x \longmapsto cos(\frac{1}{x^2})$ is bounded in the neighborhood of 0, but this function diverges at 0.

Operations on the limits

Theorem 3.1.34 :

Let f and g be two real functions defined on $D \subseteq \mathbb{R}$ having respective limits ℓ and ℓ' (finite or infinite) when $x \longrightarrow x_0$ where x_0 is a point of D or an endpoint of D and $\ell + \ell'$, $\ell \times \ell'$ and $\frac{\ell}{\ell'}$ are defined (exist) in $\overline{\mathbb{R}}$.

1 • $\lim\limits_{x \longrightarrow x_0} (f(x) + g(x)) = \lim\limits_{x \longrightarrow x_0} f(x) + \lim\limits_{x \longrightarrow x_0} g(x) = \ell + \ell'$.

2 • $\lim\limits_{x \longrightarrow x_0} (f(x) \times g(x)) = \left(\lim\limits_{x \longrightarrow x_0} f(x)\right) \times \left(\lim\limits_{x \longrightarrow x_0} g(x)\right) = \ell \times \ell'$.

3 • $\lim\limits_{x \longrightarrow x_0} \left(\dfrac{1}{f(x)}\right) = \dfrac{1}{\lim\limits_{x \longrightarrow x_0} f(x)} = \dfrac{1}{\ell}$.

Proof :

Let's write the definitions of the limits:

$$\lim_{x \longrightarrow x_0} f(x) = \ell \Longleftrightarrow \forall \varepsilon > 0,\ \exists \delta > 0; \forall x \in D,\ |x - x_0| < \delta \Longrightarrow |f(x) - \ell| < \varepsilon.$$

$$\lim_{x \longrightarrow x_0} g(x) = \ell' \Longleftrightarrow \forall \varepsilon > 0,\ \exists \delta' > 0; \forall x \in D,\ |x - x_0| < \delta' \Longrightarrow |g(x) - \ell'| < \varepsilon.$$

1• For all $x \in D$ we have :

$$|(f+g)(x) - (\ell + \ell')| = |(f(x) - \ell) + (g(x) - \ell')| \leq |f(x) - \ell| + |g(x) - \ell'|.$$

and

$$\forall \varepsilon > 0,\ \exists \delta'' = min\{\delta, \delta'\} > 0;\ |x - x_0| < \delta'' \Longrightarrow |f(x) - \ell| < \varepsilon \text{ and } |g(x) - \ell'| < \varepsilon.$$

Then

$$\forall \varepsilon > 0,\ \exists \delta'' > 0;\ |x - x_0| < \delta'' \Longrightarrow |f(x) - \ell| + |g(x) - \ell'| < 2\varepsilon.$$

If we choose $0 < \varepsilon < \dfrac{\varepsilon'}{2}$, then we find

$$\forall \varepsilon' > 0,\ \exists \delta'' > 0;\ |x - x_0| < \delta'' \Longrightarrow |(f+g)(x) - (\ell + \ell')| < \varepsilon'.$$

Therefore, by definition, we have: $\lim\limits_{x \longrightarrow x_0} (f+g)(x) = \ell + \ell'$.

2• For all $x \in D$ we have :

$$\begin{aligned} |(f \times g)(x) - (\ell \times \ell')| &= |f(x) \times g(x) - \ell g(x) + \ell g(x) - \ell \ell'| \\ &= |g(x)(f(x) - \ell) + \ell(g(x) - \ell')| \\ &\leq |g(x)| \, |(f(x) - \ell)| + |\ell| \, |(g(x) - \ell')| \\ &\leq \varepsilon(|g(x)| + |\ell|). \end{aligned}$$

As $\ell \in \mathbb{R}$ we have $\ell' - \varepsilon < g(x) < \ell' + \varepsilon$, that is, for all $x \in \,]\, x_0 - \delta', x_0 + \delta' \,[\,$;
$|g(x)| < max \{|\ell' - \varepsilon|, |\ell' + \varepsilon|\} = M$.
We deduce that if

$$|x - x_0| < \delta'' \;\; then \; \left|(f \times g)(x) - (\ell \times \ell')\right| < \varepsilon(M + |\ell|).$$

If we choose $0 < \varepsilon < \dfrac{\varepsilon'}{M + |\ell|}$, we obtain

$$\forall \varepsilon' > 0, \; \exists \delta'' > 0; \; |x - x_0| < \delta'' \Longrightarrow \left|(f \times g)(x) - (\ell \times \ell')\right| < \varepsilon'.$$

Then, we have: $\lim\limits_{x \longrightarrow x_0} (f \times g)(x) = \ell \times \ell'$.

3• For all $x \in D$ we have :

$$\begin{aligned} \left|\left(\frac{1}{f}\right)(x) - \frac{1}{\ell}\right| &= \left|\frac{1}{f(x)} - \frac{1}{\ell}\right| \\ &= \left|\frac{\ell - f(x)}{\ell . f(x)}\right| \\ &= \frac{|\ell - f(x)|}{|\ell . f(x)|}. \end{aligned}$$

As $\ell \in \mathbb{R}$ we have $\ell - \varepsilon < f(x) < \ell + \varepsilon$, that is, for all $x \in \,]\, x_0 - \delta, x_0 + \delta \,[\,$;
$|f(x)| > min \{|\ell - \varepsilon|, |\ell + \varepsilon|\} = m$.
We deduce that if

$$|x - x_0| < \delta \;\; then \; \left|\left(\frac{1}{f}\right)(x) - \frac{1}{\ell}\right| < \frac{\varepsilon}{m\,|\ell|}.$$

If we choose $0 < \varepsilon < \dfrac{\varepsilon''}{m.\,|\ell|}$, we obtain

$$\forall \varepsilon'' > 0, \; \exists \delta > 0; \; |x - x_0| < \delta \Longrightarrow \left|\left(\frac{1}{f}\right)(x) - \frac{1}{\ell}\right| < \varepsilon''.$$

Then, we have: $\lim\limits_{x \longrightarrow x_0} \left(\dfrac{1}{f(x)}\right) = \dfrac{1}{\ell}$. ∎

3.1.8 Limits of functions and limits of sequences

Proposition 3.1.35 *:*

Let $f : D \longrightarrow \mathbb{R}$, $x_0 \in D$ and $\ell \in \overline{\mathbb{R}}$. We have $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$ if and only if for every sequence (u_n) of D converging to x_0, $\lim\limits_{n \longrightarrow +\infty} f(u_n) = \ell$.

Proof :

Suppose $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$, and (u_n) a sequence of D such that $\lim\limits_{n \longrightarrow +\infty} u_n = x_0$. Let $\varepsilon > 0$. Since f of limit ℓ in x_0, there exists $\delta > 0$ such that:

$$\forall x \in D,\ |x - x_0| < \delta \Longrightarrow |f(x) - \ell| \leq \varepsilon.$$

Since u_n converging to x_0, there exists $N \in \mathbb{N}$ such that: $\forall n \geq N,\ |x - x_0| < \delta$. We then have

$$\forall n \geq N,\ |f(u_n) - \ell| \leq \varepsilon.$$

Thus $f(u_n)$ converging to ℓ.

Conversely: Let's reason by contradiction. Suppose that for any sequence (u_n) in D such that $\lim\limits_{n \longrightarrow +\infty} u_n = x_0$, $\lim\limits_{n \longrightarrow +\infty} f(u_n) = \ell$ and that f does not have ℓ as a limit at x_0.

We have that (f does not have ℓ as a limit at x_0) can be written as:

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x \in D;\ |x - x_0| < \delta \textit{ et } |f(x) - \ell| \geq \varepsilon.$$

Let us take $\delta = \dfrac{1}{n+1}$ with $n \in \mathbb{N}$, we obtain an element u_n of D such that

$$\forall n \geq N,\ \exists u_n \in D;\ |u_n - x_0| < \frac{1}{n+1} \textit{ et } |f(u_n) - \ell| \geq \varepsilon.$$

This means that the sequence $u_n \in D$ converges to x_0, but the sequence with general term $f(u_n)$ does not converge to ℓ. This contradicts our hypothesis. ∎

Remark 3.1.36 *:*

To show that a function f has no limit at x_0, we must exhibit two sequences (u_n) and (v_n) such that:

$$\lim_{n \longrightarrow +\infty} u_n = \lim_{n \longrightarrow +\infty} v_n = x_0 \textit{ but } \lim_{n \longrightarrow +\infty} f(u_n) \neq \lim_{n \longrightarrow +\infty} f(v_n).$$

Or we find a sequence (u_n) converges to x_0 and for which the sequence with general term $f(u_n)$ diverges.

Examples 3.1.37 *:*

1) The function $f : x \longmapsto sin\dfrac{2}{x}$ has no limit at 0. This is because the sequence (u_n) with general term*

$u_n = \dfrac{4}{n\pi}$ *converges to* 0*, but the sequence with general term* $f(u_n) = (-1)^n$ *diverges.*
2) The function $g : x \longmapsto cos(2x)$ *has no limit at* $+\infty$*. Consider the sequence* (v_n) *with general term* $v_n = n\pi$ *tends to* $+\infty$ *and the sequence* (w_n) *with general term* $w_n = \dfrac{(2n+1)\,\pi}{2}$ *tends to* $+\infty$ *but the sequence with general term* $g(v_n) = cos(2n\pi)$ *converges to* 1 *and the sequence with general term* $g(w_n) = cos((2n+1)\,\pi)$ *converges to* -1*.*

3.1.9 Limit of a composite function

Let D and D' be two non-empty subsets of $\mathbb{R}$, x_0 a point or endpoint of D, ℓ a finite or infinite real number, f a function from D to $\mathbb{R}$, and g a function from D' to $\mathbb{R}$ such that $f(D) \subset D'$. If $\lim\limits_{x \longrightarrow x_0} f(x) = b$ and $\lim\limits_{x \longrightarrow b} g(x) = \ell$ where b is a point or endpoint of D', then $\lim\limits_{x \longrightarrow x_0} (gof)(x) = \ell$.

3.1.10 Comparison Theorems

We now give a very important proposition which means that we can pass to the limit in a broad inequality.

Proposition 3.1.38 *:*

Let f*,* g *and* h *be three functions from* D *to* $\mathbb{R}$*,* x_0 *a point of* D *or an endpoint of* D*, and* ℓ *and* ℓ' *two finite or infinite real numbers.*

$\otimes$ *If* $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$*,* $\lim\limits_{x \longrightarrow x_0} g(x) = \ell'$ *and for all* $x \in D$*;* $f(x) \leq g(x)$ *then* $\ell \leq \ell'$*.*
$\otimes$ *If* $\lim\limits_{x \longrightarrow x_0} f(x) = +\infty$ *and for all* $x \in D$*;* $f(x) \leq g(x)$ *then* $\lim\limits_{x \longrightarrow x_0} g(x) = +\infty$*.*
$\otimes$ *If* $\lim\limits_{x \longrightarrow x_0} g(x) = -\infty$ *and for all* $x \in D$*;* $f(x) \leq g(x)$ *then* $\lim\limits_{x \longrightarrow x_0} f(x) = -\infty$*.*
$\otimes$ *If* $\lim\limits_{x \longrightarrow x_0} g(x) = \ell$*,* $\lim\limits_{x \longrightarrow x_0} h(x) = \ell'$ *and for all* $x \in D$*;* $g(x) \leq f(x) \leq h(x)$ *then* $\ell \leq \lim\limits_{x \longrightarrow x_0} f(x) \leq \ell'$*.*

Proposition 3.1.39 *:*

If f *is a bounded function on neighborhood of* x_0 *and if* $\lim\limits_{x \longrightarrow x_0} g(x) = 0$ *then* $\lim\limits_{x \longrightarrow x_0} f(x) \times g(x) = 0$*.*
That is, f *is negligible compared to* g *in the neighborhood of* x_0*. We can also write,*

$$\forall x \in D_f \cap D_g \cap \left]\, x_0 - \alpha, x_0 + \alpha \,\right[, \ \forall \varepsilon > 0; \ |f(x)| \leq \varepsilon \, |g(x)| \, .$$

Example 3.1.40 *:*

$\lim\limits_{x \longrightarrow 0} x^2 \cos \frac{1}{x} = 0$ *because* $\lim\limits_{x \longrightarrow 0} x^2 = 0$ *and the function* $x \longmapsto \cos \dfrac{1}{x}$ *is bounded on neighborhood of* 0*.*

Remark 3.1.41 *:*

For $n \in \mathbb{N}^*$ *we have :*

When $x \longrightarrow +\infty$,

$$e^{-x} << \frac{1}{x^n} << \frac{1}{lnx} << lnx << \sqrt{x} << x^n << e^x.$$

When $x \longrightarrow 0^+$,

$$x^n << \sqrt{x} << \frac{1}{lnx} << lnx << \frac{1}{\sqrt{x}} << \frac{1}{x^n}.$$

3.2 Continuity of real functions of a real variable

3.2.1 Continuity of real functions at the point

Definition 3.2.1 *:*

Let f be a function from I to $\mathbb{R}$ and $x_0 \in I$.

• *We say that f is continuous at x_0 if it is defined on an open interval containing x_0 and if $\lim\limits_{x \longrightarrow x_0} f(x) = f(x_0)$.*

In other words, if

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \forall x \in D_f,\ |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

• *We say that f is continuous on an interval I of $\mathbb{R}$ if f is continuous at every point of I.*

We denote by $\mathcal{C}(I, \mathbb{R})$ or $\mathcal{C}^0(I, \mathbb{R})$ the set of functions defined and continuous on I with values in $\mathbb{R}$.

Examples 3.2.2 *:*

1) *The function $f(x) = \frac{x^3-1}{x^2-1}$ is discontinuous at 1. Because the function f is not defined at the point 1, that is $f(1)$ is undefined.*

2) *The function*

$$\begin{array}{rccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array}$$

is continues in 0, because $\lim\limits_{\substack{x \longrightarrow 0 \\ x \neq 0}} f(x) = \lim\limits_{\substack{x \longrightarrow 0 \\ x \neq 0}} e^{-\frac{1}{x^2}} = 0 = f(0)$.

3) *The function* $g(x) = \begin{cases} x^2 - 1 \text{ if } x < -1 \\ x^3 - 1 \text{ if } x \geq -1 \end{cases}$ *is not continuous at* -1.

Because $0 = lim_{x \underset{<}{\rightarrow} -1} g(x) \neq lim_{x \underset{>}{\rightarrow} -1} g(x) = -2 = g(-1)$.

4) *The function* $h(x) = \begin{cases} \frac{cos(x)}{x-\frac{\pi}{2}} \; if \; x \neq \frac{\pi}{2} \\ 1 \; if \; x = \frac{\pi}{2} \end{cases}$ *is not continuous at* $\frac{\pi}{2}$. *Because*

$$\lim_{x \longrightarrow \frac{\pi}{2}} h(x) = \lim_{x \longrightarrow \frac{\pi}{2}} \frac{cos(x)}{x - \frac{\pi}{2}} = \lim_{y \longrightarrow 0} \frac{cos(y + \frac{\pi}{2})}{y} = \lim_{y \longrightarrow 0} \frac{-sin(y).sin(\frac{\pi}{2})}{y} = -1 \neq h\left(\frac{\pi}{2}\right) = 1.$$

5)*However, the integer part function* E *is not continuous at the points* $x_0 \in \mathbb{Z}$, *since it does not have a limit at these points. Because* $\lim_{x \underset{>}{\longrightarrow} k} E(x) = k = E(k)$ *and* $\lim_{x \underset{<}{\longrightarrow} k} E(x) = k - 1 \neq E(k)$. *But it is continuous on* $[k, k+1[$ *for all* $k \in \mathbb{Z}$. **6)** *The function* $f : x \longmapsto \sin x$ *continues at every point* $x_0 \in \mathbb{R}$. *Indeed,*

$$\begin{aligned} |\sin x - \sin x_0| &= 2\left|\sin \frac{x - x_0}{2}\right| \left|\cos \frac{x + x_0}{2}\right| \\ &\leq 2\left|\sin \frac{x - x_0}{2}\right|, \quad because\ (\forall \alpha \in \mathbb{R};\ |cos\alpha| \leq 1) \\ &\leq |x - x_0|, \quad because\ (sinx \approx x \ \ on\ neighborhood\ of\ 0). \end{aligned}$$

Therefore, for all $\varepsilon > 0$, *there exists* $\delta = \varepsilon$ *such that*

$$|x - x_0| < \delta \Longrightarrow |\sin x - \sin x_0| \leq \varepsilon.$$

Continuity on the right and on the left

Definition 3.2.3 :

Let f *be a function defined on an interval* I *of* $\mathbb{R}$ *and* $x_0 \in I$.

• *We say that* f *is right-continuous at* x_0 *if:* $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0)$.

In other words, if

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \forall x \in I;\ 0 < x - x_0 \leq \delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

• *We say that* f *is left-continuous at* x_0 *if:* $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$.

In other words, if

$$\forall \varepsilon > 0,\ \exists \delta > 0;\ \forall x \in I;\ 0 < x_0 - x \leq \delta \Longrightarrow |f(x) - f(x_0)| \leq \varepsilon.$$

Remark 3.2.4 :

The function f *is continuous from the right and from the left at the point* x_0 *if and only if* f *is continuous at* x_0.

Examples 3.2.5 :

1) *The integer part function E is continuous from the right at the points $x_0 \in \mathbb{Z}$, and discontinuous from the left at the points $x_0 \in \mathbb{Z}$. Because* $\lim\limits_{\substack{x \overset{>}{\longrightarrow} x_0 \\ x_0 \in \mathbb{Z}}} E(x) = x_0 = E(x_0)$ *and* $\lim\limits_{\substack{x \overset{<}{\longrightarrow} x_0 \\ x_0 \in \mathbb{Z}}} E(x) = x_0 - 1 \neq E(x_0)$.

2) *The function g defined on* $]-\frac{\pi}{2}, \frac{\pi}{2}[$ *by*

$$g(x) = \begin{cases} \dfrac{tanx}{x} & if\, x \neq 0 \\ 1 & if\, x = 0 \end{cases}$$

it is continuous in 0, *since* $\lim\limits_{x \overset{>}{\longrightarrow} x_0} \frac{tanx}{x} = \lim\limits_{x \overset{<}{\longrightarrow} x_0} \frac{tanx}{x} = 1 = g(0)$.

3.2.2 Operations and composite continuous functions

Theorem 3.2.6 :

$\otimes$ *Let f and g be two real functions defined on D and continuous at a point $x_0 \in D$, where λ is a real constant.*

$*$ *λf, $f + g$ et $f \times g$ are continuous at x_0.*

$*$ *If $f(x_0) \neq 0$, then $\frac{1}{f}$ is continuous at x_0.*

$\otimes$ *Let f and g be two real functions defined on I and J respectively such that $f(I) \subseteq J$. If f is continuous at a point $x_0 \in I$ and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof : The proof is a result of the properties of the limits. ■

Theorem 3.2.7 :*(Weierstrass's Theorem)*

Every real function f continuous on an interval $[a, b] \subset \mathbb{R}$ is bounded and attains its bounds, in other words, it has a minimum and a maximum. That is to say: $\exists c, d \in [a, b]\,;\, \forall x \in [a, b], f(c) \leq f(x) \leq f(d)$ or $f(d) \leq f(x) \leq f(c)$.

3.2.3 Intermediate Value Theorem

Theorem 3.2.8 :

Let f be a continuous function on $[a, b] \subset \mathbb{R}$, then for every real k between $f(a)$ and $f(b)$, there exists at least one real $c \in [a, b]$ such that $f(c) = k$.

Proof : :

Let's prove the theorem in the case where $f(a) < f(b)$. Suppose that k is a real number such that

$f(a) \leq k \leq f(b)$.

1. The set $F = \{x \in [a,b] \ / \ f(x) \geq k\}$ is non-empty (since $b \in F$) and it is bounded below (by a): it therefore has a lower bound c such that $c > a$. Let's show that $f(c) = k$.

2. First, let's show that $f(c) \geq k$. Since $c = infF$, then

$$\forall \varepsilon > 0, \exists x_\varepsilon \in F; \ c \leq x_\varepsilon < c + \varepsilon, \tag{3.1}$$

From (3.1), for $n \in \mathbb{N}^*$, taking $\varepsilon = \dfrac{1}{n}$, we have:

$$\exists x_n \in F; \ c \leq x_n < c + \frac{1}{n}.$$

We deduce that the sequence $(x_n)_{n \in \mathbb{N}^*}$ converges to c. Since f is continuous at c, the sequence $(f(x_n))_{n \in \mathbb{N}^*}$ converges to $f(c)$. Now, for all $n \in \mathbb{N}^*$, we have $f(x_n) \geq k$ because $x_n \in F$. Therefore, by taking the limit, we find $f(c) \geq k$.

3. Let's now show that $f(c) \leq k$. Consider the sequence $(y_n)_{n \in \mathbb{N}^*}$ defined by $y_n = c - \frac{c-a}{n}$. This sequence is strictly increasing and converges to c, since $c > a$. Since f is continuous at c, the sequence $(f(y_n))_{n \in \mathbb{N}^*}$ converges to $f(c)$. Since, for all $n \in \mathbb{N}^*$, we have $f(y_n) \leq k$ because ($c = infF$, which implies $y_n \notin F$). Therefore, by taking the limit, we find $f(c) \leq k$. Finally, we have $f(c) \geq k$ and $f(c) \leq k$. This implies $f(c) = k$. ■

Examples 3.2.9 *:*

a) *We will demonstrate that this equation* $2x^3 - 6x^2 + 7x - 4 = 0$ *has at least one solution in the interval* $[1;2]$.

We can use the Intermediate Value Theorem.

We consider the function $f(x) = 2x^3 - 6x^2 + 7x - 4$ *over the closed interval* $[1;2]$. *The function* f *is a polynomial, therefore it is continuous over* $[1;2]$. *We have* $f(1) = -1$ *and* $f(2) = 2$. *Since* $f(1) < 0 < f(2)$, *then by the Intermediate Value Theorem: there exists at least one value k in the interval* $]1,2[$ *such that* $f(k) = 0$, *that is a solution for the equation* $f(x) = 0$ *in the interval* $]1,2[$.

b) *The function* $f : x \longmapsto tanx + x - 1$ *has at least one zero on the interval* $\left[0, \dfrac{\pi}{4}\right]$. *This is because the function* f *is continuous on* $\left] -\dfrac{\pi}{2}, \dfrac{\pi}{2} \right[$, *and* $f(0) = -1$, $f(\frac{\pi}{4}) = \frac{\pi}{4}$, *and* 0 *lies between* $f(0)$ *and* $f(\frac{\pi}{4})$. *According to the Intermediate Value Theorem, there exists at least one* $c \in \left[0, \dfrac{\pi}{4}\right]$ *such that* $f(c) = 0$.

3.2.4 Results of monotonic continuous functions

Theorem 3.2.10 *:*

If f *is a continuous, strictly monotonic function on an interval* $I = [a,b]$ *with finite or infinite bounds,* *then:*

1) *$f(I)$ (the direct image of the interval I by the function f) is an interval of the same type as I, and whose boundaries are the limits of f at the boundaries of the interval I.*

2) *f establishes a bijection from I to $f(I)$.*

3) *Moreover, its inverse function $f^{-1} : f(I) \longrightarrow I$ is continuous and has the same monotonicity as f.*

Proof :

1) • Let $y_1, y_2 \in f(I)$, $y_1 \leq y_2$. Let's show that if $y \in [y_1, y_2]$, then $y \in f(I)$. By hypothesis, there exist $x_1, x_2 \in I$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, and therefore y is between $f(x_1)$ and $f(x_2)$. By the Intermediate Value Theorem, since f is continuous, there exists $x \in I$ such that $y = f(x)$, and thus $y \in f(I)$.

• We assume that $I = \,]\,a, b\,[[\, \subset \mathbb{R}$ and f is strictly increasing on I. Since f is strictly monotonically continuous on I, then the limits of f at a and b exist in $\overline{\mathbb{R}}$ and $\lim\limits_{x \longrightarrow a} f(x) = \underset{x \in]a,b[}{inf}\ f(x)$ and $\lim\limits_{x \longrightarrow b} f(x) = \underset{x \in]a,b[}{sup}\ f(x)$.

Then, for all $x \in]a, b[$; $\lim\limits_{x \longrightarrow a} f(x) < f(x) < \lim\limits_{x \longrightarrow b} f(x)$ because f is strictly increasing. So

$$f\left(]a, b[\right) \subset] \lim_{x \longrightarrow a} f(x), \lim_{x \longrightarrow b} f(x)[.$$

Vice versa:

Let $y \in] \lim\limits_{x \longrightarrow a} f(x), \lim\limits_{x \longrightarrow b} f(x)[$. There exists $x_1, x_2 \in]a, b[$ such that $f(x_1) < y < f(x_2)$. Applying the Intermediate Value Theorem on the interval $[x_1, x_2]$, we find $x \in [x_1, x_2] \subset]a, b[$ such that $y = f(x)$. So

$$] \lim_{x \longrightarrow a} f(x), \lim_{x \longrightarrow b} f(x)[\subset f\left(]a, b[\right).$$

Eventually

$$f\left(]a, b[\right) =] \lim_{x \longrightarrow a} f(x), \lim_{x \longrightarrow b} f(x)[.$$

In the case where $I = [a, b[\subset \mathbb{R}$.

The continuity to the right from f to a imposes $\lim\limits_{x \overset{>}{\longrightarrow} a} f(x) = f(a)$. And like

$$\begin{aligned} f\left([a\ , b\,[\right) &= f\left(\{a\} \cup]a, b[\right) \\ &= \{f(a)\} \cup f\left(]a, b[\right) \\ &= \{f(a)\} \cup \left] \lim_{x \overset{>}{\longrightarrow} a} f(x), \lim_{x \longrightarrow b} f(x) \right[\\ &= \left[f(a)\ , \lim_{x \longrightarrow b} f(x) \right[\end{aligned}$$

3) Let's assume, for the sake of clarity, that f is strictly increasing on an interval I. The same applies to the case where f is strictly decreasing on I. Let $(x_1, x_2) \in I^2$ be such that $x_1 \neq x_2$. Then we have $x_1 < x_2$ or $x_1 > x_2$. Since f is strictly increasing on I, this implies that $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, in all cases we have $f(x_1) \neq f(x_2)$. Therefore, f is injective. According to the Intermediate Value Theorem, for every real number between $f(a)$ and $f(b)$, f has at least one preimage under f, meaning that the function f is surjective. This clearly shows that f is bijective.

