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Study of polyconvexity in some problems in the calculus of variations

Thesis

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Abstract

This study seeks to investigate the concept of symmetric polyconvex functions in higher-dimensional spaces. By advancing the methodology introduced by Boussaid et al. for two-dimensional and three-dimensional cases, we introduce an innovative characterization of symmetric polyconvex functions in higher dimensions. Our principal finding reveals that the requisite condition for symmetric polyconvexity of a function f *is its ability to be formulated as a convex function that incorporates the matrix and its second-order minors, exhibiting a non-increasing tendency in a specific sense with respect to the second-order minor variable. Additionally, we propose and scrutinize the concept of S-positive semi-definite matrices, which is crucial to our characterization. This new characterization also enables the identification of the class of symmetric polyconvex quadratic forms and demonstrates the absence of non-trivial symmetric poly-affine functions.*

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Ibrahim Merabet

Contents

Notations

- $r = \min\{n, m\}$, where n and m are any positive integers.
- $N(n, m) = \sum_{i=1}^{r} \binom{m}{i} \binom{n}{i}$, where $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for any integers a and b such that $1 \leq b \leq a$.
- $T_J = \left\{ \alpha = (\alpha_1, \dots, \alpha_J) : \sum_{i=1}^{J} \alpha_i = 1, \; \alpha_i \geq 0 \right\}$.
- $\mathbb{R}^+ = \{y \in \mathbb{R} : y \geq 0\}$.
- $\mathbb{R}^{++} = \{y \in \mathbb{R} : y > 0\}$.
- $\mathbb{R}^d$ represents the d-dimensional Euclidean space. In particular, $\mathbb{R}^{\binom{d}{2}}$ represents the $\binom{d}{2}$-dimensional Euclidean space.
- $\mathbb{R}^{d \times d}$ is the real $d \times d$ matrix space.
- $\mathcal{S}^{d \times d}$ is the real $d \times d$ symmetric matrix space.
- "T" denotes the transpose.
- ξ^T signifies the transpose of the matrix ξ.
- $M_i(\xi)$ represents the matrix containing all minors of order i, for each i within the range $2 \leq i \leq r$
- $M(\xi) = (\xi, M_2(\xi), \dots, M_r(\xi))$.
- $\mathbb{R}^{d \times d}_+ = \{\xi \in \mathbb{R}^{d \times d} : \det \xi > 0\}$.
- $\mathbb{R}^{d \times d}_- = \{\xi \in \mathbb{R}^{d \times d} : det\xi < 0\}$.
- $\mathbb{R}^{d \times d}_D$ represent the subspace of diagonal matrices of $\mathbb{R}^{d \times d}$.
- $\operatorname{tr}(\xi) = \sum_{i=1}^{r} \xi_{ii}$, for every $\xi \in \mathbb{R}^{m \times n}$.
- $\operatorname{cof} \xi$ indicates the cofactor matrix.
- $\det \xi$ stands for the determinant of the matrix ξ.
- $\xi : \eta = \operatorname{tr}(\xi^T \eta)$, for any $\eta, \xi \in \mathbb{R}^{m \times n}$.
- $|\xi| = \sqrt{\xi : \xi}$, representing the Euclidean norm for any $\xi \in \mathbb{R}^{m \times n}$.