4) • Let's show that if f is strictly increasing on an interval I, then f^{-1} is strictly increasing on $f(I)$. Let $(y_1, y_2) \in f(I)^2$ be such that $y_1 < y_2$, and let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then we have $f(x_1) = y_1$ and $f(x_2) = y_2$. If $f(x_1) < f(x_2)$, since f is strictly increasing on I, then we have $x_1 < x_2$. This shows that f^{-1} is strictly increasing on $f(I)$.

• Let's show that f^{-1} is continuous at $y_0 \in f(I)$.

We need to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in f(I)$;

$$|y - y_0| < \delta \Longrightarrow \left| f^{-1}(y) - f^{-1}(y_0) \right| \leq \varepsilon.$$

Let $x_0 - \varepsilon < x < x_0 + \varepsilon$ since f is continuous and strictly increasing on I we have $f(x_0 - \varepsilon) < f(x) < f(x_0 + \varepsilon)$ [taking ε small enough so that $x_0 - \varepsilon$ and $x_0 + \varepsilon$ remain in I]. Consider $\alpha = y_0 - f(x_0 - \varepsilon)$ and $\beta = f(x_0 + \varepsilon) - y_0$. We have $f^{-1}(y_0 - \alpha) = x_0 - \varepsilon$ and $f^{-1}(y_0 + \beta) = x_0 + \varepsilon$. For $y \in [y_0 - \beta, y_0 + \beta]$ where $\delta = \min\{\alpha, \beta\}$, we have $y_0 - \alpha < y < y_0 + \beta$. Since f^{-1} is strictly increasing

$$\begin{aligned} f^{-1}(y_0 - \alpha) < f^{-1}(y) < f^{-1}(y_0 + \beta) \quad &\Longleftrightarrow x_0 - \varepsilon < f^{-1}(y) < x_0 + \varepsilon \\ &\Longleftrightarrow f^{-1}(y_0) - \varepsilon < f^{-1}(y) < f^{-1}(y_0) + \varepsilon. \end{aligned}$$

Which shows that

$$\forall \varepsilon > 0, \exists \delta = \min\{\alpha, \beta\} > 0, \forall y \in f(I); |y - y_0| < \delta \Longrightarrow \left| f^{-1}(y) - f^{-1}(y_0) \right| \leq \varepsilon.$$

■

3.2.5 Extension by continuity

Definition 3.2.11 *:*

Let I be an interval of $\mathbb{R}$, x_0 a point of I, and f be a real function defined on $I - \{x_0\}$. We say that f is extendable by continuity at x_0 if f has a finite limit at x_0. Let $\ell = \lim\limits_{x \longrightarrow x_0} f(x)$. We then define the function

$\widetilde{f}$ by

$$\begin{array}{rcl} \widetilde{f}: I & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \begin{cases} \widetilde{f}(x) & = f(x) \quad \text{if } x \in I - \{x_0\} \\ \widetilde{f}(x) & = \ell \quad\quad \text{if } x = x_0. \end{cases} \end{array}$$

Then $\widetilde{f}$ is continuous at x_0 and we call it the continuous extension of f at x_0.

Examples 3.2.12 :

1. *Let f be the function defined on $\mathbb{R} - 0$ by $f(x) = \frac{cos(x)-1}{x^2}$. Then the function f is not defined at 0, but the limit of f at 0 is $\frac{1}{2}$. Because $lim_{x\to 0} \frac{cos(2\frac{x}{2})-1}{x^2} = lim_{x\to 0} \frac{sin^2(\frac{x}{2})}{2(\frac{x}{2})^2} = \frac{1}{2}$.*

The continuous extension of f at 0 is the function g defined on $\mathbb{R}$ by $g(x) = \begin{cases} \frac{cos(x)-1}{x^2} & if \quad x \neq 0 \\ \frac{1}{2} & if \quad x = 0. \end{cases}$

2. *Consider the function f defined on $\mathbb{R}^*$ by $f(x) = xcos(1/x)$. Let's see if f has a continuous extension at 0. Since for all $x \in \mathbb{R}^*$ we have $|f(x)| \leq |x|$, we deduce that f tends to 0 at 0. It is therefore extendable by continuity at 0 and its extension is the function $\tilde{f}$ defined on $\mathbb{R}$ by:*

$$\tilde{f}(x) = \begin{cases} xcos\frac{1}{x} & if \quad x \neq 0 \\ 0 & if \quad x = 0. \end{cases}$$

3. *The function h defined on $\mathbb{R} - \{1\}$ by $h(x) = (x-1)\, ln\,|x-1|$, extendable by continuity at 1 and its extension is the function $\tilde{h}$ defined on $\mathbb{R}$ by:*

$$\tilde{h}(x) = \begin{cases} (x-1)\, ln\,|x-1| & if \quad x \in \mathbb{R} - \{1\} \\ 0 & if \quad x = 1. \end{cases}$$

3.2.6 Exercises

Exercise 3.2.13 :

a) *Study the continuity of the function f defined on $\mathbb{R}$ by:*

$$f(x) = \left| x - 2E\left(\frac{x+1}{2}\right) \right|.$$

b) *Show that the function f is periodic with period 2 and even.*

c) *Represent graphically f on the interval $[-2, 2]$.*

Exercise 3.2.14 :

Let f and g be two functions from $\mathbb{N}$ to $\mathbb{N}$ defined by:

$$f(x) = 2x \text{ and } g(x) = \begin{cases} x & \text{if } x \text{ is even} \\ \dfrac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}.$$

1. *Are the maps f and g injective? surjective?*

2. *Are the maps $(f \circ g)$ and $(g \circ f)$ injective? surjective?.*

Exercise 3.2.15 :

Consider the application f defined as follows:

$$\begin{array}{rccl} f: & \mathbb{N} & \longrightarrow & \mathbb{N} \\ & n & \longmapsto & f(n) = n + (-1)^n \end{array}$$

1. *Is f injective?.*

2. *Is f surjective ?.*

3.*Calculate $(f \circ f)(n)$.*

Exercise 3.2.16 :

Consider the following two applications:

$$\begin{array}{rccl} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x_1, x_2) & \longmapsto & f(x_1, x_2) = x_1 + x_2 \end{array} \quad \text{and} \quad \begin{array}{rccl} g: & \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ & x & \longmapsto & (x^2, x^2) \end{array}$$

1. *Is f injective? Is f surjective?.*

2. *Is g injective? Is g surjective?.*

3. *Calculate $(f \circ g)(x)$.*

4. *Is $(f \circ g)(x)$ injective?, is $(f \circ g)(x)$ surjective?.*

Exercise 3.2.17 :

1. *Are the following functions extendable by continuity on $\mathbb{R}$?*

$$a)\ x \longmapsto cosx.cos\frac{1}{x - \frac{\pi}{2}} \qquad b)\ x \longmapsto \frac{1}{x-1} ln\frac{e^{2x-2} + e^{2-2x}}{2}$$

2. *We consider the function f defined by*

$$f : x \longmapsto \left(\alpha x^2 + \pi x - \pi^2\right) tan(2x).$$

a. Determine its domain of definition.

b. For what value of α can we extend f by continuity to $x = \dfrac{\pi}{4}$?.

Exercise 3.2.18 :

For what value(s) of the real number λ is the function g defined by:

$$\begin{cases} g(x) = \dfrac{sin(\lambda x)}{x} & if \quad x < 0 \\ g(x) = \dfrac{ln(1+2x)}{x} & if \quad x > 0, \end{cases}$$

Is it extendable by continuity at 0? We will then specify this extension.

Exercise 3.2.19 :

1) Show that for every integer $n \in \mathbb{N}$, the equation $\tan x = x$ has a unique solution denoted u_n in the interval $]n\pi - \dfrac{\pi}{2}, n\pi + \dfrac{\pi}{2}[$. 2) Let $v_n = u_n - n\pi$. Calculate $f(u_{n+1} - \pi)$. Deduce that the sequence $(v_n)_{n\geq 0}$ is strictly increasing, then that it converges, and find its limit.

3.3 Differentiability of real functions of a real variable

3.3.1 Derivatives of a real function

Differentiability deals with the instantaneous rate of change of a given function, while its rules form a set of principles that facilitate the calculation of derivatives in specific ways and provide important information about the properties of numerical functions.

Definition 3.3.1 :

Let f be a real-valued function defined on an open interval I and x_0 a point of I. We say that f is differentiable at x_0 if and only if this limit $\lim\limits_{x \longrightarrow x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists and is finite. This limit, denoted $f'(x_0)$, is called the derivative of f at x_0.

We say that f is differentiable on an open interval $J \subset I$ if it is differentiable at every point $x \in J$.

The function $x \longmapsto f^{'}(x)$ is called the derivative function, denoted by $f^{'}$ or $\frac{df}{dx}$.

Examples 3.3.2 :

1) *The function f defined on $[-1,1]$ by $f(x) = \sqrt{1-x^2}$ is differentiable at every point $x_0 \in\,]-1,1[$. Indeed, for all $x_0 \in\,]-1,1[$,*

$$\begin{aligned}
\lim_{x \longrightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \longrightarrow x_0} \frac{\sqrt{1-x^2} - \sqrt{1-x_0^2}}{x - x_0} \\
&= \lim_{x \longrightarrow x_0} \frac{\left(\sqrt{1-x^2} - \sqrt{1-x_0^2}\right)\left(\sqrt{1-x^2} + \sqrt{1-x_0^2}\right)}{(x-x_0)\left(\sqrt{1-x^2} + \sqrt{1-x_0^2}\right)} \\
&= \lim_{x \longrightarrow x_0} \frac{x + x_0}{\left(\sqrt{1-x^2} + \sqrt{1-x_0^2}\right)} \\
&= \frac{2x_0}{2\sqrt{1-x_0^2}} \in \mathbb{R}.
\end{aligned}$$

2) *The function g defined on $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$ by $g(x) = tanx$ is differentiable at every point $x_0 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Indeed, for all $x_0 \in]-\frac{\pi}{2}, \frac{\pi}{2}[$,*

$$\begin{aligned}
\lim_{\substack{x \longrightarrow x_0 \\ x_0 \in I}} \frac{g(x) - g(x_0)}{x - x_0} &= \lim_{\substack{x \longrightarrow x_0 \\ x_0 \in I}} \frac{tanx - tanx_0}{x - x_0} \\
&= \lim_{\substack{x \longrightarrow x_0 \\ x_0 \in I}} \frac{sinx.cosx_0 - sinx_0.cosx}{(x - x_0)(cosx.cosx_0)} \\
&= \lim_{\substack{x \longrightarrow x_0 \\ x_0 \in I}} \frac{sin(x - x_0)}{(x - x_0)(cosx.cosx_0)} \\
&= \frac{1}{(cosx_0)^2} \in \mathbb{R}.
\end{aligned}$$

3) *The function h defined on $\mathbb{R}$ by $h : x \longmapsto \begin{cases} h(x) = xcos\left(\frac{1}{x}\right) & if\ x \neq 0 \\ h(x) = 0 & if\ x = 0, \end{cases}$*

is not differentiable at 0. Because

$$\begin{aligned}
\lim_{x \longrightarrow 0} \frac{h(x) - h(0)}{x} &= \lim_{x \longrightarrow 0} \frac{xcos\left(\frac{1}{x}\right)}{x} \\
&= \lim_{x \longrightarrow 0} cos\left(\frac{1}{x}\right) \quad (\text{ has no limit }).
\end{aligned}$$

4. *The function k defined on $\mathbb{R}$ by $k(x) = x^n$ where $n \in \mathbb{N}^*$ is differentiable at every point $x_0 \in \mathbb{R}$, and that $f'(x_0) = n.x_0^{n-1}$. Because*

$$\begin{aligned}
\lim_{x \longrightarrow x_0} \frac{k(x) - k(x_0)}{x - x_0} &= \lim_{\substack{x \longrightarrow x_0 \\ x \neq x_0}} \frac{x^n - x_0^n}{x - x_0} \\
&= \lim_{\substack{x \longrightarrow x_0 \\ x \neq x_0}} \left(x^{n-1} + x^{n-2}.x_0 + \cdots + x_0^{n-2}.x + x_0^{n-1}\right) \\
&= n.x_0^{n-1}.
\end{aligned}$$

Remark 3.3.3 *:*

The derivative of the function f at x_0 is sometimes written $\frac{df}{dx}(x_0)$ and by the change of variable $x - x_0 = h$, we have:

$$f'(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3.3.2 Geometric interpretation of the derivative

The line passing through the distinct points $M_0(x_0, f(x_0))$ and $M(x, f(x))$ has a direction coefficient which is the value $f^{'}(x_0)$ when x aproches x_0. This line is called the tangent at the point $M_0(x_0,\ f(x_0))$ of C_f, and its equation is defined as: $y = f^{'}(x_0)(x - x_0) + f(x_0)$.

Proposition 3.3.4 *:*

Let $f :]a,b[\longrightarrow \mathbb{R}$ and $x_0 \in]a,b[$. There is equivalence between the following three statements:

***(i)** f is differentiable at x_0.*

***(ii)** There exists a real number ℓ which will be $f'(x_0)$ and a function ε defined on $]a,b[$ such that*

$$f(x) = f(x_0) + \ell.(x - x_0) + (x - x_0)\varepsilon(x) \text{ and } \lim_{x \longrightarrow x_0} \varepsilon(x) = 0.$$

***(iii)** There exists a real ℓ such that $\lim\limits_{x \longrightarrow x_0} \dfrac{f(x) - f(x_0) - \ell.(x - x_0)}{x - x_0} = 0$, and $\ell = f'(x_0)$.*

Proof ::

(i) $\Longrightarrow$ **(ii)**. It suffices to set $\begin{cases} \varepsilon(x) & = \dfrac{f(x) - f(x_0) - f'(x_0).(x - x_0)}{x - x_0} & \text{if } x \neq x_0 \\ \varepsilon(x) & = 0 & \text{if } x = x_0. \end{cases}$

(ii) $\Longrightarrow$ **(iii)** and **(iii)** $\Longrightarrow$**(i)**. They are obvious. ■

3.3.3 Differentiability and continuity

Theorem 3.3.5 *:*

***(1)** Any function f that is differentiable at a point x_0, then it is continuous at that point.*

***(2)** If a function f is not continuous at a point x_0, then it is not differentiable at that point.*

Proof : :

(1) Suppose f is differentiable at x_0 and let's show that it is also continuous at this point.

For $x \neq x_0$, we have

$$f(x) - f(x_0) = \frac{f(x) - f(x_0)}{x - x_0}(x - x_0).$$

Taking the limit in this identity, we find

$$\lim_{x \longrightarrow x_0} (f(x) - f(x_0)) = \lim_{x \longrightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right)(x - x_0) = f'(x_0).0 = 0.$$

That is, $\lim\limits_{x \longrightarrow x_0} f(x) = f(x_0)$. Thus, f is continuous at x_0. ■

3.3.4 Derived from the right and from the left at a point

Definition 3.3.6 *:*

Let f be a real-valued function defined on an open interval I and x_0 a point of I. We say that f is right-differentiable (resp. left-differentiable) at x_0 if: $\lim\limits_{x \overset{>}{\longrightarrow} x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f_d'(x_0)$

(resp. $\lim\limits_{x \overset{<}{\longrightarrow} x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f_g'(x_0)$) exists and is finite.

Examples 3.3.7 *:*

•**)** *The function* $f : [-1, 1] \longrightarrow \mathbb{R}$ *defined by:* $f(x) = \sqrt{x^2(1-|x|)}$ *is differentiable from the right and from the left at* 0 *and* $f'_d(0) = 1$ *and* $f'_g(0) = -1$

•**)** *The function* $g : [0, +\infty[\longrightarrow \mathbb{R}$ *defined by:* $g(x) = \left|1-x^2\right| \sqrt{x}$ *is differentiable from the right and from the left at* 1 *and* $g'_d(1) = 2$ *and* $g'_g(1) = -2$.

Property 3.3.8 *:*

f is differentiable at x_0 if and only if it is differentiable from the right and from the left at x_0 and $f'_d(x_0) = f'_g(x_0)$.

Examples 3.3.9 *:*

a). *We show that* $f\ (x) = \ x^2 + |x-1|$ *is not differentiable at the point* 1*. The right-hand derivative at the point* 1 *is :* $lim_{x \underset{>}{\rightarrow} 1} \frac{f(x)-f(1)}{x-1} = lim_{x \underset{>}{\rightarrow} 1} \frac{x^2+x-1-1}{x-1} = lim_{x \underset{>}{\rightarrow} 1} \frac{(x-1)(x+2)}{x-1} = 3.$

The left-hand derivative at the point 1 *is :* $lim_{x \underset{<}{\rightarrow} 1} \frac{f(x)-f(1)}{x-1} = lim_{x \underset{<}{\rightarrow} 1} \frac{x^2-x+1-1}{x-1} = 1.$

Since the right-hand derivative and the left-hand derivative are different that is $f'_d(1) \neq f'_g(1)$, *then the function f is not differentiable at point* 1.

b). *We show that* $g\ (x) = \begin{cases} x sin\left(\frac{1}{x}\right), & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ *is not differentiable at the point* 0.

Since $lim_{x \longrightarrow 0} \frac{g(x)-g(0)}{x} = lim_{x \longrightarrow 0}\, sin\left(\frac{1}{x}\right)$ *, which does not exist.*

This means the function g is not differentiable at the point 0.

c). *We show that* $h\ (x) = \begin{cases} x^2 sin\left(\frac{1}{x}\right), & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ *is differentiable at the point* 0.

Since $lim_{x \longrightarrow 0} \frac{h(x)-h(0)}{x} = lim_{x \longrightarrow 0}\, x sin\left(\frac{1}{x}\right) = 0$, *because the function sin is bounded. This means the function h is differentiable at the point* 0.

3.3.5 Derivative function

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function on $J \subseteq I$. The derivative of f is the function denoted f' which, for each $x \in J$, associates $f'(x)$ the number derivative of f at x, that is

$$\begin{aligned} f' : \ J \subset \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f'(x). \end{aligned}$$

Examples 3.3.10 *:*

a) *The function tan is differentiable on* $I_k = \left] -\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi \right[$ *with* $k \in \mathbb{Z}$ *and its derivative is* tan'

defined on I_k by :

$$tan^{'}(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x.$$

b) *The function cot differentiable on $j_k =]k\pi, (k+1)\pi[$ with $k \in \mathbb{Z}$ and its derivative is cot' defined on j_k by:*

$$cot^{'}(x) = -\frac{1}{sin^2 x} = -1 - cot^2 x.$$

Operations on differentiable functions

Theorem 3.3.11 *:*

Let f and g be two functions differentiable at a point x_0 on an interval I of $\mathbb{R}$ and $\lambda \in \mathbb{R}$, then, the functions $\lambda.f$, $f+g$ and $f.g$ are also differentiable at x_0 and $(\alpha.f)^{'}(x_0) = \alpha.f^{'}(x_0)$, $(f+g)^{'}(x_0) = f^{'}(x_0) + g^{'}(x_0)$ and $(f.g)^{'}(x_0) = f^{'}(x_0)\, g(x_0) + g^{'}(x_0)\, f(x_0)$.

• *If $g(x_0) \neq 0$, then the function $\frac{f}{g}$ differentiable at x_0 and*

$$\left(\frac{f}{g}\right)^{'}(x_0) = \frac{f^{'}(x_0)\, g(x_0) - g^{'}(x_0)\, f(x_0)}{(g(x_0))^2}.$$

Proof : :

First, we show that $(f \times g)' = f' \times g + f \times g'$.

Let $x_0 \in I$,

$$\begin{aligned} \bullet \lim_{x \longrightarrow x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x_0)}{x - x_0} &= \lim_{x \longrightarrow x_0} \frac{f(x).g(x) - f(x_0)\, g(x) + f(x_0)\, g(x) - f(x_0)\, g(x_0)}{x - x_0} \\ &= \lim_{x \longrightarrow x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}\right] \\ &= f^{'}(x_0)\, g(x_0) + g^{'}(x_0)\, f(x_0). \end{aligned}$$

This implies that the function $f \times g$ is differentiable on I with derivative $f' \times g + f \times g'$.

Secondly, let's show $\left(\frac{f}{g}\right)^{'} = \frac{f^{'} \times g - f \times g^{'}}{g^2}$.

Let $x_0 \in I$, we have

$$\begin{aligned}
\lim_{x \longrightarrow x_0} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} &= \lim_{x \longrightarrow x_0} \frac{1}{x - x_0} \times \frac{f(x) \times g(x_0) - f(x_0) \times g(x)}{g(x) \times g(x_0)} \\
&= \lim_{x \longrightarrow x_0} \frac{f(x) \times g(x_0) - f(x_0) \times g(x_0) + f(x_0) \times g(x_0) - f(x_0) \times g(x)}{(x - x_0)\, g(x) \times g(x_0)} \\
&= \lim_{x \longrightarrow x_0} \frac{1}{x - x_0} \times \frac{g(x_0)\,[f(x) - f(x_0)] - f(x_0)\,[g(x) - g(x_0)]}{g(x) \times g(x_0)} \\
&= \lim_{x \longrightarrow x_0} \left[\frac{1}{g(x)} \times \frac{f(x) - f(x_0)}{x - x_0} - \frac{f(x_0)}{g(x) \times g(x_0)} \times \frac{g(x) - g(x_0)}{x - x_0}\right] \\
&= \frac{1}{g(x_0)} \times f'(x_0) - \frac{f(x_0)}{(g(x_0))^2} \times g'(x_0) \quad \textit{because } \frac{1}{g} \textit{ is continuous at } x_0 \\
&= \frac{f'(x_0) \times g(x_0) - f(x_0) \times g'(x_0)}{(g(x_0))^2}.
\end{aligned}$$

This implies that the function $\frac{f}{g}$ is differentiable at every point x_0 of I with derivative $\frac{f' \times g - f \times g'}{g^2}$.

■

3.3.6 Derivative of a composite function

Theorem 3.3.12 :

Let f be a function defined on an interval I and g a function defined on an interval J containing $f(I)$, and x_0 a point of I.

If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = f'(x_0)\, g'(f(x_0)).$$

Proof : Let $y_0 = f(x_0)$. According to our hypotheses, f is differentiable at x_0 and g is differentiable at y_0. Let G be the function defined on J by:

$$\begin{cases} G(y) &= \dfrac{g(y) - g(y_0)}{y - y_0} \quad \textit{if } y \neq y_0 \\ G(y) &= g'(y_0) \qquad\quad\; \textit{if } y = y_0. \end{cases}$$

Since the function g is differentiable at y_0, the function G is continuous at y_0. The function f is continuous at x_0, therefore the function $G \circ f$ is continuous at x_0, so its limit at x_0 is $g'(y_0)$.

For all $x \in I - \{x_0\}$ we have:

$$\begin{aligned}
\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} &= \frac{g(f(x)) - g(f(x_0))}{x - x_0} \\
&= G(f(x))\, \frac{f(x) - f(x_0)}{x - x_0} \quad \textit{by definition of } G.
\end{aligned}$$

By taking the limit, when x tends towards x_0, we therefore have $g \circ f$ is differentiable at x_0 with derivative number $(g \circ f)^{'}(x_0) = f^{'}(x_0)\, g^{'}(f(x_0))$. ■

Examples 3.3.13 :

1). *Let I be an open interval of $\mathbb{R}$ and f a function defined from I to $\mathbb{R}$. Let h be the function defined on I by $h(x) = (f(x))^n$.*

If f is differentiable at x_0, then the function h is also differentiable and $h^{'}(x_0) = n.f^{'}(x_0) \times (f(x_0))^{n-1}$.

Indeed, $h = g \circ f$ with $g(y) = y^n$.

2). *Let I be an open interval of $\mathbb{R}$ and g be a function defined on I with values in $\mathbb{R}_+^*$.*

If g is differentiable at $x_0 \in I$, then the function k defined on I by $k(x) = ln(g(x))$ is differentiable at x_0 and $(ln \circ g)^{'}(x_0) = \dfrac{g^{'}(x_0)}{g(x_0)}$.

3.3.7 Bijections and Inverse Functions

Definition 3.3.14 *The function $f : D(f) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is **injective** on D_f if*

$$\forall x_1, x_2 \in D_f;\ f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

Equivalently:

$$\forall x_1, x_2 \in D_f\ ; x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2).$$

Remark 3.3.15 *f is **not injective** if $\exists x_1, x_2 \in D_f; x_1 \neq x_2$ and $f(x_1) = f(x_2)$.*

Proposition 3.3.16 :

A sum of two increasing (decreasing) functions is an increasing (decreasing) function.

Theorem 3.3.17 :

Let f be strictly monotone on $D \subseteq \mathbb{R}$ then f is injective on D.

Definition 3.3.18 :

Let $f : D(f) \longrightarrow H(f) \subseteq \mathbb{R}$ be an injective function where $D(f)$ is the domain of f and $H(f)$ is the range of f. Inverse function of f (denoted f^{-1}) is defined by the relation $y = f(x) \Longleftrightarrow x = f^{-1}(y)$. Obviously the domain $D(f^{-1}) = H(f)$ and range $H(f^{-1}) = D(f)$.

Remarks 3.3.19 :

$*$ *The graph of f^{-1} is symmetric to the graph of f with respect to the line with equation $y = x$.*

$*$ $\forall x \in D(f);\ f^{-1}(f(x)) = x$.

$*$ $\forall y \in D\left(f^{-1}\right) = H(f);\ f(f^{-1}(y)) = y$.

3.3.8 Derivative of an inverse function

Here we mention a theory that is widely used in practice to prove that the function is bijective.

Theorem 3.3.20 *(Bijection Theorem).*

Let f be a function defined on an interval I of $\mathbb{R}$.

If f is continuous and strictly monotonic on I, then f is a bijection from I to $J = f(I)$ and has an inverse bijection f^{-1} from $J = f(I)$ to I.

Furthermore, the inverse function f^{-1} is continuous and strictly monotonic on J and has the same direction of variation as f.

Proof ::

Let's assume, for the sake of clarity, that f is strictly increasing on I.