- $\|\xi\| = \max\{\|\xi x\| : \|x\| = 1 \text{ for all } x \in \mathbb{R}^d\}$, is the operator norm of the matrix $\xi \in \mathbb{R}^{d\times d}$.
- $||\xi||_{\text{op}} = \max\{\lambda_1(\xi), \lambda_2(\xi)\}$. representing the spectral norm of $\xi \in \mathbb{R}_+^{2\times 2}$.
- $\mathbb{I}$ represents the identity matrix.
- $m(\Omega)$ represents the Lebesgue measure of Ω.
- $\mathbb{G}^d = \left\{y \in \mathbb{R}^d : y_1 \geq \dots y_{d-1} \geq |y_d|\right\}$.
- $V_d = \{y \in \mathbb{R}^{++^d},\ y_1 \geq \cdots \geq y_d > 0\}$.
- $O(d) \subset \mathbb{R}^{d\times d}$ represents the set of orthogonal matrices.
- $SO(d) \subset O(d)$ represents the set of proper orthogonal matrices, where $\det(\xi) = 1$ for every $\xi \in O(d)$.
- $CO(d) = \{\beta\eta : \eta \in O(d),\ \beta \in \mathbb{R}\}$ represents the set of conformal matrices.
- $CSO(d) = \{\alpha Q \in \mathbb{R}_+^{d\times d} : Q \in SO(d),\ \alpha \in \mathbb{R}^{++}\}$.
- $SL(d) = \{\xi \in \mathbb{R}^{d\times d} :\ \det \xi = 1\}$.
- ”.” represent the scalar product in $\mathbb{R}^d$
- ξ^s is the symmetric part of $\xi \in \mathbb{R}^{d\times d}$.
- ξ^a is the skew-symmetric part of $\xi \in \mathbb{R}^{d\times d}$.
- $a, b \in \mathbb{R}^d$, and $a \otimes b = ab^T$ denotes the tensor product.
- $a \odot b = (a \otimes b)^s$ represents the symmetric part of $a \otimes b$.
- $\text{Skew}(X)$ denotes the $d\times d$ anti-symmetric matrices, where for all $X \in \mathbb{R}^{\binom{d}{2}}$:

$$(\text{Skew}(X))_{ij} = x_{\sigma(i,j)}, \quad \text{where} \quad \sigma(i,j) = \left(\sum_{k=1}^{i}(d-k)\right) - (d-j), \quad \text{for } 1 \leq i < j \leq d.$$

- $X \boxtimes X = M_2(\text{Skew}(X))$ represents the S-tensor product.
- $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ is called S-positive semi-definite if and only if:

$$A : X \boxtimes X \geq 0, \quad \text{for all } X \in \mathbb{R}^{\binom{d}{2}}.$$

- $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ is called S-negative semi-definite if and only if $-A$ is S-positive semi-definite.
- $\mathcal{S}_{s^+}^{\binom{d}{2}\times\binom{d}{2}}$ denotes the set of all symmetric S-positive semi-definite matrices.
- $\mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$ denotes the set of all symmetric S-negative semi-definite matrices.

Chapter 0

Introduction

0.1 General Conceptions

The modeling of various material in linear and non linear elasticity can be characterize by the minimization of a functional I defined by:

$$I(u) = \int_{\Omega} f(y, u(y), \nabla u(y)), \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$, and $u : \Omega \to \mathbb{R}^m$ denotes the unknown function, and $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the energy density.

Minimizers of the functional I have been a subject of significant interest among researchers, with notable contributions from Morrey [52], Ball [7], and Dacorogna [19], they presented existence results in various forms: one assumes the convexity of the function f, while another suppose its quasiconvexity. These conditions are considered to ensure the weak lower semicontinuity of the functional I, which means that for every weakly convergence sequence $u_k \rightharpoonup u$ in the sobolev space $W^{1,p}$, the inequality bellow is satisfied:

$$\liminf_{k\to\infty} I(u_k) \geq I(u).$$

Tonelli [68] showed that the convexity of the function f serves as a sufficient condition for the weak lower semicontinuity of the functional I. Additionally, Serrin [58] extended Tonelli's theorem. When $m = 1$ or $n = 1$, convexity serves as a necessary condition for weak lower semicontinuity. In the vectorial case where $n, m > 1$, Morrey Jr [52], Acerbi and Fusco [1] provided the following result:

$$f \text{ is quasiconvex} \Longleftrightarrow I \text{ is weakly lower semi-continuous.}$$