1) Recall that the image of an interval under a continuous function on that interval is an interval. Therefore, J is an interval.

To show that f is injective, we assume that $x_1, x_2 \in I$ such that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$. If $x_1 < x_2$, since f is strictly increasing on I, then $f(x_1) < f(x_2)$ is impossible, because $f(x_1) = f(x_2)$, we deduce that $x_1 \geq x_2$.

$\otimes$ Similarly, if $x_1 > x_2$, we can show that $x_1 \leq x_2$. We conclude that $x_1 = x_2$. Therefore, f is injective.

By the Intermediate Value Theorem, we have $\forall y \in J = f(I)$, $\exists x \in I$; $f(x) = y$. We conclude that f is surjective. The function f thus establishes a bijection from I to $f(I)$.

2) $*$ Let's show that f^{-1} is strictly increasing on J.

Let $y_1, y_2 \in J$ be such that $y_1 > y_2$ and we denote $x_1 = f^{-1}(y_1) \in I$ and $x_2 = f^{-1}(y_2) \in I$. We then have $f(x_1) = y_1$ and $f(x_2) = y_2$ and since $y_1 > y_2$, then $f(x_1) > f(x_2)$. This implies that $x_1 > x_2$ (since f is strictly increasing on I), that is, $f^{-1}(y_1) > f^{-1}(y_2)$. This shows that f^{-1} is strictly increasing on J.

Let's show that f^{-1} is continuous on J.

Let $y_0 \in J$ be such that $x_0 = f^{-1}(y_0) \in I$ and y_0 is not an endpoint of J.

To show that f^{-1} is continuous at y_0, we need to show that: for all $\varepsilon > 0$, there exist $\delta > 0$; for all $y \in J$,

$$|y - y_0| < \delta \Longrightarrow \left| f^{-1}(y) - f^{-1}(y_0) \right| \leq \varepsilon.$$

Let $\varepsilon > 0$ such that $x \in [x_0 - \varepsilon, x_0 + \varepsilon] \subset I$, as f is strictly increasing and continuous, we therefore have $f(x) \in [f(x_0 - \varepsilon), f(x_0 + \varepsilon)]$. We note $\alpha = y_0 - f(x_0 - \varepsilon) > 0$ and $\beta = f(x_0 + \varepsilon) - y_0 > 0$.

We have:

On the one hand, $f(x_0-\varepsilon)=y_0-\alpha$, hence $x_0-\varepsilon=f^{-1}(y_0-\alpha)$,

and on the other hand, $f(x_0+\varepsilon)=\beta+y_0$, hence $x_0+\varepsilon=f^{-1}(y_0+\beta)$.

For $y\in[y_0-\delta,y_0+\delta]$, where $\delta=min\{\alpha,\beta\}$, we have $y\in[y_0-\alpha,y_0+\beta]$.

Since f^{-1} is strictly increasing, then $f^{-1}(y)\in\left[f^{-1}(y_0-\alpha),f^{-1}(y_0+\beta)\right]$ and we then have $f^{-1}(y)\in[x_0-\varepsilon,x_0+\varepsilon]$, that is, $f^{-1}(y)\in\left[f^{-1}(y_0)-\varepsilon,f^{-1}(y_0)+\varepsilon\right]$. This shows that the function f^{-1} is continuous on J. ■

Theorem 3.3.21 *:*

Let $f:I\longrightarrow J$ be a continuous bijective function differentiable at a point $x_0\in I$ and if $f(x_0)\neq 0$, then the function $f^{-1}:J\longrightarrow I$ is differentiable at $y_0=f(x_0)$ and

$$\begin{aligned}\left(f^{-1}\right)'(y_0) &= \frac{1}{f'(x_0)}\\ &= \frac{1}{f'\left(f^{-1}(y_0)\right)}.\end{aligned}$$

Proof ::

Let $y_0\in J$ and $x_0\in I$ be such that $y_0=f(x_0)$. The rate of change of the function f^{-1} with respect to y_0 is:

$$\begin{aligned}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} &= \frac{x-x_0}{f(x)-f(x_0)}\\ &= \frac{1}{\dfrac{f(x)-f(x_0)}{x-x_0}}.\end{aligned}$$

When $y\longrightarrow y_0$ (with $y\neq y_0$), $x=f^{-1}(y)\longrightarrow f^{-1}(y_0)=x_0$ because f^{-1} is continuous in y_0 and by limit composition we have: $\dfrac{1}{\dfrac{f(x)-f(x_0)}{x-x_0}}\longrightarrow\dfrac{1}{f'(x_0)}$. ■

Example 3.3.22 *:*

The function f from $\mathbb{R}_+^\star$ to $]1,+\infty[$ defined by $f(x)=x^2+1$ is a continuous bijective function differentiable at every point $x_0\in\mathbb{R}_+^\star$ and its inverse function $f^{-1}:]1,+\infty[$ defined by $f(y)=\sqrt{y-1}$ is differentiable at every point $y_0=f(x_0)\in]1,+\infty[$ and $\left(f^{-1}\right)'(y_0)=\dfrac{1}{f'\left(f^{-1}(y_0)\right)}=\dfrac{1}{2f^{-1}(y_0)}=\dfrac{1}{2\sqrt{y_0-1}}$.

3.3.9 Successive derivatives

Definition 3.3.23 *:*

Let $f:I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function, and f' its derivative.

If the function $f':I\subset\mathbb{R}\longrightarrow\mathbb{R}$ is also differentiable, we denote $f''=(f')'$ the second derivative of f.

More generally, we denote: $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, *and* $f^{(n+1)} = \left(f^{(n)}\right)'$. *If the n-th derivative of* f ($f^{(n)}$) *exists, we say that* f *is n times differentiable.*

Examples 3.3.24 *:*

By proof by induction, we can verify that the n^{th} *derivatives on* $\mathbb{R}$ *of these functions* $f : x \longmapsto cos(ax+b)$, $g : x \longmapsto sin(ax+b)$ *et* $h : x \longmapsto (ax+b)^p$ *are*

$$f^{(n)}(x) = a^n cos(ax+b+n\frac{\pi}{2}), \quad g^{(n)}(x) = a^n sin(ax+b+n\frac{\pi}{2})$$

and

$$h^{(n)}(x) = \begin{cases} 0, & \text{if} \quad n > p \\ a^n n!, & \text{if} \quad n = p \\ a^n p.(p-1)\cdots(p-n+1)(ax+b)^{p-n}, & \text{if} \quad n < p. \end{cases}$$

And the n^{th} *derivative on* $\mathbb{R} - \left\{-\frac{b}{a}\right\}$ *of the function* $J : x \longmapsto \frac{1}{ax+b}$ *is*

$$J^{(n)}(x) = (-1)^n n!.a^n \,(ax+b)^{-n-1}.$$

Similarly, we verify that the nth derivative on $]-\frac{b}{a}, +\infty[$ *of the function* $k : x \longmapsto ln(ax+b)$ *is*

$$k^{(n)}(x) = (-1)^{n-1}(n-1)!.a^{n-1}\,(ax+b)^{-n}.$$

Class of a function

Definition 3.3.25 *:*

• *A function* $f : I \longrightarrow \mathbb{R}$ *is said to be of class* C^n *(with* $n \in \mathbb{N}$*) if it is continuous and its derivative of order n exists and is continuous.*

We denote by $C^n(I,\mathbb{R})$ *the set of real functions defined on I and n times continuously differentiable.*

• *A function* $f : I \longrightarrow \mathbb{R}$ *is said to be of class* C^∞ *if it is infinitely differentiable on I.*

We denote by $C^\infty(I,\mathbb{R})$ *the set of real functions infinitely differentiable on I.*

Thus, $C^\infty(I,\mathbb{R}) = \bigcap_{n\in\mathbb{N}} C^n(I,\mathbb{R})$.

Examples 3.3.26 *:*

(1) *The function* f *defined by:*

$$\begin{array}{rccl} f: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} f(x) = x^2 cos(\frac{2}{x}), & \text{if } x \neq 0 \\ f(x) = 0, & \text{if } x = 0, \end{cases} \end{array}$$

is not of class C^1. *Because the function* f *differentiable on* $\mathbb{R}^*$ *with derivative* $f' : x \in \mathbb{R}^* \longmapsto 2x.cos(\frac{2}{x}) + 2sin(\frac{2}{x})$. *Since* $\lim_{x \longrightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \longrightarrow 0} x.cos(\frac{2}{x}) = 0 = f'(0)$, *then* f *is differentiable in* 0, *on the other hand the function* f' *has no limit in* 0, *therefore the function* f' *is not continuous in* 0.

(2) *The function* g *defined by:*

$$\begin{array}{rccl} g: & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \begin{cases} g(x) & = x^3 sin(\frac{1}{x}), \quad \text{if } x \neq 0 \\ f(x) & = 0, \quad\quad\quad\quad \text{if } x = 0, \end{cases} \end{array}$$

is of class C^1. *Because the function* g *differentiable on* $\mathbb{R}$ *with derivative* $g^{'} : x \in \mathbb{R}^* \longmapsto 3x.sin(\frac{1}{x}) - x.cos(\frac{1}{x})$, *and* $g^{'}(0) = \lim_{x \longrightarrow 0} \frac{g(x) - g(0)}{x} = \lim_{x \longrightarrow 0} x^2.cos(\frac{2}{x}) = 0$, *and since* $\lim_{x \longrightarrow 0} g^{'}(x) = 0 = g(0)$, *then* $g^{'}$ *is continuous at* 0.

(3) *The functions sin, cos, exp and polynomials are of class* C^∞ *on* $\mathbb{R}$. *The functions ln and* $x \longmapsto \sqrt{x}$ *are of class* C^∞ *on* $\mathbb{R}_+^*$.

Proposition 3.3.27 *:*

Let $n \in \mathbb{N}$ *and* $\lambda \in \mathbb{R}$. *If* $f, g : I \longrightarrow \mathbb{R}$ *are* n *times differentiable on* I, *then* $\lambda \times f$, $f + g$, *and* $f \times g$ *are* n *times differentiable on* I, *and for all* $x \in I$, *we have:*

$$(\lambda \times f)^{(n)}(x) = \lambda \times f^{(n)}(x), \qquad (f+g)^{(n)}(x) = f^{(n)}(x) + g^{(n)}(x)$$

and

$$(f \times g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) \times g^{(k)}(x) \quad (\textbf{\textit{Leibniz's formula}}),$$

where

$$C_n^k = \frac{n!}{k!\,(n-k)!}.$$

Furthermore, if g *does not vanish on* I, *then* $\frac{f}{g}$ *is* n *differentiable times on* I.

Proof :

These properties can be proven by induction. Let's show, for example, Leibniz's formula.
Let $P(n)$ be the property $(f \times g)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x) \times g^{(k)}(x)$.
For $n = 1$ the property $P(1)$ is true, because

$$(f \times g)^{'}(x) = f^{'}(x) \times g(x) + f(x) \times g^{'}(x)$$

and

$$\begin{aligned}\sum_{k=0}^{1} C_1^k f^{(1-k)}(x) \times g^{(k)}(x) &= C_1^0 f^{(1-0)}(x) \times g^{(0)}(x) + C_1^1 f^{(1-1)}(x) \times g^{(1)}(x)\\ &= f^{(1)}(x) \times g^{(0)}(x) + f^{(0)}(x) \times g^{(1)}(x)\\ &= f'(x) \times g(x) + f(x) \times g'(x).\end{aligned}$$

Suppose that the property $P(n)$ is true for $n \in \mathbb{N}$. We will show that $P(n+1)$ is true. We have

$$\begin{aligned}(f \times g)^{(n+1)}(x) &= \left((f \times g)^{(n)}\right)'(x)\\ &= \left(\sum_{k=0}^{n} C_n^k f^{(n-k)}(x) \times g^{(k)}(x)\right)' \quad \textit{because } P(n) \textit{ is verified}\\ &= \sum_{k=0}^{n} C_n^k \left(f^{(n-k)}(x) \times g^{(k)}(x)\right)'\\ &= \sum_{k=0}^{n} C_n^k \left(f^{(n+1-k)}(x) \times g^{(k)}(x) + f^{(n-k)}(x) \times g^{(k+1)}(x)\right)\\ &= \sum_{k=0}^{n} C_n^k f^{(n+1-k)}(x) \times g^{(k)}(x) + \sum_{k=0}^{n} C_n^k f^{(n+1-(k+1))}(x) \times g^{(k+1)}(x)\\ &= \sum_{k=0}^{n} C_n^k f^{(n+1-k)}(x) \times g^{(k)}(x) + \sum_{k=1}^{n+1} C_n^{k-1} f^{(n+1-k)}(x) \times g^{(k)}(x)\\ &= C_n^{n+1} f^{(0)}(x) \times g^{(n+1)}(x) + \sum_{k=0}^{n} C_n^k f^{(n+1-k)}(x) \times g^{(k)}(x)\\ &\quad + \sum_{k=1}^{n+1} C_n^{k-1} f^{(n+1-k)}(x) \times g^{(k)}(x) + C_n^{-1} f^{(n+1)}(x) \times g^{(0)}(x)\\ &= \sum_{k=0}^{n+1} C_n^k f^{(n+1-k)}(x) \times g^{(k)}(x) + \sum_{k=0}^{n+1} C_n^{k-1} f^{(n+1-k)}(x) \times g^{(k)}(x)\\ &= \sum_{k=0}^{n+1} \left[C_n^{k-1} + C_n^k\right] f^{(n+1-k)}(x) \times g^{(k)}(x)\\ &= \sum_{k=0}^{n+1} C_{n+1}^k f^{(n+1-k)}(x) \times g^{(k)}(x) \quad (\textit{car } C_n^{k-1} + C_n^k = C_{n+1}^k).\end{aligned}$$

This implies that the property is hereditary, so $P(n)$ is true for all $n \in \mathbb{N}$. ∎

Example 3.3.28 *:*

Let's calculate the n^{th} derivative of the function $f : x \longmapsto (x^2+3x)e^{2x}$. Since f is the product of two infinitely differentiable functions on $\mathbb{R}$, then it is also infinitely differentiable.

By Leibniz's formula, we have:

$$f^{(n)}(x) = \sum_{k=0}^{n} C_n^k (x^2+3x)^{(k)} \left(e^{2x}\right)^{n-k}.$$

Since, for all $k \in \mathbb{N}^* - \{1,2\}$, $(x^2+3x)^{(k)} = 0$. Then, we have

$$\begin{aligned} f^{(n)}(x) &= C_n^0(x^2+3x)^{(0)}\left(e^{2x}\right)^{(n)} + C_n^1(x^2+3x)^{(1)}\left(e^{2x}\right)^{(n-1)} + C_n^2(x^2+3x)^{(2)}\left(e^{2x}\right)^{(n-2)} \\ &= (x^2+3x)2^n e^{2x} + n(2x+3)2^{n-1}e^{2x} + n(n-1)2^{n-2}e^{2x} \\ &= 2^{n-2}\left(4x^2+(4n+12)x)+n(n+5)\right)e^{2x}. \end{aligned}$$

Remark 3.3.29 *:*

We say that a function f *is differentiable on an interval* I *when* f *is differentiable in any point of* I.

Theorem 3.3.30 *:*

If f has a derivative at x_0, then f is continuous at x_0.

Proof :

Suppose that f is differentiable at x_0, then we have:

$$\begin{aligned} \lim_{h\longrightarrow 0}\left(f(x_0+h)-f(x_0)\right) &= \lim_{h\longrightarrow 0}\tfrac{f(x_0+h)-f(x_0)}{h}\times h \\ &= \lim_{h\longrightarrow 0}\tfrac{f(x_0+h)-f(x_0)}{h}\times \lim_{h\longrightarrow 0} h \\ &= f^{'}(x_0).0 \\ &= 0, \end{aligned}$$

which implies that f is continuous at x_0. ■

Proposition 3.3.31 *:*

Let f be a differentiable real function on an Intervale $I \subseteq \mathbb{R}$ we have:

1 $\forall x \in I, f^{'}(x) = 0$ *if and only if f is constant on I.*

2 *If* $\forall x \in I, f^{'}(x) \geq 0$ ($f^{'}(x) > 0$), *then f is increasing (resp strictly increasing).*

3 *If* $\forall x \in I, f^{'}(x) \leq 0$ ($f^{'}(x) < 0$), *then f is decreasing (resp strictly decreasing).*

3.3.10 Theorem of finite increases

Extrema (Local and Absolute)

Definition 3.3.32 *:*

Let f be a real function defined on a set $D \subset \mathbb{R}$.

- *We say that $x_0 \in D$ is a* ***critical point*** *of f if $f^{'}(x_0) = 0$.*
- *We say that f admits a local maximum (resp. a local minimum) at $x_0 \in D$ if there exists an open interval $I \subset D$ containing x_0 such that for all $x \in I$; $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$).*
- *We say that f admits a* ***local extremum*** *at x_0 if f admits a local maximum or a local minimum at this*

point.

• *We say that f admits a* ***global maximum*** *(resp. a* ***global minimum*** *) at $x_0 \in D$ if for all $x \in D$; $f(x) \leq f(x_0)$ (resp. $f(x) \geq f(x_0)$).*

Proposition 3.3.33 *:*

Let f be a real differentiable function on an open interval I containing x_0.

1• *If f has a local maximum (or a local minimum) at x_0 then $f^{'}(x_0) = 0$.*

2• *If $f^{'}(x_0) = 0$ and the function $f^{'}$ changes sign at x_0, then f has a local extremum at x_0.*

Proof :

1• Suppose that f has a local maximum at x_0, then there exists an open interval I containing x_0 such that for all $x \in I, f(x) \leq f(x_0)$. It follows that, for $x \in I$;
$\otimes$ if $x < x_0$ then $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$ and $\otimes$ if $x > x_0$ then $\dfrac{f(x) - f(x_0)}{x - x_0} \leq 0$.
By passing to the limit, and since f is a differentiable function at x_0, we have
$\otimes$ $f_g^{'}(x_0) = \lim\limits_{\substack{x \longrightarrow x_0 \\ <}} \dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$ and $\otimes$ $f_d^{'}(x_0) = \lim\limits_{\substack{x \longrightarrow x_0 \\ >}} \dfrac{f(x) - f(x_0)}{x - x_0} \leq 0$ and $f_g^{'}(x_0) = f_d^{'}(x_0) = f^{'}(x_0)$.
So the only possibility is that $f^{'}(x_0) = 0$.

2• uppose that $f^{'}(x_0) = 0$ and there exists $\alpha > 0$ such that

$$\forall x \in]x_0 - \alpha; x_0];\ f^{'}(x) \leq 0\ \wedge\ \forall x \in [x_0; x_0 + \alpha[;\ f^{'}(x) \geq 0.$$

This implies, $\forall x \in]x_0 - \alpha; x_0];\ f(x) \geq f(x_0)\ \wedge\ \forall x \in [x_0; x_0 + \alpha[;\ f(x) \geq f(x_0)$, In other words, for all $x \in]x_0 - \alpha; x_0 + \alpha[,\ f(x) \geq f(x_0)$. Therefore, the function f has a local extremum at x_0. ■

3.3.11 Rolle's Theorem

Theorem 3.3.34 *:*

Let $a, b \in \mathbb{R}$ with $a < b$.

If $f : [a, b] \longrightarrow \mathbb{R}$ a function continues on $[a, b]$, differentiable on $]a, b[$ and $f(a) = f(b)$, then it exists $c \in\ ]a, b[$ such that $f^{'}(c) = 0$.

Proof :

• Firstly, if f is constant on $[a, b]$, then for all $c \in\ ]a, b[$; $f^{'}(c) = 0$.

• If f is not constant on $[a, b]$, there exists $x_0 \in [a, b]$ such that $f(x_0) \neq f(a)$.

For example, suppose $f(x_0) > f(a)$. Since f is a continuous function on a closed and bounded interval, it has a maximum at a point $c \in [a,b]$.

Since $\max\limits_{x\in[a,b]} f(x) = f(c)$, then $f(c) \geq f(x_0) > f(a)$, so $c \neq a$. Similarly, since $f(a) = f(b)$, then $c \neq b$. Thus, $c \in]a,b[$. f is therefore differentiable and admits a (local) maximum at c so, according to the proposition **3.3.33** we have $f'(c) = 0$. ■

3.3.12 Graphical interpretation of Rolle's theorem

At the point $M_0(x_0, f(x_0))$ the graphical representation of f admits a horizontal tangent.

Theorem of finite increasing

Theorem 3.3.35 *If f a continuous real function on the closed, bounded interval $[a,b] \subset \mathbb{R}$ and differentiable on the open interval $]\,a,b\,[$. Then, there exists c in $]\,a,b\,[$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof :

Consider the auxiliary function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

The function g is continuous on $[a,b]$ and differentiable on $]\,a,b\,[$. Indeed, it is the sum of two functions continuous on $[a,b]$ and differentiable on $]\,a,b\,[$.

Furthermore, we have

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \text{ and } g(a) = g(b).$$

Applying Rolle's theorem to the function g, we deduce that there exists a c in $]\,a,b\,[$such that $g'(c) = 0$, which completes the proof. ■

Example 3.3.36 *:*

Let's show that for all $x > 0$, $\frac{x}{x+1} < ln(1+x) < x$. Consider the function $t \longmapsto ln(1+t)$ on the interval $[0,x]$ with $x > 0$. This function is of class C^∞ on $]-1;+\infty[$. By the Mean Value Theorem, there exists $c \in\ ]0,x[$ such that $ln(1+x) - ln(1) = \frac{x}{1+c}$. Since $0 < c < x$, we have $\frac{1}{1+x} < \frac{1}{1+c} < 1$ and therefore $\frac{x}{1+x} < ln(1+x) < x$.

Inequalities of the finite increasing

Proposition 3.3.37 :

Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function on an open interval I. If there exists a constant M such that for all $x \in I$; $\left|f^{'}(x)\right| \leq M$ then $\forall x, y \in I$; $|f(x) - f(y)| \leq M|x - y|$.

Proof :

Let $x, y \in I$ with $x < y$. Then f is continuous on $[x, y]$ and differentiable on $]\,x, y\,[$. By the Mean Value Theorem, there exists $c \in\,]\,x, y\,[$ such that $f(x) - f(y) = f'(c)\,(x - y)$ and since $\left|f^{'}(x)\right| \leq M$, then $|f(x) - f(y)| = \left|f^{'}(c)\right| |x - y| \leq M|x - y|$. This inequality also holds for $y < x$. ■

Proposition 3.3.38 :

Let $f : I \longrightarrow \mathbb{R}$ be differentiable. If there exists $m, M \in \mathbb{R}$ such that $\forall x \in I$; $m \leq f'(x) \leq M$, then for all $a, b \in I$ such that $a < b$, we have

$$m(b - a) \leq f(b) - f(a) \leq M(b - a).$$

Proof :

Case $a = b$. This is immediate.

Case $a < b$.

Since f is differentiable on I, then the function f is continuous on $[a, b]$ and differentiable on $]a, b[$. By the Mean Value Theorem, there exists $c \in]a, b[$ such that $f(b) - f(a) = f^{'}(c)(b - a)$ and since $m \leq f^{'}(x) \leq M$ and $b - a > 0$, we deduce that $m(b - a) \leq f(b) - f(a) \leq M(b - a)$. ■

Examples 3.3.39 :

Since the derivatives of sin and cos are bounded by 1 on $\mathbb{R}$, then:

$$\forall x, y \in \mathbb{R};\ |sinx - siny| \leq |x - y| \ \text{ and } \forall x, y \in \mathbb{R};\ |cosx - cosy| \leq |x - y|\,.$$

In particular, if we fix $y = 0$, then we obtain $|sinx| \leq |x|$.

3.3.13 L'Hôpital's rule

Corollary 3.3.40 *Let f and g be two differentiable functions on an open interval I containing x_0 such that $f(x_0) = g(x_0) = 0$. If $\forall x \in I - \{x_0\}$; $g^{'}(x) \neq 0$,*

$$\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \frac{f^{'}(x_0)}{g^{'}(x_0)}.$$

Proof :

Suppose that f and g are two differentiable functions on I, $f(x_0) = g(x_0) = 0$ and $\forall x \in I - \{x_0\}$; $g^{'}(x_0) \neq 0$.

Consider the function $\psi : I \longrightarrow \mathbb{R}$ defined by $\psi(x) = g(t)f(x) - f(t)g(x)$ with $t \in I - \{x_0\}$ fixed such that $t < x_0$. Since the function ψ continues on $[t, x_0]$, is differentiable on $]t, x_0[$, and $\psi(x_0) = 0 = \psi(t)$, then, by Rolle's theorem, there exists $c_t \in]t, x_0[$ such that $\psi'(c_t) = 0$. Now, $\forall x \in I$;

$psi^{'}(x) = g(t)f^{'}(x) - f(t)g^{'}(x)$, therefore $g(t)f^{'}(c_t) - f(t)g^{'}(c_t) = 0$. Since $g^{'}$ does not vanish on $I - \{x_0\}$, this leads to $\dfrac{f(t)}{g(t)} = \dfrac{f^{'}(c_t)}{g^{'}(c_t)}$. Since $t < c_t < x_0$, when we let t tend towards x_0, we obtain $c_t \longrightarrow x_0$. This implies

$$\lim_{t \longrightarrow x_0} \frac{f(t)}{g(t)} = \lim_{c_t \longrightarrow x_0} \frac{f^{'}(c_t)}{g^{'}(c_t)}.$$

■

Example 3.3.41 *:*

Let's calculate the limit in 1 *of* $h(x) = \dfrac{e^{x^2+2x} - e^{3x}}{cos(\frac{\pi}{2}x)}$.