A map f is termed quasiconvex if:

$$m(\Omega) f(\xi) \leq \int_{\Omega} f(\xi + \nabla\phi(x))\, dx \tag{2}$$

is satisfied for each open bounded subset Ω of $\mathbb{R}^n$, each $\xi \in \mathbb{R}^{m\times n}$, and all $\phi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$. Given the non-local characteristics of this condition (see Kristensen [42]), it is challenging to decide whether a specific function is quasiconvex. This difficulty prompted Morrey Jr [52] to introduce an alternative condition, namely weakly quasiconvexity, which was shown to be necessary for quasiconvex functions, according to the definition of Acerbi and Fusco [1], $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is considered weakly quasiconvex if the maps

$$t \to f(\eta + t \otimes y), \;\; s \to f(\eta + x \otimes s),$$

are convex, for each $\eta \in \mathbb{R}^{m\times n}$, each $x \in \mathbb{R}^m$, and all $y \in \mathbb{R}^n$. Roughly speaking, the notion of weakly quasiconvex functions is often referred to as rank-one convex functions, equivalently to the above definition, a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is referred to as rank-one convex if:

$$f(\beta\eta + (1-\beta)\xi) \leq \beta f(\eta) + (1-\beta) f(\xi),$$

is satisfied for each $\eta, \; \xi \in \mathbb{R}^{m\times n}$ with $rank\{\eta - \xi\} \leq 1$, and all $0 \leq \beta \leq 1$. When the function $f \in C^2(\mathbb{R}^{m\times n})$, rank-one convexity is equivalent to the Legendre-Hadamard condition, which can be expressed as

$$\sum_{p,q=1}^{m} \sum_{k,l=1}^{n} \frac{\partial^2 f(\eta)}{\partial \eta_{pk} \partial \eta_{ql}} x_p x_q y_k y_l \geq 0.$$

This holds for all $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{m\times n}$. For further details, the reader is directed to the works of Meyers [48], Morrey Jr [53], and Dacorogna [19].

Morrey Jr [52] also provided a sufficient condition for quasiconvexity, which was named by Ball [7] as polyconvexity, and is defined as follows: a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is considered polyconvex if a convex function $g : \mathbb{R}^{m\times n} \times \mathbb{R}^{\binom{m}{2}\times\binom{n}{2}} \times \ldots \times \mathbb{R}^{\binom{m}{r}\times\binom{n}{r}} \to \mathbb{R}$ exists and

$$f(\xi) = g(\xi, M_2(\xi), \ldots, M_r(\xi)), \tag{3}$$

is satisfied, where $r = \min\{n, m\}$, and $M_j(\xi)$, $2 \leq j \leq r$, represent the matrix of all minors of order j of the matrix $\xi \in \mathbb{R}^{m\times n}$.

The aforementioned concepts are connected to convexity in the following manner:

Convexity ⇒ Polyconvexity ⇒ Quasiconvexity ⇒ Rank-One Convexity.

The relationships described above are precise in the sense that the reverse implications do not necessarily hold, the function $\xi \in \mathbb{R}^{2\times 2} \to \det(\xi)$ is polyconvex but not convex. Moreover, as Morrey's conjecture suggested, rank-one convexity does not imply quasiconvexity for all $n, m \geq 2$. This remained an open problem until Šverák [66] provided a counterexample for the case $n \geq 2$ and $m \geq 3$, proving that rank-one convexity is not sufficient for quasiconvexity. In contrast, in the case of quadratic forms, Van Hove [69, 70] showed that rank-one convexity is equivalent to quasiconvexity and, if $n = 2$ or $m = 2$, it is also equivalent to polyconvexity. However, for $n, m \geq 3$, polyconvexity is not a necessary condition for rank one convexity, for more details, refer to Dacorogna [19], Terpstra [67], and Serre [57]. In the case $n = m = 2$, Alibert, Dacorogna, and Marcellini [3, 23] provided a counterexample showing that there exist quasiconvex functions that are not polyconvex. As for Morrey's conjecture, this example did not provide a definitive answer, and therefore, the case $n = m = 2$ remains unresolved.

0.2 Symmetric Setting

Within the framework of geometrically linear elasticity, derived from nonlinear theory, the invariance of the elastic energy density under infinitesimal rotations, rather than full frame-indifference, implies that the integrand f depends exclusively on the small strain tensor:

$$e(\nabla u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

and the energy functional I can be expressed as

$$I(u) = \int\limits_{\Omega} f(e(\nabla u(x)))dx,$$

where $f : \mathcal{S}^{d\times d} \to \mathbb{R}$, see [10, 25, 26] and [41].

Polyconvexity is replaced by symmetric polyconvexity in this setting. A function $f : S^{d\times d} \to \mathbb{R}$ is considered symmetric polyconvex if its composition with the symmetric part of the matrix is polyconvex. Specifically, if the function $\widetilde{f} : \mathbb{R}^{d\times d} \to \mathbb{R}$, $\widetilde{f}(F) = f(F^s)$ is polyconvex, see [11]. Similarly, symmetric quasiconvex and symmetric rank-one convex functions are defined. The notion of symmetric polyconvexity plays a crucial role in the existence of minimizers in nonlinear elasticity in the small strain case and has been fully characterized by Boussaid et al. [11] in dimension 2 and 3.

In this work, we are concerned with the notion of symmetric polyconvexity in higher dimensions [47]. Characterizing such a class of functions presents a significant mathematical challenge, as noted in [12]. Our aim is to provide a complete characterization of these functions in arbitrary dimensions. Based on the work of Boussaid et al. [11], we demonstrate that a necessary and sufficient condition for a function defined on the space of symmetric matrices, $\mathcal{S}^{d\times d}$, of dimension d to be symmetric polyconvex is to be expressed as a convex function of the matrix itself and its second-order minors, with an S-negative semi-definite partial subdifferential with respect to the matrix of all second-order minors.

In this setting, a matrix A is said to be S-positive (or S-negative) semi-definite if $A : M_2(F)$ is non-negative (or non-positive) for any skew-symmetric matrix F. Here, $M_2(F)$ denotes the matrix of all second-order minors of F, as defined in Definition 1.3. The independence of g from higher-order minors greatly simplifies the search for symmetric polyconvex functions in higher dimensions. Additionally, the characterization of symmetric polyconvex functions will be useful for understanding the structure of symmetric polyconvex hulls for compact sets K.

We know from [11], Lemma 3.1, that, in three dimension, for a symmetric matrix A, the convexity of the quadratic form $\xi \in \mathbb{R}^{3\times 3} \to q_A(\xi) = A : \mathrm{cof}\xi^a$ is equivalent to its rank one convexity which is in turn equivalent to the positivity of the matrix A. In higher dimensions, this property is no longer true in general; we will provide in Lemma 1.4 a counter example of non positive semi-definite matrix $A \in \mathcal{S}^{6\times 6}$ such that the associated function q_A is rank-one convex but not convex; furthermore, we show that the convexity of q_A is both a necessary and sufficient condition for the S-positive semi-definiteness of the matrix A. This principle serves as the foundation for the representative behavior observed in our characterization of symmetric polyconvex functions in higher dimensions. On the other hand, an important observation states that the determinant, as well as all third- and fourth-order principal minors, are not symmetric rank-one convex in the cases $d = 4$ or $d = 5$. This observation was the key idea in expressing a symmetric polyconvex function independently of higher-order minors. The main result of this thesis is as follows:

Theorem 0.1 *Consider a function $f : \mathcal{S}^{d\times d} \to \mathbb{R}$. Then, the symmetric polyconvexity of f is equivalent to the existence of a convex function $g : \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}} \to \mathbb{R}$ and*

$$f(\varepsilon) = g(\varepsilon, M_2(\varepsilon)),$$

is satisfied for each $\varepsilon \in \mathcal{S}^{d\times d}$, and the partial sub-differential of g with respect to its second argument is S-negative semi-definite.

The concept of S-positive definite matrices introduced in Section 1.0.3 is the right notion which corresponds to the symmetric polyconvexity in higher dimensions, it is reduced to the regular positive definiteness in the case $d = 2$ or $d = 3$, which makes Theorem 0.1 a generalization of Theorem 4.1 and Theorem 5.1 stated in [11].