Let $f(x) = e^{x^2+2x} - e^{3x}$ *and* $g(x) = cos(\frac{\pi}{2}x)$, *then* $f(1) = 0 = g(1)$ *and for all* $\forall x \in \mathbb{R}$ *we have:*

$$f^{'}(x) = (2x+2)\, e^{x^2+2x} - 3e^{3x} \text{ and } g^{'}(x) = -\frac{\pi}{2}.sin(\frac{\pi}{2}x).$$

Let's take $I = \,]\,0, 2\,[$, *then* $g^{'}$ *does not vanish over* I. *Therefore*

$$\lim_{x \longrightarrow 1} \frac{e^{x^2+2x} - e^{3x}}{cos(\frac{\pi}{2}x)} = \frac{f^{'}(1)}{g^{'}(1)} = \frac{e^3}{-\frac{\pi}{2}} = \frac{-2e^3}{\pi}.$$

Theorem 3.3.42 ***L'Hôpital's[1] rule***

Let f *and* g *be two numerical functions differentiable on* I *and both tending to* 0 *(or both tend to* ∞*) when* $x \longrightarrow x_0$ *(or* ∞*). We assume that* $g^{'}(x) \neq 0$ *does not vanish in a neighborhood of* x_0 *and that:* $\lim_{x \to x_0} \dfrac{f^{'}(x)}{g^{'}(x)}$ *exists and finite . Then,*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f^{'}(x)}{g^{'}(x)}.$$

In other words, If the limit of $\frac{f(x)}{g(x)}$ *is of indeterminate type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *for :* $x \to x_0$ *or* ∞ *then we have:* $\lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \lim_{x \to x_0} \dfrac{f^{'}(x)}{g^{'}(x)}$.

[1] Guillaume de l'Hôpital 1661-1704, student of Johann Bernoulli

Remark 3.3.43 *To use the rule, you take the derivative of the numerator and the derivative of the denominator separately and find the limit of that new fraction. If the new limit is still indeterminate, you can repeat the process.*

3.3.14 Using L'Hôpital's rule to calculate limits

Limit	**Indeterminate form**	**Transformation to $\frac{0}{0}$**	**Hôpital's rule**
$lim_{x\to a}\frac{f(x)}{g(x)}$	$\frac{\infty}{\infty}$	$lim_{x\to a}\frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$	$lim_{x\to a}\frac{\left(\frac{1}{g(x)}\right)'}{\left(\frac{1}{f(x)}\right)'}$
$lim_{x\to a} f(x).g(x)$	$0.\infty$	$lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$	$lim_{x\to a}\frac{(f(x))'}{\left(\frac{1}{f(x)}\right)'}$
$lim_{x\to a}(f(x)-g(x))$	$\infty-\infty$	$lim_{x\to a}\left(\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x).g(x)}}\right)$	$lim_{x\to a}\frac{\left(\frac{1}{g(x)}-\frac{1}{f(x)}\right)'}{\left(\frac{1}{f(x).g(x)}\right)'}$
$lim_{x\to a} f(x)^{g(x)}$	$(0^{>})^{0}$	$e^{lim_{x\to a}\left(\frac{g(x)}{\frac{1}{lnf(x)}}\right)}$	$e^{lim_{x\to a}\left(\frac{(g(x))'}{\left(\frac{1}{lnf(x)}\right)'}\right)}$
$lim_{x\to a} f(x)^{g(x)}$	1^{∞}	$e^{lim_{x\to a}\left(\frac{lnf(x)}{\frac{1}{g(x)}}\right)}$	$e^{lim_{x\to a}\left(\frac{(lnf(x))'}{\left(\frac{1}{g(x)}\right)'}\right)}$
$lim_{x\to a} f(x)^{g(x)}$	∞^{0}	$e^{lim_{x\to a}\left(\frac{g(x)}{\frac{1}{lnf(x)}}\right)}$	$e^{lim_{x\to a}\left(\frac{(g(x))'}{\left(\frac{1}{lnf(x)}\right)'}\right)}$

Examples 3.3.44 :

1$*$ *Let's find, for example,* $lim_{x\to 1}(1+sin(x-1))^{\frac{1}{x-1}}$. *By replacing* x *with* 1 *into* $(1+sin(x-1))^{\frac{1}{x-1}}$ *we obtain the indeterminate form* 1^{∞} , *that is* $(1\pm\varepsilon)^{\infty}$ *where* ε *is very close to zero. To remove this indeterminate form, we can use L'Hospital's rule as follows.*

Let's put $f(x) = (1+sin(x-1))^{\frac{1}{x-1}}$ *then we caculate limit of* $lim_{x\to 1} ln(f(x))$.

We have $lim_{x\to 1} ln(f(x)) = lim_{x\to 1}\frac{ln(1+sin(x-1))}{x-1}$.

Substituting $x = 0$ *into the above limit gives the indeterminate form* $\frac{0}{0}$, *so we can now apply L'Hôpital's rule to calculate this limit.*

$$\lim_{x\to 1} ln(f(x)) = \lim_{x\to 1}\frac{(ln(1+sin(x-1)))'}{(x-1)'} = \lim_{x\to 1}\frac{\frac{cos(x-1)}{1+sin(x-1)}}{1} = 1.$$

We found that $lim_{x\to 1} ln(f(x)) = 1$, *which means* $lim_{x\to 1} ln(f(x)) = ln(lim_{x\to 1} f(x)) = 1$, *because function* ln *is continuous. This imply that* $lim_{x\to 1} f(x) = e$.

2$*$ *Some limits, which do not appear as limits of quotients, can be obtained using this rule: for example,*

$lim_{x\to-\infty}\left(x+\sqrt{x^2+x}\right)$.

By replacing x by $-\infty$ into $x+\sqrt{x^2+x}$ we obtain the indeterminate form $-\infty+\infty$. We can convert this indeterminate shape to an indeterminate shape $\frac{0}{0}$, as follows :

$lim_{x\to-\infty}\left(x+\sqrt{x^2+x}\right) = lim_{x\to-\infty}\frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}}$.

Then we use L'Hopital's rule :

$$\begin{aligned}
lim_{x\to-\infty}\frac{1-\sqrt{1+\frac{1}{x}}}{\frac{1}{x}} &= lim_{x\to-\infty}\frac{\left(1-\sqrt{1+\frac{1}{x}}\right)'}{\left(\frac{1}{x}\right)'}\\
&= lim_{y\underset{<}{\to}0}\frac{\left(1-\sqrt{1+y}\right)'}{(y)'}\\
&= lim_{y\underset{<}{\to}0}\frac{-\frac{1}{2\sqrt{1+y}}}{1} = -\frac{1}{2}.
\end{aligned}$$

3$*$ *Here is an example involving the indeterminate form $0.\infty$, which is rewritten as the form $\frac{\infty}{\infty}$, for example :*

$$\begin{aligned}
lim_{x\underset{>}{\to}0}\sin(x)\,ln(\sin(x)\,) &= lim_{x\underset{>}{\to}0}\frac{ln(sin(x))}{\frac{1}{sin(x)}}\\
&= lim_{x\underset{>}{\to}0}\frac{(ln(sin(x)\,))'}{\left(\frac{1}{sin(x)}\right)'}\\
&= lim_{x\underset{>}{\to}0}\frac{\frac{cos(x)}{sin(x)}}{\frac{-cos(x)}{sin^2(x)}} = -\,lim_{x\underset{>}{\to}0}\sin(x) = 0.
\end{aligned}$$

Caution: Note that we cannot use L'Hôpital's rule "in reverse" to deduce that $\frac{f'}{g'}$ has a limit if $\frac{f}{g}$ has a limit.

Example 3.3.45 *:*

Let $\frac{f(x)}{g(x)} = \frac{ln(x)+sin\ x}{ln(x)}$. Then limit of $\frac{f(x)}{g(x)}$ is indeterminate type $\frac{\infty}{\infty}$ as $x\to\infty$ and $lim_{x\to+\infty}\frac{f(x)}{g(x)} = lim_{x\to+\infty}\left(1+\frac{sin\ x}{ln(x)}\right) = 1$,

but the limit $lim_{x\to+\infty}\frac{f'(x)}{g'(x)} = lim_{x\to+\infty}\left(\frac{\frac{1}{x}+cos\ x}{\frac{1}{x}}\right) = lim_{x\to+\infty}(1+xcos\ x)$ does not exist.

3.3.15 Exercises

Exercise 3.3.46 *:*

1. *For what value(s) of the real number α is the function f defined by:*

$$\begin{cases} h(x) = \dfrac{x-\alpha}{4x^2+4} & if \quad x>0\\ h(x) = \beta x-4 & if \quad x<0. \end{cases}$$

Is it extendable by continuity at 0? We will then specify this extension, which we will denote by $\widetilde{h}$.

2. *For what value of β is the function $\widetilde{h}$ differentiable at 0?*

Exercise 3.3.47 :

Determine the n^{th} derivative of the given function in each of the following two cases:

$$a)x \longmapsto \frac{x^2+2}{(x-2)^2} \qquad b)x \longmapsto sin(2x+1).cos^2x.$$

Exercise 3.3.48 :

Using the theory of finite increases, prove the following:

$$\forall x > 0;\ \frac{1}{x+1} < ln\frac{x+1}{x} < \frac{1}{x}.$$

Deduce that for all strictly positive real numbers x, we have :

$$\left(1+\frac{1}{x}\right)^x < e < \left(1+\frac{1}{x}\right)^{x+1}.$$

Exercise 3.3.49 :

Let f be a twice differentiable function on the interval $[a,b]$, such that $f(a) = f(b) = 0$ and for all $x \in \,]\,a,b\,[$ we have $f''(x) \leq 0$. Show that $\forall x \in [a,b]\,;\ f(x) \geq 0$.

Exercise 3.3.50 :

By the theory of finite increases, show that:

$$1)\ \forall x > 0;\ x - \frac{x^3}{6} \leq sinx \leq x. \qquad 2)\ \forall x > 0;\ x + \frac{x^3}{3} \leq tanx.$$

$$3)\ \forall x > 0;\ x - \frac{x^2}{2} \leq ln\,(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}. \qquad 4)\ \forall x \in \mathbb{R}^*;\ e^x \geq 1+x.$$

Exercise 3.3.51 :

Let f, g be two continuous functions on $[a,b]$ differentiable on $]\,a,b\,[$, not equal to zero, and such that $f(a)g(b) = f(b)g(a)$ Show that there exists a $c \in \,]\,a,b\,[$ such that $\frac{f'(c)}{f(c)} = \frac{g'(c)}{g(c)}$.

Exercise 3.3.52 :

(1) Show that:

$$\forall x \geq 0;\ x - \frac{x^2}{2} \leq log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}.$$

(Only one derivative will be calculated for each inequality).

(2) Deduce that:

$$\lim_{x \longrightarrow 0} \frac{log(1+x)}{x} = 1.$$

Chapter 4

Application to elementary functions

4.1 Inverse trigonometric functions

4.1.1 Trigonometry formulas

For all $\alpha, \beta \in \mathbb{R}$;

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha.\cos\beta - \sin\alpha.\sin\beta. \\ \sin(\alpha+\beta) &= \sin\alpha.\cos\beta + \sin\beta.\cos\alpha. \end{aligned} \tag{4.1}$$

Remark 4.1.1 *:*

1). For $\cos(\alpha-\beta)$ *and* $\sin(\alpha-\beta)$ *by replacing* β *with* $-\beta$ *in the formulas* (4.1).

2). By taking $\alpha=\beta$ *in* (4.1), *and using*

$$\forall x \in \mathbb{R};\ \cos^2\alpha + \sin^2\alpha = 1, \tag{4.2}$$

we obtain

$$\cos(2.\alpha) = 2\cos^2\alpha - 1 = 1 - 2.\sin^2\alpha \ \ \text{and} \ \ \sin 2\alpha = 2\sin\alpha.\cos\alpha. \tag{4.3}$$

By setting $a=\alpha+\beta$ and $b=\alpha-\beta$ we obtain for all $a, b \in \mathbb{R}$,

$$\cos a + \cos b = 2.\cos\frac{a+b}{2}.\cos\frac{a-b}{2}. \tag{4.4}$$

$$\sin a + \sin b = 2.\sin\frac{a+b}{2}.\cos\frac{a-b}{2}. \tag{4.5}$$

Remark 4.1.2 :

1). For $\cos a - \cos b$*, replace* β *with* $\pi - \beta$ *in the formula* (4.4).

2). For $\sin a - \sin b$*, replace* β *with* $-\beta$ *in the formula* (4.5).

Half angle formulas

For all $x \in \mathbb{R} - \{(2k+1)\,\pi/k \in \mathbb{Z}\}$ and for $t = tan\frac{x}{2}$ we have

$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2},$$

and when $x \neq (2k+1)\,\frac{\pi}{2}/k \in \mathbb{Z}$ we have

$$\tan x = \frac{2t}{1-t^2}$$

4.1.2 arc-sinus function

The restriction of sin to $I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ to the interval $[-1,1]$, is a continuous and strictly increasing function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, therefore it establishes a bijection and consequently

$$\forall x \in [-1,1], \exists! t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \; ; \; \sin t = x. \tag{4.6}$$

Definition 4.1.3 :

For any $x \in [-1,1]$*, we denote by* $\arcsin x$ *or* $\sin^{-1} x$ *the unique angle in the interval* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *whose sine is equal* x*.*

This defines a function $\arcsin : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *which is the inverse bijection of the restriction* $\sin_{|_I} : I = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1,1]$.

Properties 4.1.4 :

• *For all* $x \in [-1,1]$*, we have the following simplifications*

$$\sin(\arcsin x) = x, \qquad \cos(\arcsin x) = \sqrt{1-x^2}.$$

And when $x \in]-1,1[$

$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}.$$

• *For all* $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, *we have :*

$$\arcsin(\sin x) = x. \tag{4.7}$$

$$\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \forall y \in [-1,1];\ \sin x = y \Longleftrightarrow x = \arcsin y. \tag{4.8}$$

Proposition 4.1.5 *:*

The function arcsin is odd, strictly increasing, continuous on $[-1,1]$, *of class* C^∞ *on* $]-1,1[$ *and* $\forall x \in]-1,1[;\ (\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}$.

Proof :

The function arcsin is odd, because $\forall y \in [-1,1], \exists -y \in [-1,1]$ and

$$\begin{aligned} \forall y \in [-1,1], \exists! x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];\ \arcsin(-y) &= \arcsin(\sin(-x)) \quad (\textit{ according to } (4.6)) \\ &= -x \quad (\textit{ according to } (4.7)) \\ &= -\arcsin(y) \quad (\textit{ according to } (4.8)). \end{aligned}$$

According to the theorems (3.3.20) and (3.3.21), the function arcsin is strictly increasing, continuous on $[-1,1]$, of class C^∞ on $]-1,1[$ and for all $x \in]-1,1[$ we have

$$\begin{aligned} \arcsin'(x) &= \frac{1}{\sin'(\arcsin(x))} \\ &= \frac{1}{\cos(\arcsin(x))} \\ &= \frac{1}{\sqrt{1-\sin^2(\arcsin(x))}} \\ &\quad (\textit{ because } \cos^2(\arcsin x) + \sin^2(\arcsin x) = 1 \textit{ since } \arcsin x \in]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[, \textit{ then } \cos(\arcsin x) > \\ &= \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

■

4.1.3 arc-cosinus function

The restriction $\cos_{|_J}: J = [0,\pi] \longrightarrow [-1,1]$, is a continuous and strictly increasing function on $[0,\pi]$, therefore it establishes a bijection and consequently

$$\forall x \in [-1,1], \exists! t \in [0,\pi];\ \cos t = x. \tag{4.9}$$

Definition 4.1.6 *:*

For any $x \in [-1,1]$, we denote by $\arccos x$ or $\cos^{-1} x$ the unique angle in the interval $[0,\pi]$ whose cosine is equal to x.

This defines a function $\arccos : [-1,1] \longrightarrow [0,\pi]$ which is the inverse bijection of the restriction $\cos_{|_J} : J = [0,\pi] \longrightarrow [-1,1]$.

Therefore, by definition of the inverse bijection:

$$\forall x \in [-1,1]\,;\ \cos(\arccos(x)) = x.$$
$$\forall x \in [0,\pi]\,;\ \arccos(\cos(x)) = x.$$

This means

$$\forall x \in [0,\pi]\,,\ \forall y \in [-1,1];\ \cos x = y \Longleftrightarrow x = \arccos y. \tag{4.10}$$

Proposition 4.1.7 *:*

The function $\arccos$ is strictly increasing, continuous on $[-1,1]$, and of class C^∞ on $]-1,1[$ and $\forall x \in]-1,1[;\ (\arccos)^{'}(x) = \dfrac{-1}{\sqrt{1-x^2}}$.

Proposition 4.1.8 *:*

$$\forall x \in [-1,1]\,;\ \arcsin x + \arccos x = \frac{\pi}{2}.$$

Proof :

Let f be the function defined on $[-1,1]$ by $f(x) = \arcsin x + \arccos x$. The function f is continuous on $[-1,1]$ and differentiable on $]-1,1[$ with derivative $f^{'}(x) = 0$. This clearly shows that f is a constant function on $[-1,1]$. Since $f(0) = \frac{\pi}{2}$, we deduce that $\forall x \in [-1,1]\,;\ f(x) = \frac{\pi}{2}$. ■

4.1.4 arc-tangente function

The restriction $\tan_{|_{I'}} : I' = \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\longrightarrow \mathbb{R}$ is a bijection. Its inverse bijection is the arctangent function

$$\begin{array}{rccl} \arctan : & \mathbb{R} & \longrightarrow & \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\\ & x & \longmapsto & \arctan x \ (\textit{ is the unique angle of the interval } \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\textit{ whose tangent is } x). \end{array}$$

And since the restriction $\tan_{|_{I'}}$ is bijective, we have:

$$\forall x \in \mathbb{R}, \exists! t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\,;\ \tan t = x. \tag{4.11}$$

Properties 4.1.9 *:*

• *By definition of the inverse bijection, we have*

$$\forall x \in \mathbb{R};\ \tan(\arctan x) = x.$$

$$x \in]-\frac{\pi}{2}, \frac{\pi}{2}[;\ \arctan(\tan x) = x. \tag{4.12}$$

This means

$$\forall x \in]-\frac{\pi}{2}, \frac{\pi}{2}[,\ \forall y \in \mathbb{R};\ \tan x = y \Longleftrightarrow x = \arctan y. \tag{4.13}$$

Proposition 4.1.10 *:*

The function arctan *is odd, strictly increasing, continuous, of class* C^∞ *on* $\mathbb{R}$ *and* $\forall x \in \mathbb{R};\ (\arctan)^{'}(x) = \dfrac{1}{1+x^2}$.

Proof :

The arctan function is odd, because for every $y \in \mathbb{R}$ there exists a unique $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ such that:

$$\begin{aligned}
\arctan(-y) &= \arctan(\tan(-x)) \quad (\textit{ according to } (4.11)) \\
&= -x \quad (\textit{ according to } (4.12)) \\
&= -\arctan(y) \quad (\textit{ according to } (4.13)).
\end{aligned}$$

According to the theorems (3.3.20) and (3.3.21) the function arctan is strictly increasing, continuous, of class C^∞ on $\mathbb{R}$, and for all $x \in \mathbb{R}$ we have

$$\begin{aligned}
\arctan^{'}(x) &= \frac{1}{\tan^{'}(\arctan(x))} \\
&= \frac{1}{1+\tan^2(\arctan(x))} \quad (\textit{ because } \forall t \in]-\frac{\pi}{2}, \frac{\pi}{2}[;\ \tan^{'}(t) = 1+\tan^2(t)) \\
&= \frac{1}{1+x^2}.
\end{aligned}$$

■

Proposition 4.1.11 *:*

For all $x \in \mathbb{R}$, *we have :*

$$a)\ \ \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}} \quad \textit{and} \quad b)\ \ \mathit{sin}(\arctan x) = \frac{x}{\sqrt{1+x^2}}.$$

Proof :

Let $x \in \mathbb{R}, \theta \in]-\dfrac{\pi}{2}, \dfrac{\pi}{2}[$ and $\theta = \arctan x$

$$\begin{aligned} a)\ \cos^2\theta &= \frac{\cos^2\theta}{\cos^2\theta + sin^2\theta} \\ &= \frac{\cos^2\theta}{\cos^2\theta\left(1+\dfrac{\sin^2\theta}{\cos^2\theta}\right)} \\ &= \frac{1}{1+\tan^2\theta} \quad (\textit{because}\ \forall\theta \in]-\frac{\pi}{2},\frac{\pi}{2}[;\ \cos^2\theta > 0). \end{aligned}$$

So $\forall x \in \mathbb{R}; \cos^2(\arctan x) = \dfrac{1}{1+x^2}$. This implies that

$$\forall x \in \mathbb{R};\ \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}.$$

$$\begin{aligned} b)\ \sin\theta &= \sqrt{1-\cos^2\theta} \\ &= \sqrt{1-\frac{\cos^2\theta}{\cos^2\theta+\sin^2\theta}} \\ &= \sqrt{1-\frac{\cos^2\theta}{\cos^2\theta\left(1+\dfrac{\sin^2\theta}{\cos^2\theta}\right)}} \\ &= \sqrt{1-\frac{1}{1+\tan^2\theta}} \\ &= \sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}. \end{aligned}$$

This shows that$\forall x \in \mathbb{R};\ sin(\arctan x) = \dfrac{x}{\sqrt{1+x^2}}$. ■

Exercise 4.1.12 *:*

Show that

$$\forall x \in \mathbb{R}_+^*;\ \arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \quad \textit{and} \quad \forall x \in \mathbb{R}_-^*;\ \arctan x + \arctan\left(\frac{1}{x}\right) = -\frac{\pi}{2}.$$

Exercise solution 4.1.12 :

Consider the function $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ defined by $f(x) = \arctan x + \arctan\left(\dfrac{1}{x}\right)$.

The function f is differentiable on $\mathbb{R}^*$ and $\forall x \in \mathbb{R}^*; f'(x) = \dfrac{1}{1+x^2} - \dfrac{\dfrac{1}{x^2}}{1+\dfrac{1}{x^2}} = 0.$

Therefore, the function f is constant on each interval of $\mathbb{R}^*$.

Since $f(1) = 2\arctan(1) = 2.\dfrac{\pi}{4} = \dfrac{\pi}{2}$, then $\forall x \in \mathbb{R}_+^*;\ \arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$.

And since f is odd and $f(-1) = 2\arctan(-1) = -\dfrac{\pi}{2}$ we then have $\forall x \in \mathbb{R}^*_-;\ \arctan x + \arctan\left(\dfrac{1}{x}\right) = -\dfrac{\pi}{2}$.

4.2 Hyperbolic Functions

4.2.1 Hyperbolic sine function

Definition 4.2.1 :

The hyperbolic sine function is the function denoted sinh or sh which is defined on $\mathbb{R}$ by:

$$shx = \frac{e^x - e^{-x}}{2}.$$

Properties 4.2.2 :

The function sh is odd, of class C^∞ on $\mathbb{R}$ and $sh^{'}(x) = chx$. Its image is $\mathbb{R}$.

Proof :

These properties can be shown using the properties of the exp function. ■

4.2.2 Hyperbolic cosine function

Definition 4.2.3 :

The hyperbolic cosine function is the function denoted cosh or ch which is defined on $\mathbb{R}$ by:

$$chx = \frac{e^x + e^{-x}}{2}.$$

Properties 4.2.4 :

- *The function ch is even, of class C^∞ on $\mathbb{R}$ and $ch^{'}(x) = shx$. Its image is $[1, +\infty[$*
- *For all $x \in \mathbb{R}$; $ch^2x - sh^2x = 1$.*

Proof :

These properties can be demonstrated using the properties of the exp function. ■

4.2.3 Hyperbolic tangent function

Definition 4.2.5 :

$$th(x) = \frac{sh(x)}{ch(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ or } th(x) = \frac{e^{2x} - 1}{e^{2x} + 1}.$$

Properties 4.2.6 :
The function th is odd, differentiable of class C^∞ on $\mathbb{R}$ and for all $x \in \mathbb{R}$; $th^{'}(x) = 1 - th^2x = \frac{1}{ch^2x}$. Its image is $]-1,1[$ that is $th(\mathbb{R}) =]-1,1[$.

Proof :

• The function th is of class C^∞ on $\mathbb{R}$. Because it is composed of functions also of class C^∞ on $\mathbb{R}$. And it is odd, Because for all $x \in \mathbb{R}$,

$$th(-x) = \frac{sh(-x)}{ch(-x)} = \frac{-sh(x)}{ch(x)} = -th(x).$$

• According to **theorem** (3.3.11), the function th differentiable on $\mathbb{R}$, and for all $x \in \mathbb{R}$, we have

$$th^{'}(x) = \left(\frac{sh}{ch}\right)^{'}(x) = \frac{ch^2(x) - sh^2(x)}{ch^2(x)} = \frac{1}{ch^2(x)} = 1 - th^2(x).$$

• For all $x \in \mathbb{R}$ we have

$$th(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{e^{2x} - 1}{1 + e^{2x}}.$$

And on the one hand, we have for all $x \in \mathbb{R}$

$$-\infty < 1 - e^{-2x} < 1 \quad and \quad 0 < \frac{1}{1 + e^{-2x}} < 1.$$

This implies that, for all $x \in \mathbb{R}$; $th(x) < \frac{1}{1 + e^{-2x}} < 1$.
And on the other hand, we have for all $x \in \mathbb{R}$

$$-1 < e^{2x} - 1 < +\infty \quad et \quad 0 < \frac{1}{1 + e^{2x}} < 1.$$

This implies that, for all $x \in \mathbb{R}$; $th(x) > -\frac{1}{1 + e^{2x}} > -1$.
This means that $x \in \mathbb{R}$; $-1 < th(x) < 1$. ■

4.2.4 Properties of Hyperbolic Trigonometry

For all $\alpha,\ \beta \in \mathbb{R}$ we have:

$$ch(\alpha + \beta) = ch\alpha.ch\beta + sh\alpha.sh\beta$$
$$ch(2\alpha) = ch^2\alpha + sh^2\alpha = 2ch^2\alpha - 1 = 1 + 2sh^2\alpha$$
$$sh(\alpha + \beta) = sh\alpha.ch\beta + sh\beta.ch\alpha$$
$$sh(2\alpha) = 2sh\alpha.ch\alpha$$
$$th(\alpha + \beta) = \frac{th\alpha + th\beta}{1 + th\alpha.th\beta}$$

Proof :

These relationships can be shown by simple calculation from the expression of hyperbolic functions. Let us show for example $sh(\alpha+\beta) = sh\alpha.ch\beta + sh\beta.ch\alpha$.

$$\begin{aligned} sh\alpha.ch\beta + sh\beta.ch\alpha &= \frac{e^{\alpha}-e^{-\alpha}}{2}.\frac{e^{\beta}+e^{-\beta}}{2} + \frac{e^{\beta}-e^{-\beta}}{2}.\frac{e^{\alpha}+e^{-\alpha}}{2} \\ &= \frac{e^{\alpha+\beta}+e^{\alpha-\beta}-e^{-\alpha+\beta}-e^{-\alpha-\beta}}{4} + \frac{e^{\alpha+\beta}+e^{\beta-\alpha}-e^{\alpha-\beta}-e^{-\alpha-\beta}}{4} \\ &= \frac{2e^{\alpha+\beta}-2e^{-\alpha-\beta}}{4} \\ &= sh(\alpha+\beta). \end{aligned}$$

■

4.3 Inverse hyperbolic functions

4.3.1 Inverse function of the hyperbolic sine

The function $sh : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and strictly increasing on $\mathbb{R}$. Based on the results of the Mean Value Theorem, we conclude that it has an inverse function, denoted by $argsh : \mathbb{R} \longrightarrow \mathbb{R}$.