The main difficulty in proving Theorem 0.1 is to demonstrate the independence of the representative g on different order minors greater than 2. This involves a systematic classification of minors based on their common elements with the diagonal of the matrix. Special matrices are employed to eliminate m-diagonal minors of order $p \geq 3$ for $2 \leq m \leq p$, utilizing the sub-differential property of convex functions. The proof then extends to m-diagonal minors with $m = 1$ or $m = 0$, starting with third-order minors and generalizing through induction to all orders.

Our new characterization enables the identification of the class of symmetric polyconvex quadratic forms and symmetric poly-affine functions. Specifically, for quadratic forms, we show that:

Theorem 0.2 *Consider a quadratic form $q : \mathcal{S}^{d\times d} \to \mathbb{R}$. Then, the symmetric polyconvexity of q is equivalent to the existence of a convex quadratic map $h : \mathcal{S}^{d\times d} \to \mathbb{R}$ and an S-positive semi-definite matrix $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, for which*

$$q(\varepsilon) = h(\varepsilon) - A : M_2(\varepsilon),$$

is satisfied.

In other words, a necessary and sufficient condition for the quadratic form q to be symmetric polyconvex is the existence of an S-positive semi-definite matrix $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ such that $q(\varepsilon) + A : M_2(\varepsilon) \geq 0$. This characterization refines the classical case where the matrix A is arbitrary, as indicated in [19].

Since S-positive semi-definiteness is equivalent to standard positive semi-definiteness in the case $d = 3$, we recover the characterization of quadratic forms stated in [11] for $2d$ and $3d$ dimensions.

Additionally, it is worth noting that in three dimensions, symmetric polyconvexity of the form $\varepsilon \to A : -\mathrm{cof}\varepsilon$ is equivalent to its symmetric rank-one convexity, which in turn is equivalent to A being positive semi-definite, as shown in [11, Corollary 5.9, p. 446]. However, in higher dimensions, having the matrix A S-positive semi-definite is only a sufficient condition for the form $\varepsilon \to -A : M_2(\varepsilon)$ to be symmetric polyconvexity; a necessary condition requires the existence of a matrix B such that the sum of A and B is S-positive semi-definite and $B : M_2(\varepsilon) = 0$, as exhibited in Corollary3.1. It is important to emphasize that this condition does not guarantee that A itself is S-positive semi-definite. This distinction is particularly illustrated by the counterexample provided in Remark 3.2, where we present a non S-positive definite matrix A such that the associated quadratic form $\varepsilon \to -A : M_2(\varepsilon)$ is symmetric polyconvex. Furthermore, in higher dimensions, it is currently unknown whether the symmetric polyconvexity of the form $\varepsilon \to -A : M_2(\varepsilon)$ is equivalent to its symmetric rank-one convexity.

Analogously, in the $3d$ case where a counterexample illustrating the non-equivalence between symmetric polyconvexity and symmetric rank-one convexity for quadratic forms was provided in [11, Theorem 5.7, p. 443]. The adapted counterexample to higher dimensions,

$$\begin{aligned} f(\varepsilon) = {} & (\varepsilon_{12} - \varepsilon_{13})^2 + (\varepsilon_{12} - \varepsilon_{23})^2 + (\varepsilon_{13} - \varepsilon_{23})^2 + \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 \\ & - \eta(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2)), \quad \forall \varepsilon = (\varepsilon_{ij}) \in \mathcal{S}^{d\times d}, \end{aligned}$$

where, η is chosen as in [11], is symmetric rank one convex but not symmetric polyconvex, see Remark 3.3. Moreover, we note that the form

$$F_{11}^2 + F_{22}^2 + F_{33}^2 + F_{12}^2 + F_{31}^2 + F_{23}^2 - 2(F_{11}F_{22} + F_{11}F_{33} + F_{22}F_{33}) + \sum_{i=4}^{d} F_{ii}^2,$$

analyzed by Harutyunyan et al. [34, 33], see also [14], which presented as a counterexample of a rank-one convex, non-polyconvex quadratic form, cannot be applied to the symmetric setting. This is because the altered form

$$\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 - 2(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) + \sum_{i=4}^{d} \varepsilon_{ii}^2$$

is symmetric polyconvex (see Remark 3.3).