Definition 4.3.1 *:*

The inverse function of the hyperbolic sine function is called ***argument hyperbolic sine*** *and is denoted argsinh (or argsh). For any real number x, argshx is the unique real number y such that* $\sinh y = x$*, that is:*

$$\forall x \in \mathbb{R}, \exists! t \in \mathbb{R};\ sht = x. \tag{4.14}$$

Therefore, by definition of the inverse bijection:

$$\forall x \in \mathbb{R},\ \forall y \in \mathbb{R};\ shx = y \Longleftrightarrow x = argshy. \tag{4.15}$$

And for all $x \in \mathbb{R}$,

$$sh(argshx) = x, \qquad argsh(shx) = x.$$

Properties 4.3.2 *:*

• *For all* $x \in \mathbb{R}$*, we have :*

$$ch(argshx) = \sqrt{1+x^2}, \qquad th(argshx) = \frac{x}{\sqrt{1+x^2}} \quad et \quad argsh(x) = ln(x+\sqrt{x^2+1}).$$

• *The hyperbolic sine argument function argsh is differentiable, of class C^∞ on $\mathbb{R}$, and for all real numbers x we have:*

$$argsh'(x) = \frac{1}{\sqrt{1+x^2}}.$$

Proof :

• ⊗ Since for all $x \in \mathbb{R}$; $ch(x) = \sqrt{1+sh^2(x)}$, we therefore have

$$ch(argsh(x)) = \sqrt{1+sh^2\left(argsh(x)\right)} = \sqrt{1+x^2}.$$

⊗ According to the definition of th, we have for all $x \in \mathbb{R}$,

$$th(argsh(x)) = \frac{sh(argsh(x))}{ch(argsh(x))} = \frac{x}{\sqrt{1+sh^2\left(argsh(x)\right)}} = \frac{x}{\sqrt{1+x^2}}.$$

⊗ According to (4.15) we have:

$$\begin{aligned} y = shx \Longleftrightarrow y = \frac{e^x - e^{-x}}{2} &\Longleftrightarrow e^x = y + \sqrt{1+y^2} \\ &\Longleftrightarrow x = ln\left(y + \sqrt{1+y^2}\right) \\ &\Longleftrightarrow x = argsh(y). \end{aligned}$$

Finally

$$\forall y \in \mathbb{R};\ argsh(y) = ln\left(y + \sqrt{1+y^2}\right).$$

• ⊗ The function sh is differentiable on $\mathbb{R}$ and $sh'(t) = ch(t) \geq 1$, therefore, according to the theorem 3.3.21, the function *argsh* is differentiable on $\mathbb{R}$ and for all $x \in \mathbb{R}$ we have

$$argsh'(x) = \frac{1}{ch(argsh(x))} = \frac{1}{\sqrt{1+sh^2(argsh(x))}} = \frac{1}{\sqrt{1+x^2}}.$$

⊗ Finally, since the function $x \longmapsto \dfrac{1}{\sqrt{1+x^2}}$ is infinitely differentiable on $\mathbb{R}$, then, the function *argsh* is also differentiable. ∎

4.3.2 Inverse function of the hyperbolic cosine

Since the restriction of ch to $\mathbb{R}_+$ to the interval $[1,+\infty[$, continuous and strictly increasing on $\mathbb{R}_+$, it therefore realizes an inverse bijection, denoted $argch : [1,+\infty[\longrightarrow \mathbb{R}_+$.

Definition 4.3.3 *:*

The inverse function of the function $ch_{|_{\mathbb{R}_+}}$ is called the hyperbolic cosine argument and is denoted argcosh (or argch). For any positive real number x, argchx is the unique real number $y \geq 1$ such that $chy = x$, that is:

$$\forall x \in [1,+\infty[,\ \exists! t \in \mathbb{R}_+;\ cht = x. \tag{4.16}$$

Therefore, by definition of the inverse bijection:

$$\forall x \in \mathbb{R}_+,\ \forall y \in [1,+\infty[;\ ch\,(x) = y \Longleftrightarrow x = argch\,(y)\,. \tag{4.17}$$

And for all $x \in \mathbb{R}_+$, $argsh(\sinh(x)) = x$.

And for all $x \in [1,+\infty[$, $\sinh(argsh\,(x)) = x$.

Properties 4.3.4 *:*

• *For all $x \in [1,+\infty[$, we have :*

$$sh(argch\,(x)) = \sqrt{x^2-1}, \qquad th\,(argch\,(x)) = \frac{\sqrt{x^2-1}}{x} \quad \text{and} \quad argch(x) = ln(x+\sqrt{x^2-1}).$$

• *The hyperbolic cosine argument function is differentiable on $]\,1,+\infty\,[$ and for all real numbers $x>1$ we have:*

$$argch'(x) = \frac{1}{\sqrt{x^2-1}}$$

Proof :

• ⊗ Since for all $x \in \mathbb{R}$, we have $coh^2x \geq 1$ and $|\sin x| = \sqrt{coh^2x-1}$, we therefore have for all $x \in [\,1,+\infty\,[$

$$sh(argch(x)) = \sqrt{ch^2\,(argch\,(x)) - 1} = \sqrt{x^2-1}.$$

⊗ According to the definition of th and what precedes it, we have for all $x \in [\,1,+\infty\,[$,

$$th(argchx) = \frac{sh(argch(x))}{ch(argch(x))} = \frac{\sqrt{x^2-1}}{x}.$$

⊗ According to (4.17), for all $x \in \mathbb{R}_+$ and $y \in [\,1,+\infty\,[$ we have,

$$\begin{aligned} y = ch\,(x) \Longleftrightarrow y = \frac{e^x+e^{-x}}{2} \quad &\Longleftrightarrow e^x = y + \sqrt{y^2-1} \\ &\Longleftrightarrow x = ln\left(y+\sqrt{y^2-1}\right) \\ &\Longleftrightarrow x = argch\,(y)\,. \end{aligned}$$

Finally

$$\forall y \in [1,+\infty[;\ argch\,(y) = ln\left(y+\sqrt{y^2-1}\right).$$

• ⊗ The function ch is differentiable on $\mathbb{R}_+$ and $\forall x \in \mathbb{R}_+^*;\ ch'(t) = \sinh(t) > 0$ therefore, according to the theorem 3.3.21, the function $argch$ is differentiable on $]\,1,+\infty\,[$ and by differentiation of the equality $ch(argch(x)) = x$ for all $x \in\ ]\,1,+\infty\,[$ we have ,

$$argch'\,(x)\,.sh\,(argch\,(x)) = 1 \Longleftrightarrow argch'\,(x)\,.\sqrt{x^2-1} = 1 \Longleftrightarrow argch'\,(x) = \frac{1}{\sqrt{x^2-1}}.$$

⊗ Finally, since the function $x \longmapsto \dfrac{1}{\sqrt{x^2-1}}$ is infinitely differentiable on $]\,1,+\infty\,[$, the function $argch$ is also differentiable. ■

4.3.3 Inverse function of the hyperbolic tangent

Since the function $th : \mathbb{R} \longrightarrow]-1,1[$ is continuous and strictly increasing on $\mathbb{R}$, it therefore establishes an inverse bijection, denoted $argth :]-1,1[\longrightarrow \mathbb{R}$.

Definition 4.3.5 *:*

The inverse function of the hyperbolic tangent function is called the argument of the hyperbolic tangent function and is denoted argtanh (or argth). For any real number x strictly between -1 and 1, $argthx$ is the unique real number $y \in \mathbb{R}$ such that $thy = x$, that is:

$$\forall x \in \,]-1,1[\,,\ \exists! y \in \mathbb{R}_+;\ th\,(y) = x. \tag{4.18}$$

Therefore, by definition of the inverse bijection:

$$\forall x \in \mathbb{R},\ \forall y \in \,]-1,1[\,;\ th\,(x) = y \Longleftrightarrow x = argth\,(y)\,. \tag{4.19}$$

And for all $x \in \mathbb{R}$, $argsh(th(x)) = x$.
And for all $x \in \,]-1,1[\,$, $th(argsh(x)) = x$.

Properties 4.3.6 *:*

• *For all $x \in \,]-1,1[$, we have:*

$$argth(x) = \frac{1}{2}ln\frac{1+x}{1-x}.$$

• *The hyperbolic tangent argument function is differentiable, of class C^∞ on $]-1,1[$ and for all real $|x| < 1$ we have:*

$$argth^{'}(x) = \frac{1}{1-x^2}$$

Proof :

• According to (4.19), for all $x \in \mathbb{R}$ and $y \in \,]-1,1[$ we have,

$$y = thx \Longleftrightarrow y = \frac{e^{2x}-1}{e^{2x}+1} \Longleftrightarrow e^{2x} = \frac{y+1}{1-y} \Longleftrightarrow x = \frac{1}{2}ln\frac{y+1}{1-y} \Longleftrightarrow x = argthy.$$

From this we deduce

$$\forall y \in \,]-1,1[\,;\ argthy = ln\frac{y+1}{1-y}.$$

• The function th is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R};'(x) = \dfrac{1}{ch^2x} = 1 - th^2x > 0$, therefore, according to the theorem 3.3.21, the function $argth$ is differentiable on $]-1,1[$ and for all $x \in \,]-1,1[$ we have,

$$argth^{'}\,(x) = \frac{1}{th^{'}\,(argth\,(x))} = \frac{1}{1-th^2\,(argth\,(x))} = \frac{1}{1-x^2}.$$

⊗ Finally, since the function $x \longmapsto \dfrac{1}{1-x^2}$ is infinitely differentiable on $]-1,1[$, it is therefore the function *argth* of class C^∞ on $]-1,1[$. ∎

4.3.4 Exercises

Exercise 4.3.7 :

Show that for all $\alpha,\ \beta\ \in \mathbb{R}$

$$sh(\alpha-\beta) = sh(\alpha)ch(\beta) - sh(\beta)ch(\alpha), \qquad th(\alpha+\beta) = \frac{th(\alpha)+th(\beta)}{1+th(\alpha)\times th(\beta)}.$$

Exercise 4.3.8 :

Use two different methods to demonstrate the following:

$$\forall x \in [-1,1]\,;\ \sin^2\left(\frac{1}{2}arcsin x\right) = \frac{1}{2}\left(1-\sqrt{1-x^2}\right).$$

Exercise 4.3.9 :

Simplify the following expressions:

$a)\ ch(argshx)$ $\quad b)\ th(argshx),$ $\quad c)\ sh(2argshx),$ $\quad d)\ argch(2x^2-1),$

$e)\ sh(argchx),$ $\quad f)\ th(argchx),$ $\quad g)\ ch(argthx),$ $\quad h)\ argsh(2x\sqrt{1+x^2})$

$j)\ argth\left(\dfrac{x^2-1}{x^2+1}\right),$ $\quad k)\ argsh\left(\dfrac{x^2-1}{2x}\right),$ $\quad l)\ argch\left(\sqrt{\dfrac{1+chx}{2}}\right).$

Exercise 4.3.10 :

1. *Calculate :* $arccos(\cos\frac{13\pi}{3}),$ $\quad arcsin(\sin\frac{13\pi}{3})$ $\quad$ *and* $\quad arctan(\tan\frac{13\pi}{3}).$

2. *Write the following expressions in terms of x (that is, without the trigonometric functions)*

$$\cos(arctanx), \qquad \cos(arcsinx), \qquad \sin(3arctanx). \qquad \text{and} \qquad \tan(arcsinx).$$

Exercise 4.3.11 :

1. Calculate the derivative of $f(x) = arctan(x/sqrt1-x^2).$

Deduce that for all $x \in\ ]-1,1[\ ,\ f(x) = arcsin(x).$

2. Show that for all $x \in\ ]-1,1[\ ,\ arccos(x) + arcsin(x) = \frac{\pi}{2}.$

Exercise 4.3.12 :

Let f be a real-valued function of a real variable defined by:

$$f(x) = \frac{1+\tan x}{1-\tan x}$$

1) Determine D_f *and* $D_{\frac{1}{f}}$ *the domains of f and* $\dfrac{1}{f}$ *respectively.*

2) Show that for all integers n and for all $x \in D_{\frac{1}{f}}$*, we have* $(f(x))^n = f(nx).$

Exercise 4.3.13 :

Let f and g be two real-valued functions of a real variable defined by:

$$f(x) = \cos^2\left(\frac{1}{2}arcsin(x)\right) \quad et \quad g(x) = \frac{1}{2}\left(1+\sqrt{1-x^2}\right)$$

1). Determine D_f and D_g, the domains of f and g respectively.

2). Study the continuity and differentiability of f and g on their domains.

3). Find the expressions $f'(x)$ and $g'(x)$ for all $x \in]-1,1[$.

4). Deduce that for all $x \in [-1,1]$ we have $f(x) = g(x)$.

Exercise 4.3.14 :

1°) *Let f be a real function of a real variable defined by: $f(x) = arcsin\left(\dfrac{2x}{1+x^2}\right)$.*

Plot the variation table of f on its domain of definition.

2°) *Show that*

$$\forall x \in \mathbb{R};\ -1 < \frac{1-x^2}{1+x^2} \leq 1.$$

3°) *Let g be a real-valued function of a real variable defined by: $g(x) = arcsin(\dfrac{1-x^2}{1+x^2})$.*

Study the function g on its domain of definition.

Exercise 4.3.15 :

1° *Show that*

$$\forall x \in]0, \frac{\pi}{2}[;\ \sin x = \frac{\tan x}{1+\tan^2 x} \quad \text{and} \quad \forall x \in]0, \frac{\pi}{2}[;\ \cos x = \frac{1}{\sqrt{1+\tan^2 x}}.$$

2° *Show that*

$$0 < arctan\left(\frac{3}{4}\right) + arctan\left(\frac{5}{12}\right) < \frac{\pi}{2}.$$

3° *Solve the following equation in the interval $[-1,1]$*

$$arcsin\,(x) = arctan\left(\frac{3}{4}\right) + arctan\left(\frac{5}{12}\right).$$

Exercise 4.3.16 :

Solve the following equations:

$$arcsinx = arcsin\left(\frac{2}{5}\right) - arcsin\left(\frac{3}{5}\right), \quad arccosx = 2.arccos\left(\frac{3}{4}\right) \quad \text{and} \quad arctanx = 2.arctan\left(\frac{1}{2}\right).$$

$$arctan(2x) + arctan(3x) = \frac{\pi}{4}, \quad \text{and} \quad arctan\left(\frac{x-1}{x-2}\right) + arctan\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}.$$

Exercise 4.3.17 *:*

Study the changes in the following two functions:
$f : x \longmapsto x + arcsin\left(\dfrac{x+1}{x}\right)$.
$g : x \longmapsto arcsin\left(\dfrac{2x}{1+x^2}\right) - 2arctanx$.

Chapter 5

Limited Development

4-1 *Limited development in the neighborhood of a point*
4-2 *Taylor's Formula*
4-3 *Limited development of useful functions*
4-4 *Applications of limited development*

This chapter aims to simplify the calculation of limits of real-valued functions of a single real variable and to examine the position of their curves relative to their tangents or asymptotes. It also explores the possibility of replacing any function with a simple asymptote at or near a given point.

We will therefore explore the possibility of replacing any function with a polynomial of degree n at or near a given point, such that the latter best approximates our function.

5.1 Limited development in the neighborhood of a point

Definition 5.1.1 *:*

Let f be a function defined on $V = V_{x_0}$ (the neighborhood of x_0), or on $V = V_{x_0} - \{x_0\}$. We say that f admits a limited development of order n in the neighborhood of x_0 (abbreviated, we note $LD_n(x_0)$), if there exist real numbers $a_0, a_1, \cdots, a_n$ and a function ε defined on V such that for every $x \in V$,

$$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots + a_n (x - x_0)^n + (x - x_0)^n \varepsilon (x - x_0),$$

and $\lim\limits_{x \longrightarrow x_0} \varepsilon (x - x_0) = 0$.
We can also write

$$f(x) = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots + a_n (x - x_0)^n + O(x - x_0).$$

The polynomial function $x \longmapsto a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$ *is called the regular part of the* $DL_n(x_0)$ *of* f*, and the function* $x \longmapsto (x - x_0)^n \varepsilon(x - x_0)$ *is called the remainder (rest) of the* $DL_n(x_0)$ *of* f*.*

Example 5.1.2 *:*
1. Let $f(x)\dfrac{1}{1-x}$*. The function* f *admits* $LD_n(0)$ *as follows:*
We observe that $1 + x + x^2 + \cdots + x^n$ *is the sum of* $n+1$ *first terms of the geometric sequence with common ratio* x*. Therefore, we have*

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n &= \frac{1 - x^{n+1}}{1-x} \\ &= \frac{1}{1-x} - x^n \frac{x}{1-x}. \end{aligned}$$

This imply that

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + \cdots + x^n + x^n \frac{x}{1-x} \\ &= 1 + x + x^2 + \cdots + x^n + x^n \varepsilon(x). \end{aligned}$$

This means there exist $a_0, a_1, \cdots, a_n$ *all equal* 1 *and the function* ε *defined on the neighborhood of* 0 *by* $\varepsilon(x) = \dfrac{x}{1-x}$ *such that*

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \varepsilon(x) \text{ and } \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0.$$

This is the limited development of $\dfrac{1}{1-x}$ *of order* n *in* 0*.*

Theorem 5.1.3 *:*
If a function f *admits a limited development of order* n *at* x_0 *then it is unique.*

Proof :
By contradiction, suppose that f admits two distinct limited developments of order n at x_0. There exist two polynomials of degree at most n, two neighborhoods V_1, V_2 of x_0, and two functions $\varepsilon_1, \varepsilon_2$ defined on V_1 *or* $V_1 - \{x_0\}$, and V_2 *or* $V_2 - \{x_0\}$ respectively, such that:
On the one hand

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon_1(x) \quad \text{and} \lim_{x \longrightarrow x_0} \varepsilon_1(x) = 0 \tag{5.1}$$

and the other hand

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots + b_n(x - x_0)^n + (x - x_0)^n \varepsilon_2(x) \quad \text{and} \lim_{x \longrightarrow x_0} \varepsilon_2(x) = 0. \tag{5.2}$$

Let $V = V_1 \cap V_2$, then we have, for all $x \in V - \{x_0\}$,

$$(a_0 - b_0) + (a_1 - b_1)(x - x_0) + \cdots + (a_n - b_n)(x - x_0)^n + (x - x_0)^n [\varepsilon_1(x) - \varepsilon_2(x)]. \tag{5.3}$$

By calculating the limit of this equality as $x \longrightarrow x_0$, we find $a_0 = b_0$.

Therefore, the equality (5.3) becomes, for all $x \in V - \{x_0\}$

$$(a_1 - b_1)(x - x_0) + (a_2 - b_2)(x - x_0)^2 + \cdots + (a_n - b_n)(x - x_0)^n + (x - x_0)^n [\varepsilon_1(x) - \varepsilon_2(x)]. \tag{5.4}$$

Next, we can divide this equality by $x - x_0$, and by calculating the limit as $x \longrightarrow x_0$, we find $a_1 = b_1$, and so on.

In the end, we find $a_0 = b_0$, $a_1 = b_1$, and so on, $a_n = b_n$. The regular parts are equal, and therefore the remainders are also equal. ■

Corollary 5.1.4 *:*

Let f be a function admitting a limited development of order n at 0.

If f is even (resp. odd), then its regular part contains monomials of even (resp. odd) degree.

Proof :

Let f admit a limited development of order n at 0, as follows:

$$f(x) = a_0 + a_1(x) + a_2(x)^2 + \cdots + a_n(x)^n + (x)^n \varepsilon_1(x) \quad \text{and} \lim_{x \longrightarrow 0} \varepsilon_1(x) = 0.$$

Since V is a neighborhood of 0, then $\exists \delta > 0;\ ]-\delta, \delta[\subseteq V$.

For all $x \in V - \{0\}$, we have $-x \in V - \{0\}$ and

$$\begin{aligned} f(-x) &= a_0 + a_1(-x) + a_2(-x)^2 + \cdots + (-1)^n a_n(x)^n + (-1)^n (x)^n \varepsilon_1(-x) \\ &= a_0 - a_1 x + a_2 x^2 + \cdots + (-1)^n a_n x^n + x^n \varepsilon_2(x). \end{aligned}$$

where $\varepsilon_2 = (-1)^n \varepsilon_1$ and verified $\lim_{x \longrightarrow 0} \varepsilon_2(x) = 0$.

Case f even : We have $\forall x \in V$;

$f(x) = f(-x)$. By the uniqueness of the limited development to order n of f at 0, we obtain

$$\forall k \in \{0, 1, 2, \cdots, n\};\ a_k = (-1)^k a_k.$$

We deduce that for any odd index k, we have $a_k = 0$. This shows that the regular part of f contains only monomials of even degree.

Case f odd : We have $\forall x \in V$;

$f(-x) = -f(x)$. By the uniqueness of the limited development to order n of f at 0, we obtain

$$\forall k \in \{0, 1, 2, \cdots, n\}; \; -a_k = (-1)^k a_k.$$

We deduce that for any even index k, we have $a_k = 0$. This shows that the regular part of f contains only monomials of odd degree. ■

5.2 Taylor formula

5.2.1 Taylor-Lagrange formula

Let f be a function of class C^{n+1} on an interval I with $n \in \mathbb{N}$ and $x_0, x \in I$. Then there exists a value c between x_0 and x such that:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

This relationship is called the Taylor-Lagrange formula, and the term $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is called the Lagrange remainder.

5.2.2 Taylor-Young formula

Let f be defined on an interval I, admitting at a point $x_0 \in I$ derivatives up to order n with $n \in \mathbb{N}$. Then there exists a neighborhood V of x_0 and a function $\varepsilon : V \longrightarrow \mathbb{R}$ such that:

$$\begin{aligned} \forall x \in V;\; f(x) \; &= f(x_0) + \tfrac{f'(x_0)}{1!}(x - x_0) + \tfrac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \tfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n + (x - x_0)^n \varepsilon(x) \\ &\text{and } \lim_{x \longrightarrow x_0} \varepsilon(x) = 0. \end{aligned}$$

This is Taylor's formula with a Young's remainder $((x - x_0)^n \varepsilon(x))$.

5.2.3 Maclaurin formula

This is the Taylor-Lagrange formula with $a = 0$ and $c = \theta x$ with $0 < \theta < 1$, that is:

$$\begin{aligned} \forall x \in I,\; \exists \theta \in]0, 1[;\; f(x) \; &= f(x_0) + \tfrac{f'(x_0)}{1!}(x - x_0) + \tfrac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \tfrac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &+ \tfrac{f^{(n+1)}(\theta x)}{(n+1)!}(x - x_0)^{n+1}. \end{aligned}$$

5.2.4 Existence of limited development

Corollary 5.2.1 :

If a function f of class C^n on an interval I then f admits a limited development of order n at $x_0 \in I$ of the form:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o((x-x_0)^n).$$

That is, if

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + (x-x_0)^n \varepsilon(x) \quad \text{with} \lim_{x \longrightarrow x_0} \varepsilon(x) = 0,$$

then

$$\frac{f^{(k)}(x_0)}{k!} = a_k \text{ for all } k \in \{0, 1, 2, \cdots, n\}$$

Remarks 5.2.2 :

1. *If f admits a LD at a point x_0 of order $n > 0$ with regular part $a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n$, then f is continuous at x_0 or is extendable by continuity at x_0 and $f(x_0) = a_0$.*

2. *If f admits a LD at a point x_0 of order $n \geq 1$, then f is differentiable at x_0 and we have $f(x_0) = a_0$ and $f'(x_0) = a_1$, and the equation of the tangent to the graph of f at this point is $y = a_0 + a_1(x-x_0)$.*

Every polynomial function has a limited development to any order at 0. If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and $n \geq m$, then f has a limited development of order n at 0 where the regular part is the function itself and the remainder is the zero function. If $n \leq m$, the regular part of the limited development of f at 0 is $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and the remainder is $R(x) = a_n + 1x^(n+1) + \cdots + a_mx^m = x^n\varepsilon(x)$. where $\varepsilon(x) = a_{n+1}x + \cdots + a_mx^{m-n}$.

As by **example:** The function $f : x \longmapsto 3x^5 - 2x^3 + x^2 - x + 4$ has a limited development of order 4 at 0 whose regular part is $4 - x + x^2 - 2x^3$ and whose remainder is $x^4\varepsilon(x)$ where $\varepsilon(x) = 3x$.

The function f also has a limited development of order 6 at 0 whose regular part is $4 - x + x^2 - 2x^3 + 3x^5$ and whose remainder is the zero function.

Proposition 5.2.3 *If $f : D \longrightarrow \mathbb{R}$ admits a Taylor series expansion of order n at x_0 of the form:*

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + O(x-x_0).$$

Therefore, for all $m \leq n$, the function f admits a $LD_m(x_0)$ obtained by truncation as follows:

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_m(x-x_0)^m + O((x-x_0)^m).$$

5.3 Limited developments of usual functions

The Taylor-Young formula allows us to obtain the limited development of order n at 0 of several usual functions.

1• Consider the function $exp : x \in \mathbb{R} \longmapsto e^x$. The function exp is infinitely differentiable in $\mathbb{R}$ and for all $k \in \mathbb{R}, \exp^{(k)}(0) = 1$.

We deduce that the LD of order n at 0 of the function exp is,

$$\boxed{e^x = 1 + \tfrac{x}{1!} + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + x^n \varepsilon(x) \quad with\, \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0.}$$

2• The functions sin and cos are infinitely differentiable in $\mathbb{R}$ and for all $x \in \mathbb{R}$ we have:

$$\sin^{(n)} x = \sin\left(x + n\frac{\pi}{2}\right) \quad and\; cos^{(n)} x = \cos\left(x + n\frac{\pi}{2}\right).$$

We deduce that the LD of order $2n+1$ at 0 of sin and the LD of order $2n$ at 0 of cos are,

$$\boxed{\begin{array}{c} \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + x^{2n+2}\varepsilon(x) \quad with\, \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + x^{2n+1}\varepsilon(x) \quad with\, \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \end{array}}$$

3• The function $g : x \in \,]-1, +\infty[\, \longmapsto (1+x)^\alpha$ (with $\alpha \in \mathbb{R}$ fixed) is infinitely differentiable in $]-1, +\infty[$ and for all $x \in \,]-1, +\infty[$ we have: $\left((1+x)^\alpha\right)^{(n)} = [\alpha \times (\alpha - 1) \times \cdots \times (\alpha - n + 1)]\,(1+x)^{\alpha - n}$.

It follows that the LD of order n at 0 of g is,

$$\boxed{\begin{array}{c} (1+x)^\alpha = 1 + \dfrac{\alpha}{1!}x + \dfrac{\alpha \times (\alpha-1)}{2!}x^2 + \cdots + \dfrac{\alpha \times (\alpha-1) \times \cdots \times (\alpha - n + 1)}{n!}x^n + x^n \varepsilon(x) \\ with\, \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \end{array}} \tag{5.5}$$

4• The function $h : x \in \,]-1, +\infty[\, \longmapsto ln(1+x)$ is infinitely differentiable in $]-1, +\infty[$ and for all $x \in \,]-1, +\infty[$ we have : $ln^{(n)}(1+x) = \left((1+x)^{-1}\right)^{(n+1)}$.

It follows that the LD order n at 0 of h is,

$$\boxed{ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + (-1)^{n-1}\frac{1}{n}x^n + x^n\varepsilon(x) \quad with\, \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0.} \tag{5.6}$$

Remarks 5.3.1 *:*

• *By replacing x by $-x$ in the LD of $x \longmapsto ln(1+x)$ and $x \longmapsto (1+x)^\alpha$, we deduce the LD of $x \longmapsto ln(1-x)$ and of $x \longmapsto (1-x)^\alpha$ of order n in 0.*

• *Let f be a function that admits LD in the neighborhood of 0 and in the neighborhood of x_0. To determine a LD of f at x_0, we determine the LD at 0 of the function $g \longmapsto f(x_0 + h)$ by making the change of variable $x = x_0 + h$.*

Examples 5.3.2 *:*

1). *The* $LD_n(0)$ *of* $x \longmapsto \dfrac{1}{1-x}$ *is obtained as follows:*
According to (5.5) *with* $\alpha = -1$*, We have :*

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + (-1)^n x^n \varepsilon(x) \quad \text{with}\ \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \tag{5.7}$$

By replacing x *with* $-x$ *in* (**??**) *we obtain*

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \varepsilon(x) \quad \text{with}\ \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0.$$

2). *The* $LD_3(0)$ *of* $x \longmapsto \dfrac{1}{\sqrt{1-x}}$ *is obtained as follows:*
According to (5.5) *with* $\alpha = -\dfrac{1}{2}$*, We have :*

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{8}x^3 + x^3\varepsilon(x) \quad \text{with}\ \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \tag{5.8}$$

By replacing x *with* $-x$ *in* (5.8) *we obtain*

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3 + x^3\varepsilon(x) \quad \text{with}\ \varepsilon(x) \underset{x \longrightarrow 0}{\longrightarrow} 0.$$

3). *The* $LD_n(2)$ *of* $x \longmapsto e^x$ *is obtained as follows:*

Let's put $h = x - 2 \underset{x \longrightarrow 2}{\longrightarrow} 0.$

$$\begin{aligned}
e^x &= e^2 e^{x-2} \\
&= e^2 e^h \ \ \text{with}\ h = x - 2 \\
&= e^2 \left(1 + \frac{h}{1!} + \frac{h^2}{2!} + \cdots + \frac{h^n}{n!} + h^n\varepsilon(h)\right) \quad \text{with}\ \varepsilon(h) \underset{h \longrightarrow 0}{\longrightarrow} 0 \\
&= e^2 \left(1 + \frac{x-2}{1!} + \frac{(x-2)^2}{2!} + \cdots + \frac{(x-2)^n}{n!} + (x-2)^n \varepsilon(x-2)\right) \quad \text{with}\ \varepsilon(x-2) \underset{x \longrightarrow 2}{\longrightarrow} 0.
\end{aligned}$$

4). *The* $LD_4(1)$ *of* $x \longmapsto lnx$ *is obtained as follows:*

Let's put $h = x - 1 \underset{x \longrightarrow 1}{\longrightarrow} 0.$

$$\begin{aligned}
lnx &= ln(h+1) \quad \text{with}\ h = x - 1 \\
&= h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + h^4\varepsilon(h) \quad \text{with}\ \ h = x - 1 \ \text{and}\ \varepsilon(h) \underset{h \longrightarrow 0}{\longrightarrow} 0 \\
&= -1 + x - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + (x-1)^4 \varepsilon((x-1)) \quad \text{with}\ \ \varepsilon(x-1) \underset{x \longrightarrow 1}{\longrightarrow} 0.
\end{aligned}$$

5). *The* $LD_3(1)$ *of* $x \longmapsto \sqrt{2+x}$ *is obtained as follows:*

Let's put $h = x - 1 \underset{x \longrightarrow 1}{\longrightarrow} 0.$

$$\begin{aligned}
\sqrt{2+x} &= \sqrt{3+h} \ \ \text{with } h = x-1 \\
&= \sqrt{3}\sqrt{1+\frac{h}{3}} \\
&= \sqrt{3}\left(1+\frac{h}{3}\right)^{\frac{1}{2}} \\
&= \sqrt{3}\left(1-\frac{1}{2}\frac{h}{3}+\frac{3}{8}\left(\frac{h}{3}\right)^2-\frac{5}{8}\left(\frac{h}{3}\right)^3+\left(\frac{h}{3}\right)^3\varepsilon\left(\frac{h}{3}\right)\right) \quad \text{with } \varepsilon\left(\frac{h}{3}\right) \underset{h\longrightarrow 0}{\longrightarrow} 0 \\
&= \sqrt{3}\left(1-\frac{1}{6}h+\frac{1}{24}h^2-\frac{5}{216}h^3+h^3\varepsilon(h)\right) \quad \text{with } \varepsilon(h) \underset{h\longrightarrow 0}{\longrightarrow} 0 \\
&= \sqrt{3}\left(1-\frac{1}{6}(x-1)+\frac{1}{24}(x-1)^2-\frac{5}{216}(x-1)^3+(x-1)^3\varepsilon((x-1))\right) \quad \text{with } \varepsilon(x-1) \underset{x\longrightarrow 1}{\longrightarrow} 0.
\end{aligned}$$

5.3.1 Operations on limited developments

In this paragraph, we assume that f and g are two functions that admit LD at 0 of order n, where:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon_1(x) \ \text{ with } \lim_{x\longrightarrow 0}\varepsilon_1(x) = 0,$$

$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + x^n\varepsilon_2(x) \ \text{ with } \lim_{x\longrightarrow 0}\varepsilon_2(x) = 0,$$

limited developments of a sum and a product of functions

• $f+g$ admits a LD of order n in the neighborhood of 0 which is:

$$f(x)+g(x) = a_0+b_0+(a_1+b_1)x+(a_2+b_2)x^2+\cdots+(a_n+b_n)x^n+x^n\varepsilon(x) \ \text{ with } \lim_{x\longrightarrow 0}\varepsilon(x) = 0.$$

• $f\times g$ admits a LD of order n in the neighborhood of 0 which is:

$$f(x)\times g(x) = a_0b_0+(a_0b_1+a_1b_0)x+\cdots+(a_0b_n+a_1b_{n-1}+\cdots a_nb_0)x^n+x^n\varepsilon(x) \ \text{ with } \lim_{x\longrightarrow 0}\varepsilon(x) = 0.$$

Example 5.3.3 :

Calculate the LD of $e^x ln(1+x)$ at 0 of order 3. We know that

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+x^3\varepsilon(x) \quad \text{with } \varepsilon(x) \underset{x\longrightarrow 0}{\longrightarrow} 0.$$

$$\text{and } ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+x^3\varepsilon(x) \quad \text{with } \varepsilon(x) \underset{x\longrightarrow 0}{\longrightarrow} 0.$$

On note $a_0 = 1$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{6}$, $b_0 = 0$, $b_1 = 1$, $b_2 = -\frac{1}{2}$ and $b_3 = \frac{1}{3}$. Then

$$\begin{aligned}
e^x ln(1+x) &= a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+(a_0b_3+a_1b_2+a_2b_1+a_3b_0)x^3+x^3\varepsilon(x) \\
&= x+\frac{1}{2}x^2+\frac{1}{3}x^3+x^3\varepsilon(x).
\end{aligned}$$

Limited development of a composite function

Let I and J be two open intervals containing 0, g a function from I to J, and f a function defined on J. If $g(0) = 0$ (that is, $b_0 = 0$), then the function $f \circ g$ admits a LD of order n in a neighborhood of 0 whose regular part is the truncated polynomial of order n of $P(x) = a_0 + a_1 g(x) + a_2 g^2(x) + ... + a_n g^n(x)$.

Examples 5.3.4 :

1∗ *Let's calculate the LD of* $ln\left(1+x+x^2\right)$ *at* 0 *to order* 3.

Here we set $u(x) = x + x^2 \underset{x\longrightarrow 0}{\longrightarrow} 0$. *We know that*

$$ln\,(1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 + u^3\varepsilon(u) \quad with\, \varepsilon(u) \underset{u\longrightarrow 0}{\longrightarrow} 0.$$

We will need to calculate a LD of order 3 *at* 0 *of* u^2 *and* u^3.

$$u^2(x) = a_0.a_0 + (a_0.a_1 + a_1.a_0)\,x + (a_0.a_2 + a_1.a_1 + a_2.a_0)\,x^2 + (a_0.a_3 + a_1.a_2 + a_2.a_1 + a_3.a_0)\,x^3,$$

where $a_0 = 0$, $a_1 = 1$, $a_2 = 1$, $a_3 = 0$.

Then, the DL of order 3 *at* 0 *of* u^2 *is*

$$u^2(x) = x^2 + 2x^3 + x^3\varepsilon(x).$$

$$\begin{aligned} u^3(x) &= u(x).u^2(x) \\ &= a_0.b_0 + (a_0.b_1 + b_1.a_0)\,x + (a_0.b_2 + a_1.b_1 + a_2.b_0)\,x^2 + (a_0.b_3 + a_1.b_2 + a_2.b_1 + a_3.b_0)\,x^3 + x^3\varepsilon(x), \end{aligned}$$

where $b_0 = 0$, $b_1 = 0$, $b_2 = 1$, $b_3 = 2$.

Then, the DL of order 3 *at* 0 *of* u^3 *is*

$$u^3(x) = x^3 + x^3\varepsilon(x).$$

From this we deduce,

$$\begin{aligned} ln\left(1+x+x^2\right) &= x + x^2 - \frac{1}{2}\left(x^2+2x^3\right) + \frac{1}{3}x^3 + x^3\varepsilon\left(x+x^2\right) \quad with\, \varepsilon\left(x+x^2\right) \underset{x\longrightarrow 0}{\longrightarrow} 0 \\ &= x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + x^3\varepsilon(x) \quad with\, \varepsilon(x) \underset{x\longrightarrow 0}{\longrightarrow} 0. \end{aligned}$$

2∗ *Let's calculate the LD of* $\sqrt{1+2e^x}$ *at* 0 *of order* 3.

We know the LD of e^x *at* 0 *to order* 3 *as:*

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3\varepsilon'(x) \quad with\, \varepsilon'(x) \underset{x\longrightarrow 0}{\longrightarrow} 0.$$

So

$$\begin{aligned} \sqrt{1+2e^x} &= \sqrt{1 + 2\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^3\varepsilon'(x)\right)} \\ &= \sqrt{3 + 2x + x^2 + \frac{1}{3}x^3 + x^3\varepsilon'(x)} \\ &= \sqrt{3}\sqrt{1 + \left(\frac{2}{3}x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \frac{x^3}{3}\varepsilon'(x)\right)} \\ &= \sqrt{3}\sqrt{1+u(x)} \quad with\, u(x) = \frac{2}{3}x + \frac{1}{3}x^2 + \frac{1}{9}x^3 + \frac{x^3}{3}\varepsilon'(x) \underset{x\longrightarrow 0}{\longrightarrow} 0. \end{aligned}$$

We know the LD of $\sqrt{1+u(x)}$ *at* 0 *of order* 3 *:*

$$\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+\frac{1}{16}u^3+u^3\varepsilon_0(u)\quad \text{with } \varepsilon_0(u)\underset{u\longrightarrow 0}{\longrightarrow}0.$$

We will only need to calculate the monomials of degree p *such that* $p\leq 3$ *of* u^2 *and* u^3.

$$\begin{aligned}u^2(x)&=a_0.a_0+(a_0.a_1+a_1.a_0)\,x+(a_0.a_2+a_1.a_1+a_2.a_0)\,x^2+(a_0.a_3+a_1.a_2+a_2.a_1+a_3.a_0)\,x^3+x^3\varepsilon_1(x)\\&=\frac{4}{9}x^2+\frac{4}{9}x^3+x^3\varepsilon_1(x)\quad \text{with } \varepsilon_1(x)\underset{x\longrightarrow 0}{\longrightarrow}0,\end{aligned}$$

where $a_0=0$, $a_1=\frac{2}{3}$, $a_2=\frac{1}{3}$, $a_3=\frac{1}{9}$.

$$\begin{aligned}u^3(x)&=u(x).u^2(x)\\&=a_0.b_0+(a_0.b_1+b_1.a_0)\,x+(a_0.b_2+a_1.b_1+a_2.b_0)\,x^2+(a_0.b_3+a_1.b_2+a_2.b_1+a_3.b_0)\,x^3+x^3\varepsilon_2(x)\\&=\frac{8}{27}x^3+x^3\varepsilon_2(x)\quad \text{with } \varepsilon_2(x)\underset{x\longrightarrow 0}{\longrightarrow}0,\end{aligned}$$

where $b_0=0$, $b_1=0$, $b_2=\frac{4}{9}$, $b_3=\frac{4}{9}$.

Then we have,

$$\begin{aligned}\sqrt{1+2e^x}&=\sqrt{3}\left(1+\frac{1}{2}\left(\frac{2}{3}x+\frac{1}{3}x^2+\frac{1}{9}x^3+\frac{x^3}{3}\varepsilon'(x)\right)-\frac{1}{8}\left(\frac{4}{9}x^2+\frac{4}{9}x^3+x^3\varepsilon_1(x)\right)\right.\\&\quad\left.+\frac{1}{16}\left(\frac{8}{27}x^3+x^3\varepsilon_2(x)\right)+x^3\varepsilon(x)\quad \text{with } \varepsilon(x)\underset{x\longrightarrow 0}{\longrightarrow}0.\right)\\&=\sqrt{3}+\frac{\sqrt{3}}{3}x+\frac{\sqrt{3}}{9}x^2+\frac{\sqrt{3}}{54}x^3+x^3\varepsilon(x)\quad \text{with } \varepsilon(x)\underset{x\longrightarrow 0}{\longrightarrow}0.\end{aligned}$$

Limited development of an inverse

Here is how to calculate the LD of $\frac{1}{f}$ where f has a $LD_n(0)$ as follows:

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n+x^n\varepsilon_1(x)\quad\text{with}\quad \lim_{x\longrightarrow 0}\varepsilon_1(x)=0.$$

• If $a_0\neq 0$ we set $u(x)=\frac{a_1}{a_0}x+\frac{a_2}{a_0}x^2+\cdots+\frac{a_n}{a_0}x^n+\frac{x^n}{a_0}\varepsilon_1(x)$. Then, $\frac{1}{f}$ is written $\frac{1}{f}=\frac{1}{a_0}\frac{1}{1+u}$.

• If $a_k\neq 0$ for some $k\in\mathbb{N}^*$ with $k<n$ and $\forall p\in\{0,1,2,\cdots,k-1\}$; $a_p=0$, the function $\frac{1}{f}$ does not admit a Taylor series expansion in a neighborhood of 0. But the function $x\longmapsto\frac{x^m}{f(x)}$ with $m\geq k$ admits a LD of order $n-k$ in a neighborhood of 0, because for all $x\in D_{\frac{1}{f}}-\{0\}$

$$\begin{aligned}\frac{x^m}{f(x)}&=\frac{x^{m-k}}{a_k+a_{k+1}x^1+\cdots+a_nx^{n-k}+x^{n-k}\varepsilon_1(x)}\\&=\frac{x^{m-k}}{a_k}.\frac{1}{1+v(x)},\end{aligned}$$

where $v(x)=\frac{a_{k+1}}{a_k}x+\cdots+\frac{a_n}{a_k}x^{n-k}+\frac{x^{n-k}}{a_k}\varepsilon_1(x)$.

Examples 5.3.5 *:*

1)$*$ *Let's calculate the LD of* $x \longmapsto \dfrac{1}{2-x}$ *at* 0 *of order* 4.

$$\begin{aligned}
\frac{1}{2-x} &= \frac{1}{2}\frac{1}{1-\frac{x}{2}} \\
&= \frac{1}{2}\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^3+\left(\frac{x}{2}\right)^4+O\left(\left(\frac{x}{2}\right)^4\right)\right) \\
&= \frac{1}{2}+\frac{x}{4}+\frac{x^2}{8}+\frac{x^3}{16}+\frac{x^4}{32}+O\left(x^4\right).
\end{aligned}$$

2)$*$ *Let's calculate the LD of* $x \longmapsto \dfrac{1}{\cos x}$ *at* 0 *of order* 5.

$$\begin{aligned}
\frac{1}{\cos x} &= \frac{1}{1-\frac{1}{2}x^2+\frac{1}{24}x^4+O\left(x^5\right)} \\
&= \frac{1}{1-u} \quad \textit{with} \;\; u=\frac{1}{2}x^2-\frac{1}{24}x^4-O\left(x^5\right) \\
&= 1+u+u^2+O\left(u^2\right) \quad \textit{with} \;\; u^2=\frac{1}{4}x^4+O\left(x^5\right) \;\; \textit{and } O\left(u^2\right)=O\left(x^5\right) \\
&= 1+\left(\frac{1}{2}x^2-\frac{1}{24}x^4-O\left(x^5\right)\right)+\left(\frac{1}{4}x^4+O\left(x^5\right)\right)+O\left(x^5\right) \\
&= 1+\frac{1}{2}x^2-\frac{5}{24}x^4+O\left(x^5\right).
\end{aligned}$$

3)$*$ *Let's calculate the LD of* $x \longmapsto \dfrac{x}{\sin x}$ *in the neighborhood of* 0 *of order* 4.

$$\begin{aligned}
\frac{x}{\sin x} &= x.\frac{1}{\sin x} \\
&= x.\frac{1}{x-\frac{1}{6}x^3+\frac{1}{120}x^5+O\left(x^5\right)} \\
&= x.\frac{1}{x\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+O\left(x^4\right)\right)} \\
&= \frac{1}{1+u} \quad \textit{with } u=-\frac{1}{6}x^2+\frac{1}{120}x^4+O\left(x^4\right) \\
&= 1-u+u^2+O\left(u^2\right) \quad \textit{with } u^2=\frac{1}{36}x^4+O\left(x^4\right) \\
&= 1-\left(-\frac{1}{6}x^2+\frac{1}{120}x^4+O\left(x^4\right)\right)+\left(\frac{1}{36}x^4+O\left(x^4\right)\right)+O\left(x^4\right) \\
&= 1+\frac{1}{6}x^2+\frac{7}{360}x^4+O\left(x^4\right).
\end{aligned}$$

Method: If $g(0) \neq 0$, then the LD of $\dfrac{f}{g}$ is defined via the quotient of the division of the regular part of f by the regular part of g according to increasing powers.

Examples 5.3.6 *:*

1)$\bullet$ *Let's calculate the LD of* $x \longmapsto \dfrac{2-x+2x^2-x^3}{1+x-x^2}$ *at* 0 *of order* 3.

The algorithm for calculating the quotient of the division, according to increasing powers						
2	$-x$	$+2x^2$	$-x^3$			$1+x-x^2$
$-(2$	$+2x$	$-2x^2)$				$2-3x+7x^2-11x^3$
$=$	$-3x$	$+4x^2$	$-x^3$			
	$-(-3x$	$-3x^2$	$+3x^3)$			
	$=$	$7x^2$	$-4x^3$			
		$-(7x^2$	$+7x^3$	$-7x^4)$		
		$=$	$-11x^3$	$+7x^4$		
			$-(-11x^3$	$-11x^4$	$+11x^5)$	
			$=$	$18x^4$	$-11x^5$	

2)• *Let's calculate the LD of* $x \longmapsto \dfrac{e^x}{x-lnx}$ *at* 1 *of order* 3.

Let's put $h = x-1 \underset{x\longrightarrow 1}{\longrightarrow} 0$.

We have

$$
\begin{aligned}
\frac{e^x}{x-lnx} &= \frac{e^{h+1}}{h+1-ln\,(1+h)}\\
&= \frac{e.e^h}{h+1-ln\,(1+h)}\\
&= \frac{e.\left(1+h+\dfrac{h^2}{2}+\dfrac{h^3}{6}+O\left(h^3\right)\right)}{h+1-\left(h-\dfrac{1}{2}h^2+\dfrac{1}{3}h^3+O\left(h^3\right)\right)}\\
&= e.\frac{1+h+\dfrac{h^2}{2}+\dfrac{h^3}{6}+O\left(h^3\right)}{1-h+\dfrac{1}{2}h^2-\dfrac{1}{3}h^3+O\left(h^3\right)}.
\end{aligned}
$$

By calculating the division of $1+h+\dfrac{h^2}{2}+\dfrac{h^3}{6}$ *by* $1-h+\dfrac{1}{2}h^2-\dfrac{1}{3}h^3$ *according to increasing powers*

1	$+h$	$+\frac{1}{2}h^2$	$+\frac{1}{6}h^3$				$1-h+\frac{1}{2}h^2-\frac{1}{3}h^3$
$-(1$	$-h$	$+\frac{1}{2}h^2$	$-\frac{1}{3}h^3)$				$1+2h+2h^2+\frac{3}{2}h^3$
$=$	$2h$		$+\frac{1}{2}h^3$				
	$-(2h$	$-2h^2$	$+h^3$	$-\frac{2}{3}h^4)$			
	$=$	$2h^2$	$-\frac{1}{2}h^3$	$+\frac{2}{3}h^4$			
		$-(2h^2$	$-2h^3$	$+h^4$	$+\frac{2}{3}h^5)$		
		$=$	$\frac{3}{2}h^3$	$-\frac{1}{3}h^4$	$-\frac{2}{3}h^5$		
			$-(\frac{3}{2}h^3$	$-\frac{3}{2}h^4$	$+\frac{3}{4}h^5$	$-\frac{1}{2}h^6)$	
			$=$	$\frac{7}{6}h^4$	$-\frac{17}{12}h^5$	$+\frac{1}{2}h^6$	

5.3.2 Integration of limited developments

Theorem 5.3.7 :

Let f be a differentiable function on an interval I containing x_0 and $f^{'}$ a function of class C^n on I whose LD at x_0 to order n is:

$$f^{'}(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + o((x-x_0)^n),$$

then f admits a LD to order $n+1$ at x_0 of the form:

$$f(x) = f(x_0) + a_0(x-x_0) + \cdots + a_{n+1}(x-x_0)^{n+1} + o((x-x_0)^{n+1}).$$

This means that we integrate the polynomial part of the DL of $f^{'}(x)$ term by term to obtain the DL of $f(x) - f(x_0)$ at x_0.

Proof :

Suppose that $f^{'}$ admits a LD of order $n-1$ at 0 with regular part P. Then there exists $P \in \mathbb{R}[X]$ of degree at most $n-1$, a neighborhood V of 0, and a function ε defined on $V - \{0\}$ such that:

$$\text{for all } x \in V - \{0\}\,;\; f^{'}(x) = P(x) + x^{n-1}\varepsilon(x) \quad \text{and} \quad \lim_{x \longrightarrow 0} \varepsilon(x) = 0.$$

It follows that:

$$f(x) = f(0) + \int_0^x P(t)dt + x^n\varepsilon(x).$$

Since the integral of a polynomial of degree at most $n-1$ is a polynomial of degree at most n, then by the uniqueness of the limited development we obtain:

$$Q : x \longmapsto f(0) + \int_0^x P(t)dt \quad \text{is the primitive of } P \text{ which equals } f(0) \text{ at } 0.$$

Hence the result. ■

Examples 5.3.8 :

1) We can find the LD of $x \longmapsto ln(1-x)$ at 0 using this tool.

Indeed, $\forall x \in \,]-\infty, 1[\; (ln(1-x))^{'} = \dfrac{-1}{1-x}$.

We know that the LD of the function $x \longmapsto \dfrac{1}{1-x}$ at 0 is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + O(x^n).$$

Therefore, by integrating this limited expansion, we obtain

$$\int_0^x \frac{-1}{1-t}dt = ln(1-x) - ln(1) = -x - \frac{1}{2}x^2 - \cdots \frac{1}{n+1}x^{n+1} + O(x^{n+1}).$$

2) Let's calculate the LD of $x \longmapsto arcsinx$ *at* 0 *to order* 7. *The function* $x \longmapsto arcsinx$ *is of class* C^{∞} *on* $I =]-1,1[$ *and* $(arcsinx)' = \dfrac{1}{\sqrt{1-x^2}}$.