Regarding symmetric poly-affine functions, our characterization permits us to show in Proposition 3.2, that there is no non-trivial poly-affine functions.

0.3 The Structure of The Thesis

The thesis is organized as follows. Chapter 1 reviews essential notations and preliminaries related to the properties of minors, introducing a classification of minors based on their diagonal elements, which will be crucial for the proof of Theorem 3.5. The chapter also defines the S-tensor product and S-positive semi-definite matrices and discusses sub-differentials of convex functions and their relation to S-positivity.

Chapter 2 delves into the theory of polyconvexity, presenting general properties and characterizations for various function classes.

The main result of the thesis is presented and proved in Chapter 3. This chapter begins with a review of a characterizations of symmetric polyconvexity in dimensions 2 and 3, followed by the proof of the main theorem. The final sections explore symmetric polyconvex quadratic forms and symmetric poly-affine functions.

Chapter 1

Preliminaries and intermediate results

Let e_i denote the standard basis vectors of $\mathbb{R}^d$, where $d, i \in \mathbb{N}$. We define $\mathbb{R}^{d\times d}$ as the space of real $d \times d$ matrices, and $\mathcal{S}^{d\times d}$ as the space of symmetric $d \times d$ matrices. For any matrix $A \in \mathbb{R}^{d\times d}$, we refer to its trace as $\operatorname{tr}(A)$, and $M_k(A) \in \mathbb{R}^{\binom{d}{k}\times\binom{d}{k}}$, $1 \leq k \leq d$ refers to the matrix composed of the k-order minors of A. in particularly, we note that $M_1(A) = A$ denotes the matrix itself, and $M_d(A) = \det(A)$, denotes the determinant of A. Moreover, we consider $M(A)$ to be the vector containing all minors of A, defined as

$$M(A) = (A, M_2(A), \dots, \det(A)) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}} \times \cdots \times \mathbb{R}^{\binom{d}{d}\times\binom{d}{d}}.$$

In addition, we denote the symmetric and skew-symmetric parts of A as A^s and A^a, respectively. Any matrix A can be naturally decomposed into the sum of two components:

$$A = A^s + A^a.$$

For a, b vectors in $\mathbb{R}^d$, the tensor product of the vectors a and b is denoted by $a \otimes b$. It is the matrix whose entries are $a_i b_j$. We also denote by $a \odot b$ the symmetric part of the matrix $a \otimes b$. Finally, for two matrices $A, B \in \mathbb{R}^{d\times d}$, $A : B$ stands for the scalar product of A and B. We also recall the spectral decomposition of symmetric matrices, which states that any symmetric matrix $\varepsilon \in \mathcal{S}^{d\times d}$ can be represented as a linear combination of symmetric rank one matrices:

$$\forall \varepsilon \in \mathcal{S}^{d\times d},\ \exists\ \lambda_i \in \mathbb{R},\ X_i \in \mathbb{R}^d : \varepsilon = \sum_{i=1}^{d} \lambda_i X_i \otimes X_i \tag{1.1}$$

1.0.1 Some minors properties

According to [65] we call for $1 \leq p \leq d$, a p-tuple any multi-index $(i_1, i_2, ..., i_p)$, such that $1 \leq i_1 < i_2 < \cdots < i_p \leq d$. The set of all p-tuples defined on the set $\{1,2,\ldots,d\}$ is denoted by $\mathbb{I}_p$. The cardinal of $\mathbb{I}_p$ is $\binom{d}{p}$, it is endowed by the lexicographic order: two p-tuples $I=(i_1,i_2,...,i_p)$, $J=(j_1,j_2,...,j_p)$ are defined to satisfy $I\leq J$ if there is an index r within $\{1,2,\ldots,p\}$ such that $i_r\leq j_r$ and for every $s\in\{1,2,\ldots,r-1\}, i_s=j_s$. For all matrix $\xi\in\mathcal{S}^{d\times d}$ and for each $2\leq p\leq d$, we define $M_p(\xi)$ as the $\binom{d}{p}\times\binom{d}{p}$ matrix of all p-order minors of ξ. Specifically, $M_p(\xi)$ is a matrix such that each entry $(M_p(\xi))_{IJ}$ represent the determinant of the $p\times p$ submatrix of ξ. This submatrix is formed by selecting the rows and columns of ξ indexed by the p-tuples I and J, respectively. The position of $(M_p(\xi))_{IJ}$ in the matrix $M_p(\xi)$ is determined according to the lexicographically order. For more details, see [37] and [65].