We know that the LD of the function $x \longmapsto \dfrac{1}{\sqrt{1-x^2}} = (1-y)^{-\frac{1}{2}}$ *at* 0 *where* $y = x^2$ *is*

$$\frac{1}{\sqrt{1-y}} = 1 + \frac{1}{2}y + \frac{3}{8}y^2 + \frac{5}{16}y^3 + y^3\varepsilon(x) \quad \text{with } \varepsilon(y) \underset{y\longrightarrow 0}{\longrightarrow} 0. \tag{5.9}$$

Replacing y *with* x^2 *in the above LD, we have*

$$\left(1-x^2\right)^{-\dfrac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + O\left(x^6\right).$$

Therefore, by integrating the above result, we obtain

$$arcsinx - arcsin0 = arcsinx = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{114}x^7 + O\left(x^7\right).$$

Attention: A limited development can be integrated but not differentiated. This is because the function f can admit a limited development, but f' does not. For example:

Example 5.3.9 :

The function

$$f : x \in \mathbb{R} \longmapsto \begin{cases} 2x + x^2 + x^3 + x^3\cos(\frac{2}{x^2}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

This function is differentiable on $\mathbb{R}$ *with derivative*

$$f' : x \in \mathbb{R} \longmapsto \begin{cases} 2 + 2x + 3x^2 + 3x^2\cos(\frac{2}{x^2}) + 4\sin(\frac{2}{x^2}), & \text{if } x \neq 0 \\ 2, & \text{if } x = 0. \end{cases}$$

Since for all x *in a neighborhood of* 0 *we have* $f(x) = x + x^2 + x^2\varepsilon(x)$ *where* $\varepsilon(x) = x\left(1+\cos(\frac{2}{x^2})\right) \underset{x\longrightarrow 0}{\longrightarrow} 0$, *then* f *admits a Taylor series expansion of order* 2 *at* 0. *But* f' *has no limit at* 0, *so it cannot have a LD of order* 1 *at* 0. f' *has a regular part of* $2+2x$.

Since for all x *in a neighborhood of* 0 *we have* $f(x) = x + x^2 + x^2\varepsilon(x)$ *where* $\varepsilon(x) = x\left(1+\cos(\frac{2}{x^2})\right) \underset{x\longrightarrow 0}{\longrightarrow} 0$, *then* f *admits a Taylor series expansion of order* 2 *at* 0. *But* f' *has no limit at* 0, *so it cannot have a Taylor series expansion of order* 1 *at* 0. f' *has a regular part of* $2+2x$.

5.3.3 Asymptotic development

Definition 5.3.10 :

Let I *be an interval and* $x_0 \in I$ *be an endpoint of* I, *and let* f *be a real function defined on* I *or on* $I - \{x_0\}$. *We say* *that asymptotic development of f at x_0 if there exist simple functions $g_0, g_1, \cdots, g_n$ such that in the neighborhood of x_0:*

$$f(x) = a_0 g_0(x) + a_1 g_1(x) + \cdots + a_n g_n(x) + o(g_n(x)) \quad \text{and} \quad g_0 \geq g_1 \geq \cdots \geq g_n,$$

where $a_0, a_1, \cdots, a_n$ are real constants.

Remark 5.3.11 :

- *In an asymptotic development of a function at x_0 (finite or infinite), each term of the sum is negligible compared to the previous one in the neighborhood of x_0.*
- *To form an asymptotic development, we exploit the computational techniques of limited developments.*
- *We can perform a limited development at ∞. We introduce an auxiliary variable $X = \frac{1}{x} \underset{x \to \infty}{\longrightarrow} 0$.*

Examples 5.3.12 :

1. *Let's calculate the asymptotic development of $x \longmapsto ln(1-x)$ with 5 terms at $-\infty$. When x is in the neighborhood of $-\infty$, we have*

$$\begin{aligned}
ln(1-x) &= ln(-x(1-\frac{1}{x})) \\
&= ln(-x) + ln(1-\frac{1}{x}) \\
&= ln(-x) + ln(1-u(x)) \quad \text{with } u(x) = \frac{1}{x} \underset{x \to -\infty}{\longrightarrow} 0 \\
&= ln(-x) - u(x) - \frac{1}{2}u^2(x) - \frac{1}{3}u^3(x) - \frac{1}{4}u^4(x) - O\left(u^4(x)\right) \quad \text{with } u(x) = \frac{1}{x} \\
&= ln(-x) - \frac{1}{x} - \frac{1}{2x^2} - \frac{1}{3x^3} - \frac{1}{4x^4} - O\left(\frac{1}{x^4}\right).
\end{aligned}$$

2. *Let's calculate the asymptotic development of $x \longmapsto \sqrt{1+x^2}$ with 4 terms at $+\infty$. When x is in the neighborhood of $+\infty$, we have*

$$\begin{aligned}
\sqrt{1+x^2} &= x\sqrt{1+\frac{1}{x^2}} \\
&= x\sqrt{1+u(x)} \quad \text{with } u(x) = \frac{1}{x^2} \\
&= x\left(1 + \frac{1}{2}u(x) - \frac{1}{8}u^2(x) + \frac{1}{16}u^3(x) - O\left(u^3(x)\right)\right) \quad \text{with } u(x) = \frac{1}{x^2} \\
&= x + \frac{1}{2x} - \frac{1}{8x^3} + \frac{1}{16x^5} - O\left(\frac{1}{x^5}\right).
\end{aligned}$$

5.4 Applications of limited development

Determining equivalents

Limited developments are an effective tool for finding the equivalent of a given function in the neighborhood of a point.

★ The first non-zero term of a limited development or an asymptotic expansion of a function at a point provides a simple equivalent of the function studied at the point considered.

Examples 5.4.1 :

I. *From the limited developments of usual functions, we therefore obtain the following equivalents in the neighborhood of* 0:

$$\sin x \approx x,\ \tan x \approx x,\ ln(1+x) \approx x,\ e^x - 1 \approx x,\ 1-\cos x \approx \frac{x^2}{2},\ (1+x)^\alpha - 1 \approx \alpha x.$$

II. *Let's determine a simple equivalent of* $ln(1-x) + arcsinx$ *in the neighborhood of* 0.

We know that

$$ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O(x^3) \ \text{ and } \ arcsinx = x + \frac{1}{6}x^3 + O\left(x^3\right).$$

Then $ln(1-x) + arcsinx = -\frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^3)$. *This shows the following equivalent* $ln(1-x) + arcsinx \approx -\frac{1}{2}x^2$ *when* $x \longrightarrow 0$.

III. *Let's determine a simple equivalent of* $\frac{\sin x - x}{x\sin x}$ *in the neighborhood of* 0.

From the Taylor series expansion of order 4 *at* 0 *of the function* $\sin$, *we deduce that* $\sin x - x \underset{0}{\approx} -\frac{x^3}{6}$.

We conclude that

$$\frac{\sin x - x}{x \sin x} \underset{0}{\approx} -\frac{\frac{x^3}{6}}{x^2} = -\frac{x}{6} \ \text{ because } \ \sin x \underset{0}{\approx} x.$$

Limit determination

Obtaining an equivalent allows us to obtain the limit of the function under consideration.

Examples 5.4.2 :

$1*$ *Let's calculate the limit at* 0 *of the function* $x \longmapsto \frac{2x\cos x}{ln\frac{2+x}{2-x}}$.

We can calculate this limit as follows:

$$ln(2+x) = ln2 + ln(1+\frac{x}{2}) = ln2 + \frac{x}{2} + o(\frac{x}{2}) \ \text{ and } \ ln(2-x) = ln2 + ln(1-\frac{x}{2}) = ln2 - \frac{x}{2} + o(\frac{x}{2}).$$

Then we have

$$\begin{aligned} ln\frac{2+x}{2-x} &= ln(2+x) - ln(2-x) \\ &= x + O(\frac{x}{2}) \underset{0}{\approx} x. \end{aligned}$$

We can therefore deduce that

$$\begin{aligned} \lim_{x\longrightarrow 0} \frac{2x\cos x}{ln\frac{2+x}{2-x}} &= \lim_{x\longrightarrow 0} \frac{2x\cos x}{x} \\ &= \lim_{x\longrightarrow 0} 2.\cos x \\ &= 2. \end{aligned}$$

2∗ *Let's calculate the limit at* 0 *of the function* $x \longmapsto \dfrac{x.\tan^2 x - x^3}{2x^2.\tan^3 x}$.
From the limited development of order 3 *at* 0 *of the function* tan*, we obtain*

$$\tan x \underset{0}{\approx} x, \quad \tan x - x \underset{0}{\approx} \frac{1}{3}x^3 \quad \text{and} \quad \tan x + x \underset{0}{\approx} 2x.$$

Since

$$\begin{aligned}\frac{x.\tan^2 x - x^3}{2x^2.\tan^3 x} &= x.\frac{\tan^2 x - x^2}{2x^2.\tan^3 x} \\ &= x.\frac{(\tan x - x)(\tan x + x)}{2x^2.\tan^3 x}.\end{aligned}$$

We can therefore deduce that

$$\begin{aligned}\lim_{x \longrightarrow 0} \frac{x.\tan^2 x - x^3}{2x^2.\tan^3 x} &= \lim_{x \longrightarrow 0} x.\frac{\frac{1}{3}x^3.2x}{2x^5} \\ &= \frac{1}{3}.\end{aligned}$$

Remark 5.4.3 :

Taylor series expansions are very effective for calculating limits with indeterminate forms! It is enough to simply note that if $f(x) = c_0 + c_1(x - x_0) + \cdots$, *then* $\lim_{x \longrightarrow x_0} f(x) = c_0$.

Extension of a function at a point

Proposition 5.4.4 :

Let x_0 *be a point in an interval* I *or a finite endpoint of* I, *and let* f *be a real-valued function defined on* $I - \{x_0\}$.
If, in a neighborhood of x_0, *the function* f *admits a Taylor expansion of the form* $f(x) = c_0 + c_1(x - x_0) + o(x - x_0)$, *then we can extend* f *by continuity at* x_0 *by setting* $f(x_0) = c_0$. *This extension is differentiable at a with* $f'(x_0) = c_1$.

Proof :

Since $\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} (c_0 + c_1(x - x_0) + o(x - x_0)) = c_0$ we extend f by continuity at x_0 by setting $f(x_0) = c_0$. Moreover $\lim_{x \longrightarrow x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim_{x \longrightarrow x_0} \dfrac{(c_0 + c_1(x - x_0) + o(x - x_0) - f(x_0)}{x - x_0} = c_1$ and therefore this extension is differentiable at x_0 and $f'(x_0) = c_1$. ■

Example 5.4.5 :

Let's show that the function $f : x \longmapsto \dfrac{\tan x}{x}$ *extendable at* 0 *and its extension is a function of class* C^1 *on* $]-\frac{\pi}{2}, \frac{\pi}{2}[$.
The function f *is obviously defined and of class* C^1 *on the intervals* $]-\frac{\pi}{2}, 0[$ *and* $]0, \frac{\pi}{2}[$.
In the neighborhood of 0 *we have* $f(x) = 1 + \frac{1}{3}x^2 + o(x^2)$.
Therefore, f *can be extended by continuity at 0 by setting* $f(0) = 1$.
Furthermore, this extension is differentiable and $f'(0) = 0$.
Let's now examine the continuity of f' *at* 0.

For all $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[- \{0\}$, *we have*

$$\begin{aligned} f'(x) &= \frac{\frac{x}{\cos^2 x} - \tan x}{x^2} \\ &= \frac{x\left(1+\tan^2 x\right) - \tan x}{x^2}. \end{aligned}$$

In the neighborhood of 0 *we have*

$$\begin{aligned} f'(x) &= \frac{x\left(1+x^2+o(x^2)\right) - \left(x + \frac{1}{3}x^3 + o(x^3\right)}{x^2} \\ &= \frac{\frac{2}{3}x^3 + o(x^3)}{x^2} \\ &= \frac{2}{3}x + o(x) \underset{x \longrightarrow 0}{\longrightarrow} 0. \end{aligned}$$

Therefore, f' *is continuous at* 0.

Local position of a curve relative to its tangent

Let f be a real-valued function defined on an interval I containing x_0.

Let f admit a limited development of order n at x_0 of the form

$$f(x) = c_0 + c_1(x - x_0) + c_k(x - x_0)^k + \cdots c_n(x - x_0)^n + o((x - x_0)^n),$$

where c_k is the first non-zero coefficient after c_1 with $1 < k \leq n$.

Since the function f is defined at x_0, we then have that $c_0 = f(x_0)$.

According to proposition **5.4.4**, It is also established that $c_1 = f'(x_0)$.

The equation of the tangent T to the curve of f at the point with abscissa x_0 is: $y = c_0 + c_1(x - x_0)$.

Let's now study the position of the graph of f relative to T in the neighborhood of x_0.

To do this, let's determine the sign of $f(x) - (c_0 + c_1(x - x_0))$, that is, the sign of $c_k(x - x_0)^k$.

There are three possible cases.

• If the sign of $c_k(x - x_0)^k$ is positive, then the curve is above the tangent T.

• If the sign of $c_k(x - x_0)^k$ is negative, then the curve is below the tangent T.

• If the sign of $c_k(x - x_0)^k$ changes when moving from left to right at x_0, then the curve crosses the tangent line T at the point with abscissa x_0. This point is called the **inflection point**.

Examples 5.4.6 :

a) *Let's determine the tangent* (T) *at* 0 *of the* (Γ) *graph of* $f : \mathbb{R}^* \longrightarrow \mathbb{R}$ *defined by* $f(x) = \frac{2x}{e^x - 1}$ *and specify the position of the graph* (Γ) *relative to* (T).

In the neighborhood of 0 *with* $x \neq 0$*, we have*

$$\begin{aligned}
f(x) &= 2x.\frac{1}{x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+O\left(x^4\right)} \\
&= 2.\frac{1}{1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+O\left(x^3\right)} \\
&= 2.\frac{1}{1+u\left(x\right)} \quad with \quad u\left(x\right)=\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+O\left(x^3\right) \\
&= 2.\left(1-u+u^2-u^3+O\left(u^3\right)\right) \quad with \quad u^2\left(x\right)=\frac{x^2}{4}+\frac{x^3}{6}+O\left(x^3\right) \;\; and \;\; u^3\left(x\right)=\frac{x^3}{8}+O\left(x^3\right) \\
&= 2.\left(1-\left(\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+O\left(x^3\right)\right)+\left(\frac{x^2}{4}+\frac{x^3}{6}+O\left(x^3\right)\right)-\left(\frac{x^3}{8}+O\left(x^3\right)\right)+O\left(x^3\right)\right) \\
&= 2-x+\frac{1}{12}x^2+O\left(x^2\right).
\end{aligned}$$

Therefore, we can extend f *by continuity at* 0 *by setting* $f(0)=2$*. Furthermore, this extension is differentiable at* 0 *and* $f'(0)=-1$*. The equation of the tangent* (T) *is* $y=2-x$*.*

Since $f(x)-(2-x)\underset{0}{\approx}\frac{1}{12}x^2 \geq 0$*, then in the neighborhood of* 0 *the curve* (Γ) *lies above its tangent* (T)*.*

b) *Let's determine the tangent* (Δ) *at* 0 *of the* (C) *graph of* $g:\mathbb{R}-\{-2\}\longrightarrow\mathbb{R}$ *defined by:* $g(x)=ln(1/x^2+4x+4)$ *and specify the position of the graph* (C) *relative to* (Δ)*. In the neighborhood of* 0 *we have:*

$$\begin{aligned}
g(x) &= ln\left((x+2)^{-2}\right) \\
&= -2ln(x+2) \quad with \quad x>-2 \\
&= -2\left(ln2+ln(1+\frac{x}{2})\right) \quad with \quad x>-2 \\
&= -2\left(ln2-\frac{x}{2}+\frac{x^2}{8}-\frac{x^3}{24}+o\left(x^3\right)\right) \\
&= -2ln2+x-\frac{1}{4}x^2+\frac{1}{12}x^3+o\left(x^3\right).
\end{aligned}$$

The equation of the tangent line (Δ) *is* $y=-2ln2+x$*.*

Since $g(x)-(2-x)\underset{0}{\approx}-\frac{1}{4}x^2\leq 0$*, then in the neighborhood of* 0 *the curve* (C) *lies below its tangent line* (Δ)*.*

c) *We seek to determine the position of the graph* $(\mathfrak{C})$ *of* $h:\mathbb{R}\longrightarrow\mathbb{R}$ *defined by* $h(x)=x^4-4x^3+2$ *with respect to its tangent* (D) *at the point with abscissa* 2*.*

We note that the equation of the tangent at the point with abscissa 0 *is* $y=2$*. Since* $-4x^3$ *changes sign at* 0*, the point with abscissa* 0 *is an inflection point of* h*.*

Because the limited development of h *at* 0 *is* $h(x)=2-4x^3+x^4$*.*

• *For the LD of* $h(x)$ *at* 2*, we set* $u=x-2$

$$\begin{aligned}
h(x) &= h(u+2) \quad with \quad x=u+2 \\
&= (u+2)^4-4\left(u+2\right)^3+2 \\
&= \left(u^4+8u^3+24u^2+32u+16\right)-4\left(u^3+6u^2+12u+8\right)+2 \\
&= -14-16u+4u^3+u^4 \\
&= -14-16\left(x-2\right)+4\left(x-2\right)^3+\left(x-2\right)^4.
\end{aligned}$$

The equation of the tangent (D) *is* $y = -14 - 16(x-2)$. *Since* $4(x-2)^3$ *changes sign at* 2, *the point with abscissa* 2 *is an inflection point of* h. *Furthermore, for* x *close enough to* 2 *on the left* $(x \in]2-\varepsilon, 2[$ with $\varepsilon > 0)$, *the graph* (C) *is below its tangent* (D), *and for* x *close enough to* 2 *on the right* $(x \in]2, 2+\varepsilon[)$, *the graph* (C) *is above its tangent* (D).

Position of a curve relative to its asymptote

Let f be a function defined in the neighborhood of ∞ and a and b be two real constants. An asymptotic development of f at ∞ of the form $f(x) = ax + b + \frac{c_k}{x^k} \cdots \frac{c_n}{x^n} + o(\frac{1}{x^n})$ where k is the smallest integer such that $1 < k \leq n$ and the coefficient of $\frac{1}{x^k}$ is non-zero, allows us to conclude that the line with equation $y = ax + b$ is an asymptote to the curve of f at ∞.
Furthermore, studying the sign of $\frac{c_k}{x^k}$ allows us to position the curve of f relative to this asymptote.

Examples 5.4.7 :

1°) *Let's determine the equation of the asymptote* (Δ) *of the graph of* (C), *defined by:*
$f(x) = e^{\frac{1}{x}}.\sqrt{x^2+2x+1}$ *and specify the position of the graph* (C) *relative to* (Δ) *at* $+\infty$.
In the neighborhood of ∞ *we have*

$$\begin{aligned}
f(x) &= e^{\frac{1}{x}}.(x+1) \\
&= e^u.\left(\frac{1}{u}+1\right) \quad \text{with} \quad u = \frac{1}{x} \underset{x \longrightarrow +\infty}{\longrightarrow} 0 \\
&= \left(1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o\left(u^3\right)\right)\left(\frac{1}{u}+1\right) \\
&= 2 + \frac{1}{u} + \frac{3}{2}u + \frac{2}{3}u^2 + o\left(u^2\right) \\
&= 2 + x + \frac{3}{2x} + \frac{2}{3x^2} + o\left(\frac{1}{x^2}\right).
\end{aligned}$$

Therefore, the equation of (Δ), *the asymptote of* f *at* $+\infty$, *is* $y = 2 + x$.
Since $f(x) - (2+x) \underset{+\infty}{\approx} \frac{3}{2x} > 0$, *then in the neighborhood of* $+\infty$, *the curve* (C) *remains above* (Δ).

2°) *Let's determine the equation of the asymptote* (Δ') *of* (C'), *the graph of* $g : \mathbb{R} \longrightarrow \mathbb{R}$, *defined by:*
$g(x) = \dfrac{1}{\sqrt{x^2+x+1}}$ *and specify the position of the graph* (C') *relative to* (Δ') *at* $-\infty$.

In the neighborhood of $-\infty$, we have

$$\begin{aligned} g(x) &= \frac{1}{-x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} \\ &= -\frac{1}{x}.\frac{1}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} \\ &= -\frac{1}{x}.\frac{1}{\sqrt{1+u}} \quad with \quad u=\frac{1}{x}+\frac{1}{x^2} \underset{x\longrightarrow-\infty}{\longrightarrow} 0 \\ &= -\frac{1}{x}.\left(1-\frac{1}{2}u+\frac{3}{8}u^2+o\left(u^2\right)\right) \quad with \quad u^2=\frac{1}{x^2}+o\left(\frac{1}{x^2}\right) \\ &= -\frac{1}{x}.\left(1-\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x^2}\right)+\frac{3}{8}\left(\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)+o\left(\frac{1}{x^2}\right)\right). \\ &= -\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{8x^3}+o\left(\frac{1}{x^3}\right). \end{aligned}$$

Therefore, the equation of (Δ'), the asymptote of (C') at $-\infty$, is $y=0$. Since $g(x) \underset{-\infty}{\approx} -\frac{1}{x} > 0$, then in the neighborhood of $-\infty$, the curve (C') remains above the asymptote (Δ').

5.4.1 Exercises

Exercise 5.4.8 :

Calculate the limited development in the neighborhood of 0 of

a) $x \longmapsto \dfrac{1}{1+\sqrt{1+x}+\sqrt{1-x}}$ *of order* 3.

b) $x \longmapsto ln\left(\dfrac{ln\,(1+x)}{x}\right)$ *of order* 4.

c) $x \longmapsto \dfrac{1}{1-arctanx}$ *of order* 5.

Exercise 5.4.9 :

1* *Determine the limited development to order 4, in the neighborhood of $\frac{\pi}{3}$, of the function: $x \longmapsto \cos x$*

2* *Calculate the limited development to order 4, at $\frac{\pi}{3}$, of the function $h : x \longmapsto ln(\sin x)$.*

3* ***a)*** *Give a limited development to the order 2 of $f(x)=\dfrac{\sqrt{1+x^2}}{1+x+\sqrt{1+x^2}}$ at 0.*

b) *Deduce an asymptotic development with 2 terms in $+\infty$ of f.*

c) *Calculate an asymptotic expansion with one term in $-\infty$.*

Exercise 5.4.10 :

1. *Write the limited development of order 6 at 0 of the functions $\longmapsto shx$ and $\longmapsto chx$.*

2. *Calculate, by performing the multiplication, the limited development of order 6 at 0 of the functions:*

$$x \longmapsto sh^2x, \quad x \longmapsto ch^2x \quad \text{and} \quad x \longmapsto shx.chx.$$

3. *Find the results from the previous question, using the formulas:*

$$sh^2x = \frac{ch(2x)-1}{2}, \quad ch^2x = \frac{ch(2x)+1}{2} \quad \text{and} \quad shx.chx = \frac{sh(2x)}{2}.$$

Exercise 5.4.11 *:*

We assume that the limited development of the function argsh to order 5 *at* 0 *is*

$$argshx = ax + bx^3 + cx^5 + o(x^5),$$

where a, b, and c are real constants.

Find a, b, and c using two different methods.

Recall that for all $x \in \mathbb{R}$, we have

$$argshx = ln\left(x + \sqrt{x^2+1}\right), \ argsh'x = \frac{1}{\sqrt{x^2+1}}, \ ch\left(argshx\right) = \sqrt{x^2+1} \ \text{and} \ sh\left(argshx\right) = x.$$

Exercise 5.4.12 *:*

(**1**) *Calculate the asymptotic development of $x \longmapsto cothx$ at* 0 *to the precision of x^3.* (**2**) *Calculate the asymptotic development with* 3 *terms in the neighborhood of $+\infty$ of $x \longmapsto x.arctan\left(\frac{1}{x+1}\right)$.*

Exercise 5.4.13 *:*

Determine the following limits:

$$\lim_{x \longrightarrow a} \frac{x^a - a^x}{arctanx - arctana} \ \text{with} \ a > 0, \qquad \lim_{x \longrightarrow 0} \frac{arctanx - x^2}{\cos\left(x^2\right) - 1},$$

$$\lim_{x \longrightarrow 0} \frac{shx - 2sh2x + sh3x}{ln\left(1 + x + 2x^2 + \sqrt{1-2x} - 1 - x^2\right)}, \qquad \lim_{x \longrightarrow +\infty} \left(\frac{ln(x+1)}{lnx}\right)^{xlnx}.$$

Exercise 5.4.14 *:*

1) *Specify the position of the graph $(\mathcal{C})$ of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = e^x + \sin x$ with respect to its tangent (T) at the point with abscissa* 0.

2) *Specify the position of the graph $(-)$ of the function $g :]-1, +\infty \longrightarrow \mathbb{R}$ defined by $g(x) = ln(x^2 + 2x + 1)$ with respect to its tangent (Δ) at the point with abscissa* 1.

Exercise 5.4.15 *:*

a) *Calculate the asymptotic development with* 5 *terms of $x \longmapsto \frac{x}{x^2-1}$ in $+\infty$.*

b) *Calculate the asymptotic development with* 3 *terms of $x \longmapsto \left(1 + \frac{1}{x}\right)^x$ in $+\infty$.*

c) *Calculate the asymptotic development with* 3 *terms of $f : x \longmapsto \sqrt[3]{x^3+1} - \sqrt{x^2+|x|}$ in $+\infty$ and in $-\infty$. Deduce the limits of f as x approaches $+\infty$ and $-\infty$.*

d) *Calculate the asymptotic development in* 3 *terms of g: $x \longmapsto \sqrt{\frac{x^3+1}{x+1}}$ as x approaches $+\infty$. Then determine the equation of the asymptote as x approaches $+\infty$ and the position of the graph of g relative to this asymptote.*

Exercise 5.4.16 *:*

Note that $f' = g \circ h$ *where:*

$$g : x \in]-1,1[\longmapsto \frac{-1}{1+x} \quad \text{and} \quad h : x \in \mathbb{R} \longmapsto x^2 + 2x.$$

1* *Find the expression for the limited development of order* 4 *at* 0 *of* f'.

2* *Deduce the expression for the limited development of order* 5 *at* 0 *of* f.

Exercise 5.4.17 *:*

Show that the limited development of order 3 *at* 0 *of* $x \longmapsto \cos x . ln\,(1+x)$ (*resp.* $x \longmapsto (1+x)^{\frac{1}{1+x}}$) *has a regular part* $x - \frac{x}{2} - \frac{x^3}{6}$ (*resp.* $1 + x - x^2 - \frac{x^3}{2}$).

Exercise 5.4.18 *:*

For $n \in \mathbb{R}$, *we denote by* f_n *the function defined on* $\mathbb{R}_+^*$ *by*

$$f_n(x) = \begin{cases} \dfrac{x^n lnx}{x^2 - 1} & if\, x \neq 1 \\ \dfrac{1}{2} & if x = 1 \end{cases} .$$

a^*) *Determine a Taylor series expansion to order* 2 *at* 1 *of the function* f_n.

Deduce the equation of the tangent (T_n) *of the graph* (C_n) *of* f_n *at the point with abscissa* 1, *as well as the position of the curve* (C_0) *of* f_0 *relative to* (T_0) *in the neighborhood of* 1.

Exercise 5.4.19 *:*

Let f *be the function defined on* $]-1,1[$ *by* $f(x) = \dfrac{arcsin(x)}{1-x^2}$.

1.) *Determine the function* $g :]-1,1[\longrightarrow \mathbb{R}$ *such that, for all* $x \in]-1,1[$; $f'(x) + g(x)f(x) = \dfrac{1}{1-x^2}$.

2.) *Determine a limited development to order* 4 *at* 0 *of the function* g.

3.) *Deduce a Taylor series expansion to order* 5 *at* 0 *of* f.

Chapter 6

Simple Integrals

5-1 *Riemann integral.*
5-2 *Calculus of primitives.*
The aim of this chapter is twofold: First, to understand and know integration and its main properties. Most importantly, it teaches how to quickly calculate integrals using antiderivatives or the two effective tools: integration by parts and by changing variables.

6.1 Riemann integral

Integral of a Step Function

Definition 6.1.1 *Let $[a, b]$ be a closed bounded interval of $\mathbb{R}$ such that $(a < b)$. A subdivision of $[a, b]$ is a finite sequence of real numbers $\{x_0, x_1, \cdots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.*

Abusing notation, we will denote a partition P either by its intervals

$$P = \{I_1, I_2, \cdots, I_n\},$$

or by the set of endpoints of the intervals

$$P = \{x_0, x_1, \cdots, x_n\}.$$

⊗ We denote the length of an interval $I = [a, b]$ by $|I| = b - a$ and $|I_k| = |[x_{k-1}, x_k]| = x_k - x_{k-1}$.
⊗ Note that $\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} (x_k - x_{k-1}) = |I|$.
For example, the set of intervals

$$P = \left\{ \left[0, \frac{1}{6}\right], \left[\frac{1}{6}, \frac{1}{3}\right], \left[\frac{1}{3}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{2}{3}\right], \left[\frac{2}{3}, \frac{5}{6}\right], \left[\frac{5}{6}, 1\right] \right\}$$

is a partition of $[0,1]$.

The corresponding set of endpoints is

$$P=\left\{0,\frac{1}{6},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{5}{6},1\right\}.$$

Suppose that $P=\{I_1,I_2,\cdots,I_n\}$ is a partition of the compact interval $I=[a,b]$ and $f:I\longrightarrow\mathbb{R}$ is a bounded function on I with $M=\underset{[a,b]}{sup}f$ and $m=\underset{[a,b]}{inf}f$. Then $M_k=\underset{I_k}{sup}f$ and $m_k=\underset{I_k}{inf}f$.

Moreover, since f is bounded on I, then these suprema and infima are finite real numbers and $m\le m_k\le M_k\le M$.

$\otimes$ We define the **upper** (**lower**) **Riemann sum** of f with respect to the partition P by

$$U(f;P)=\sum_{k=1}^{n}M_k\,|I_k|=\sum_{k=1}^{n}M_k\,(x_k-x_{k-1}),$$

$$L(f;P)=\sum_{k=1}^{n}m_k\,|I_k|=\sum_{k=1}^{n}m_k\,(x_k-x_{k-1}).$$

$\otimes$ Geometrically, $U(f;P)$ $(L(f;P))$ is the sum of the areas of rectangles based on the intervals I_k that lie above (below) the graph of f.

Note that

$$m(b-a)\le L(f;P)\le U(f;P)\le M(b-a).$$

Let $\Pi([a,b])$, denote the collection of all partitions of $[a,b]$.

Now, we define the upper (lower) Riemann integral of f on $[a,b]$ by

$$U(f)=\underset{P\in\Pi([a,b])}{inf}U(f;P)\quad \text{and}\, L(f)=\underset{P\in\Pi([a,b])}{sup}U(f;P).$$

Definition 6.1.2 *A bounded function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be integrable (in the Riemann sense) if $U(f)=L(f)$. This number is called the* ***Riemann integral*** *of f on $[a,b]$ and is denoted $\displaystyle\int_a^b f(x)dx$ or $\int_a^b f$ or $\int_{[a,b]} f$.*

Examples 6.1.3 *:*

1 Let $f:[0,1]\longrightarrow\mathbb{R}$ defined by*

$$f(x)=\begin{cases}\frac{1}{x}, & if\, 0<x\le 1\\ 0, & if\, x=0.\end{cases}$$

Then $\int_0^1\frac{1}{x}dx$ isn't defined as a Riemann integral becuase f is unbounded in the interval $[0,1]$. In fact, if $P=\{x_0,x_1,\cdots,x_n\}$ is a partition of $[0,1]$, where $0<x_1<x_2<\cdots<x_{n-1}<1$, then $\underset{[0,x_1]}{sup}f=+\infty$, so the upper Riemann sums of f are not well-defined.

2 Let c areal constant and f the constant function $f(x)=c$ on $[0,2]$ is Riemann integrable. Then $\int_0^2 cdx=2c$.*

To show this, let $P = \{I_1, I_2, \cdots, I_n\}$ *be any partition of* $[0,2]$ *with endpoints* $P = \{0, x_1, \cdots, x_{n-1}, 2\}$. *Then* $M_k = \underset{I_k}{sup} f = c$, $m_k = \underset{I_k}{inf} f = c$, *for* $k = 1, \cdots, n$, *and therefore*

$$U(f;P) = L(f;P) = \sum_{k=1}^{n} c\,(x_k - x_{k-1}) = c\,(x_n - x_o) = 2c.$$

Which implies that

$$U(f) = \underset{P \in \Pi([0,2])}{inf} U(f;P) = L(f) = \underset{P \in \Pi([0,2])}{sup} U(f;P) = 2c.$$

$U(f)$ *and* $L(f)$ *are equal, and the integral of* f *on the interval* $[0,2]$ *is* $2c$.

3 Let* $g : [1,2] \longrightarrow \mathbb{R}$ *defined by*

$$g(x) = \begin{cases} 1, & if\, 1 < x \leq 2 \\ 2, & if\, x = 1. \end{cases}$$

Then g *is Riemann integrable, and* $\int_1^2 f(x)dx = 1$. *To show this, let* $P = \{I_1, I_2, \cdots, I_n\}$ *be any partition of* $[1,2]$ *with endpoints* $\{0, x_1, \cdots, x_{n-1}, 2\}$. *Then, since* $g(x) = 1$ *for* $1 < x \leq 2$, *we have*

$$M_k = \underset{I_k}{sup} g = m_k = \underset{I_k}{inf} g = 1, \text{ for } k = 2, \cdots, n.$$

The first interval in the partition is $I_1 = [1, x_1]$, *where* $1 < x_1 \leq 2$, *and* $M_1 = 2$, $m_1 = 1$, *since* $g(1) = 2$ *and* $g(x) = 1$ *for* $1 < x \leq x_1$. *It follows that* $U(g;P) = x_1$, $L(g;P) = 1$.

Thus, $L(g) = 1$ *and* $U(g) = inf\,\{x_1; 1 < x_1 \leq 2\} = 1$, *so* $U(g) = L(g) = 1$, *and the integral of* f *is* 1.

Remark 6.1.4 *This example shows that a function can be discontinuous but integrable in the Riemann sense.*

Theorem 6.1.5 *[2]Let* $f : [a,b] \longrightarrow \mathbb{R}$ *a continuous function on a compact interval, then it is Riemann integrable.*

Theorem 6.1.6 *[2]If* $f : [a,b] \longrightarrow \mathbb{R}$ *is a bounded function with finitely many discontinuities, then* f *is Riemann integrable.*

Theorem 6.1.7 *(Fundamental theorem) [2]*

If $f : [a,b] \longrightarrow \mathbb{R}$ *is continuous on* $[a,b]$ *and differentiable in* $]\,a,b\,[$ *with* $F'(x) = f(x)$ *where* f *is Riemann integrable on* $[a,b]$, *then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Examples 6.1.8 ***1).*** *Let* $F : [0,1] \longrightarrow \mathbb{R}$ *defined by*

$$F(x) = \begin{cases} x^3 \sin(\frac{1}{x}), & if\, 0 < x \leq 1 \\ 0, & if\, x = 0. \end{cases}$$

Then F *is continuous and differentiable on* $[0,1]$, *with*

$$F'(x) = \begin{cases} 3x^3 sin(\frac{1}{x}) - x\cos(\frac{1}{x}), & if\, 0 < x \leq 1 \\ 0, & if\, x = 0. \end{cases}$$

The derivative $F^{'}$ is bounded and continuous on $[0,1]$. From Theorem 6.1.7 implies that $F^{'}$ is integrable on $[0,1]$, and we conclude that

$$\int_0^1 F^{'}(x)dx = \sin 1.$$

2). *Let $F : [1,2] \longrightarrow \mathbb{R}$ defined by: $F(x) = \sqrt{x-1}$. Then F is continuous on $[1,2]$ and differentiable in $]1,2\,]$, with $F^{'}(x) = \frac{1}{2\sqrt{x-1}} for 1 < x \leq 2$. This function is unbounded in $]1,2\,]$, this imply that $F^{'}$ is not Riemann integrable on $[1,2]$, and Theorem 6.1.7 does not apply. We can, however, interpret the integral of $F^{'}$ on $[1,2]$ as an improper Riemann integral. The function F is continuously differentiable on $[1+\varepsilon, 2]$ for every $0 < \varepsilon < 1$, so*

$$\int_{1+\varepsilon}^2 \frac{1}{2\sqrt{x-1}} dx = 1 - \sqrt{\varepsilon}.$$

Thus, we get the improper integral

$$\lim_{\epsilon \longrightarrow 0} \int_{1+\varepsilon}^2 \frac{1}{2\sqrt{x-1}} dx = 1.$$

Theorem 6.1.9 *[1]Let $f : [a,b] \longrightarrow \mathbb{R}$ a monotonic function on a compact interval, then it is Riemann integrable.*

Property 6.1.10 *If $f : [a,b] \longrightarrow \mathbb{R}$ is integrable, then the restriction of f to any interval $\left[a^{'}, b^{'}\right] \subset [a,b]$ is still integrable.*

6.1.1 Properties of the Riemann integral

Property 6.1.11 *The integral has the following three basic properties.*

(1). ***Additivity (Chasles relation).***

Let $a < c < b$ be three real numbers. If f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$. And we have:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(2). ***Positivity of the integral (Monotonicity).***

Let a and b be two real numbers and f and g be two integrable functions on $[a,b]$. If $f \leq g$ then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

$\otimes$ *In particular, the integral of a positive function is positive: That is, if $f \geq 0$ on $[a,b]$ then $\int_a^b f(x)dx \geq 0$.*

(3). ***Linearity**: Let f and g be two integrable functions on $[a,b]$. Then*

1. $f+g$ is an integrable function and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

2. For all real numbers λ, λf is integrable function and

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$$

3. fg is an integrable function on $[a,b]$ but in general

$$\int_a^b (f.g)(x)dx \neq \left(\int_a^b f(x)dx\right).\left(\int_a^b g(x)dx\right).$$

4. $|f|$ is an integrable function on $[a,b]$ and

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|\, dx.$$

Example 6.1.12 *(a). Evaluate* $\displaystyle\int_1^3 -2e^{3x} + 15 - \frac{1}{2x}\, dx.$

Answer:

$$\begin{aligned}\int_1^3 -2e^{3x} + 15 - \frac{1}{2x}\, dx &= -2\int_1^3 e^{3x}\, dx + \int_1^3 15\, dx - \tfrac{1}{2}\int_1^3 \tfrac{1}{2x}\, dx\\ &= -2\left[\tfrac{1}{3}e^{3x}\right]_1^3 + [15x]_1^3 - \tfrac{1}{2}\left[ln\,|x|\right]_3^1\\ &= -\tfrac{2}{3}\left(e^9 - e\right) + 15\,(3-1) - \tfrac{1}{2}\,(ln3 - ln1)\\ &= -\tfrac{2e}{3}\left(e^8 - 1\right) + 30 - \tfrac{1}{2}ln3.\end{aligned}$$

6.2 Calculus of primitives

Definition 6.2.1 *Let $f : I \longrightarrow \mathbb{R}$ be a function defined on any interval I. We say that $F : I \longrightarrow \mathbb{R}$ is an anti-derivative of f on I if F is a differentiable function on I satisfying $F^{'}(x) = f(x)$ for all $x \in I$.*

Examples 6.2.2 *:*

1. *Let $f : I = \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = 2x^3$. Then $F : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $F(x) = \frac{1}{2}x^4$ is an anti-derivative of f. The function defined by $F(x) = \frac{1}{2}x^4 + c$ where c a real constant, is also an anti-derivative of f.*

2. *Let $g : I = [\,0, +\infty\,[\longrightarrow \mathbb{R}$ defined by $g(x) = \sqrt{x^3}$. Then $G : [\,0, +\infty\,[\longrightarrow \mathbb{R}$ defined by $G(x) = \frac{2}{5}x^{\frac{5}{2}}$ is an anti-derivative of g on $[\,0, +\infty\,[$. For all $c \in \mathbb{R}$, the function $G + c$ is also an anti-derivative of g on $[\,0, +\infty\,[$.*

Proposition 6.2.3 *Let $f : I \longrightarrow \mathbb{R}$ be a function and let $F : I \longrightarrow \mathbb{R}$ be an anti-derivative of f. Every anti-derivative of f can be written as $G = F + c$ where c a real constant.*

Proof : First, note that if we denote by G the function defined by $G(x) = F(x) + c$, then $G^{'}(x) = F^{'}(x)$, but since $F^{'}(x) = f(x)$, then $G^{'}(x) = f(x)$, and G is indeed an anti-derivative of f.

For the converse, suppose that G is any anti-derivative of f. Then $(G - F)^{'}(x) = G^{'}(x) - F^{'}(x) = f(x) - f(x) = 0$, so the function $G - F$ has a derivative of zero on an interval; it is therefore a constant function! Therefore, there exists $c \in \mathbb{R}$ such that $(G - F)(x) = c$. In other words, $G(x) = F(x) + c$ for all $x \in I$. ∎

Notation. We will denote an anti-derivative of f by $\int f(t)dt$ or $\int f(x)dx$ or $\int f(u)du$ (the letters $t, x, u, \cdots$ are called silent letters, that is to say, interchangeable).

We can even denote an anti-derivative simply by $\int f$.

To find an anti-derivative of a function f, one might be lucky enough to recognize that f is the derivative of a well-known function. Unfortunately, this is very rarely the case, and the anti-derivatives of most functions are unknown. However, we will examine two techniques that allow us to calculate integrals and anti-derivatives: integration by parts and the change of variables.

6.2.1 Integration by parts

Suppose that $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable in $]\,a, b\,[$, and $f^{'}, g^{'}$ are integrable on $[a, b]$. Then

$$\int_a^b f g^{'} dx = [f(x)g(x)]_a^b - \int_a^b f^{'} g dx = f(b)g(b) - f(a)g(a) - \int_a^b f^{'} g dx.$$

Examples 6.2.4 ***(a).*** *Calculation of* $\int_0^1 x\cos(x)dx$.
Let $u(x) = x$ *and* $v^{'}(x) = \cos(x)$. *We will need to know that* $u^{'}(x) = 1$ *and that an anti-derivative of* $v^{'}$ *is simply* $v(x) = \sin(x)$. *The integration by parts formula gives:*

$$\begin{aligned}
\int_0^1 x\cos(x)dx &= \int_0^1 u(x)v^{'}(x)dx \\
&= [u(x)v(x)]_0^1 - \int_0^1 u^{'}(x)v(x)dx \\
&= [x.\sin(x)]_0^1 - \int_0^1 1.\sin(x)dx \\
&= [x.\sin(x)]_0^1 - [-\cos(x)]_0^1 \\
&= (1.\sin(1) - 0.\sin(0)) - ((-\cos(1)) - (-\cos(0))) \\
&= \sin(1) + \cos(1) - 1.
\end{aligned}$$

(b). *We find* $I = \int_{-1}^1 e^{3x}\sin(2x)\,dx$.
Let us make a choice: $u(x) = \sin(2x)$ *and* $v^{'}(x) = e^{3x}$.
So that $u^{'}(x) = 2\cos(2x)$ *and* $v(x) = \frac{1}{3}e^{3x}$.
The integration by parts formula gives:

$$\begin{aligned}
\int_{-1}^1 e^{3x}\sin(2x)\,dx &= \int_{-1}^1 v^{'}(x).u(x)dx \\
&= [u(x).v(x)]_{-1}^1 - \int_{-1}^1 v(x).u^{'}(x)dx \\
&= \left[\sin(2x).\frac{1}{3}e^{3x}\right]_{-1}^1 - \int_{-1}^1 \frac{1}{3}e^{3x}.2\cos(2x)dx \\
&= \frac{1}{3}\left[\sin(2x).e^{3x}\right]_{-1}^1 - \frac{2}{3}\int_{-1}^1 e^{3x}.\cos(2x)dx.
\end{aligned} \tag{6.1}$$

This is better than the original integral, but we now integrate by parts again, choosing $f(x) = \cos(2x)$ *and* $g^{'}(x) = e^{3x}$.

So that $f^{'}(x) = -2\sin(2x)$ *and* $g(x) = \frac{1}{3}e^{3x}$. *This gives*

$$\begin{aligned}
\int_{-1}^{1}\cos(2x).e^{3x}dx &= \int_{-1}^{1} f(x)g^{'}(x)dx \\
&= [f(x).g(x)]_{-1}^{1} - \int_{-1}^{1} f^{'}(x).g(x)dx \\
&= \left[\cos(2x).\frac{1}{3}e^{3x}\right]_{-1}^{1} - \int_{-1}^{1} -2\sin(2x).\frac{1}{3}e^{3x}dx \\
&= \frac{1}{3}\left[\cos(2x).e^{3x}\right]_{-1}^{1} + \frac{2}{3}\int_{-1}^{1}\sin(2x).e^{3x}dx \\
&= \frac{1}{3}\left[\cos(2x).e^{3x}\right]_{-1}^{1} + \frac{2}{3}I.
\end{aligned} \tag{6.2}$$

By inserting (6.2) *into* (6.1), *we obtain*

$$\begin{aligned}
I &= \frac{1}{3}\left[\sin(2x).e^{3x}\right]_{-1}^{1} - \frac{2}{3}\left[\frac{1}{3}\left[\cos(2x).e^{3x}\right]_{-1}^{1} + \frac{2}{3}I\right] \\
&= \frac{1}{3}\left[\sin(2x).e^{3x}\right]_{-1}^{1} - \frac{2}{9}\left[\cos(2x).e^{3x}\right]_{-1}^{1} - \frac{4}{9}I.
\end{aligned} \tag{6.3}$$

From which, we have

$$\begin{aligned}
\frac{13}{9}I &= \frac{1}{3}\left[\sin(2x).e^{3x}\right]_{-1}^{1} - \frac{2}{9}\left[\cos(2x).e^{3x}\right]_{-1}^{1} \\
&= \frac{1}{3}\left[\sin(2.1).e^{3.1} - \sin(2.(-1)).e^{3.(-1)}\right] - \frac{2}{9}\left[\cos(2.1).e^{3.1} - \cos(2.(-1)).e^{3.(-1)}\right] \\
&= \frac{1}{3}\left[\sin(2).e^{3} - \sin(-2).e^{-3}\right] - \frac{2}{9}\left[\cos(2).e^{3} - \cos(-2).e^{-3}\right] \\
&= \frac{1}{3}\sin(2)\left[e^{3} + e^{-3}\right] - \frac{2}{9}\cos(2)\left[e^{3} - e^{-3}\right] \\
&= \frac{1}{9}\left(e^{3} - e^{-3}\right)\left(3\sin(2) - 2\cos(2)\right).
\end{aligned} \tag{6.4}$$

This imply that

$$I = \int_{-1}^{1} e^{3x}\sin(2x)\,dx = \frac{1}{13}\left(e^{3} - e^{-3}\right)\left(3\sin(2) - 2\cos(2)\right)$$

6.2.2 Integration by substitution(Variable change)

Suppose that $g : I \longrightarrow \mathbb{R}$ differentiable on an open interval I and $g^{'}$ is integrable on I. Let $J = g(I)$. If $f : J \longrightarrow \mathbb{R}$ continuous, then for every $a, b \in I$,

$$\int_{a}^{b} f\left(g(x)\right)g^{'}(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 6.2.5 ***(a)*** *Find* $\int_{-1}^{2} 3x\exp(x^2)dx$.

Solution: *We transform the variable* x *into* u *by setting* u *to* x^2. *Therefore, for* $du = 2xdx$, *the lower boundary,*

$x=-1$, becomes $u=1$. The upper boundary, $x=2$, becomes $u=4$. By substitution, we have

$$\begin{aligned}
\int_{-1}^{2} 3xe^{(x^2)}dx &= \frac{3}{2}\int_{-1}^{2} e^{(x^2)}(2xdx) \\
&= \frac{3}{2}\int_{1}^{4} e^{u}du && \text{(Do the substitution.)} \\
&= \frac{3}{2}e^{u}\Big|_{1}^{4} && \text{(Find the antiderivative.)} \\
&= \frac{3}{2}e^{4}-\frac{3}{2}e^{1}=\frac{3}{2}e\left(e^{3}-1\right). && \text{(Evaluate.)}
\end{aligned}$$

(b) *Find* $J=\displaystyle\int_{1}^{2}\frac{lnx}{xlnx-x}dx.$

Solution: *Let the change of variable be $u=\phi(x)=xlnx-x$. Then $du=\phi'(x)=lnxdx$. For $x=1$ we have $u=\phi(1)=-1$ and for $x=2$ we have $u=\phi(2)=2ln2-2$. Then*

$$\begin{aligned}
J=\int_{1}^{2}\frac{lnx}{xlnx-x}dx &= \int_{1}^{2}\frac{lnxdx}{(xlnx-x)} \\
&= \int_{-1}^{2ln2-2}\frac{du}{u} && \text{(Do the substitution.)} \\
&= (ln\,|u|)|_{-1}^{2ln2-2} && \text{(Find the antiderivative.)} \\
&= ln\,|2ln2-2|-ln\,|-1| && \text{(Evaluate.)} \\
&= ln\,(2-2ln2).
\end{aligned}$$

6.2.3 Exercises

Exercise 6.2.6 *Calculate the integrals using integration by parts:*

$$\int_{0}^{\frac{\pi}{2}} t\ \sin(t)\,dt, \quad \text{and} \quad \int_{0}^{\frac{\pi}{2}} t^{2}\ \sin(t)\,dt,$$

then by induction

$$\int_{0}^{\frac{\pi}{2}} t^{n}\ \sin(t)\,dt \text{ where } n\in\mathbb{N}.$$

Exercise 6.2.7 *Calculate the integrals using changes of variables:*

$$\int_{0}^{a}\sqrt{a^{2}-t^{2}}dt \quad \text{and} \quad \int_{-\pi}^{\pi}\sqrt{1+\cos t}dt,$$

(for the latter, make two changes of variables: $u=\cos t$, then $v=1-u$).

Exercise 6.2.8 *Evaluate the following integrals:*

$$I(x)=\int\sin^{3}xdx; \qquad J(x)=\int\frac{x}{e^{x}}dx; \qquad L(x)=\int\frac{x^{3}}{(x^{2}+2)^{2}}dx.$$

Exercise 6.2.9 *Calculate the value of each of the following:*

$$I=\int_{1}^{2}x^{3}ln\,|x|\,dx; \qquad J=\int_{0}^{\frac{\pi}{4}}x^{2}\sin 2xdx; \qquad L=\int_{1}^{2}\sqrt{x^{3}}ln(\sqrt[x]{3})dx.$$

Bibliography

[1] Arnaud Bodin, Marc Bourdon, Niels Borne, Guoting Chen, Gilles Costantini, Laura Desideri, Abdellah Hanani and Jean-Louis Rouget (L'équipe Exo7), *The Riemann Integral*, université de Lille 1, Janvier 2016.

[2] Arnaud Bodin, Marc Bourdon, Sophie Chemla (L'équipe Exo7), *Anlyse. COURS DE MATHEMATIQUES PREMIÈRE ANNÉE*, université de Lille 1, Version 1.00 - Janvier 2016.

[3] ATTAR Ahmed, MIRI Sofiane Elhadi, *Algèbre et Analyse Recueil d'Exercices Corrigés*,Faculté des Sciences Département de Mathématiques, Université Abou Bekr Belkaid Tlemcen. 8 mars 2018 .

[4] Bernard Ycart, *Développements limités*, Université Joseph Fourier, Grenoble.

[5] David Delaunay, *Cours mathématiques MPSI* , http://mp.cpgedupuydelome.fr. 26 août 2013.

[6] David Delaunay, *Algèbre linéaire 1 : Résumé de cours*, Université de Bordeaux Licence de Sciences, Technologies, Santé Mathématiques, Informatique, Sciences de la Matière et Ingénierie. Version du 25 avril 2017.

[7] Cl. Gabriel, *Mathématiques avancées* , http://ww3.ac-poitiers.fr/math/mol/mole.htm, 2015-2016.

[8] Gauss, Carl Friedrich, *Carl Friedrich Gauss Werke*, Vol 2. Gottingen, Koniglichen Gesellschaft Wisenschaften Zu Gottingen, 1876. 526p.

[9] Raymond Delbert Haertel (Thesis), *Summation formulas for the greatest integer part function*, Oregon state university, August 1966.

[10] Stéphane Balac, Frédéric Sturm, *Algebre et analyse. Cours de mathématiques de première année avec exercices corrigés* , Collection des sciences appliquées de l'INSA de Liyon. Juillet (2003).