We give now some useful properties on matrix minors, we begin with the following lemma quoted from [44].

Lemma 1.1 *For any $d\times d$ skew-symmetric matrix ξ with entries in an arbitrary commutative ring, the matrix $M_k(\xi)$ of k-order minors exhibits the following property: it is skew-symmetric when k is odd and symmetric when k is even.*

For each matrix $F\in\mathbb{R}^{d\times d}$, the matrix of k-order minors satisfy the following properties:

$$M_k(F^T)=(M_k(F))^T,\quad M_k(\alpha F)=\alpha^k M_k(F),\quad \text{for all}\quad \alpha\in\mathbb{R}$$

Furthermore, for the specific case of $k=2$, we observe the following decomposition:

$$(M_2(F))^s=M_2(F^s)+M_2(F^a). \tag{1.2}$$

Particularly, for $a,b\in\mathbb{R}^d$, the fact that $M_2((a\otimes b)^a)=\Lambda\otimes\Lambda$, where $\Lambda=(\Lambda_I)_I, I=(i,j), 1\leq i<j\leq d$, $\Lambda_I=\frac{1}{2}(a_ib_j-a_jb_i)$ is a vector in $\mathbb{R}^{\binom{d}{2}}$ whose components are ordered lexicographically, combined with the application of (1.2) for $F=a\otimes b$, leads to the following result:

$$M_2(a\odot b)=-\Lambda\otimes\Lambda. \tag{1.3}$$

Moreover, for any matrices $\varepsilon,\eta\in\mathcal{S}^{d\times d}$ the following holds :

$$M_2(\varepsilon+\eta)=M_2(\varepsilon)+M_2(\eta)+A(\varepsilon,\eta), \tag{1.4}$$

such that $A:\mathcal{S}^{d\times d}\times\mathcal{S}^{d\times d}\to\mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ is a bilinear form in the sens that $A(\alpha\varepsilon+\varepsilon',\eta)=\alpha A(\varepsilon,\eta)+A(\varepsilon',\eta)$ and $A(\varepsilon,\alpha\eta+\eta')=\alpha A(\varepsilon,\eta)+A(\varepsilon,\eta')$, for all $\varepsilon,\varepsilon',\eta,\eta'\in\mathcal{S}^{d\times d}$ and $\alpha\in\mathbb{R}$. Specifically, for $\eta=\varepsilon$ we get, $M_2(\varepsilon)=\frac{1}{2}A(\varepsilon,\varepsilon)$.

Lemma 1.2 *Let $B\in\mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, a given matrix. The following statements are equivalent:*

1. *$B : M_2(\varepsilon) \leq 0$, (or $B : M_2(\varepsilon) \geq 0$) for any $\varepsilon \in \mathcal{S}^{d\times d}$,*
2. *$B : M_2(\varepsilon) = 0$, for every $\varepsilon \in \mathcal{S}^{d\times d}$.*

Proof. It suffice to demonstrate that 1) $\Rightarrow$ 2). Let $\varepsilon \in \mathcal{S}^{d\times d}$ such that $B : M_2(\varepsilon) \leq 0$, the case $B : M_2(\varepsilon) \geq 0$ follow analogously. Consider $\alpha \in \mathbb{R}$ and $X \in \mathbb{R}^d$. We then get $B : M_2(\varepsilon + \alpha X \otimes X) \leq 0$, according to (1.4) we will have,

$$B : M_2(\varepsilon) + \alpha B : A\left(\varepsilon, X \otimes X\right) \leq 0$$

Due to the arbitrariness of α, we obtain $B : A(\varepsilon, X\otimes X) = 0$ for all vector $X \in \mathbb{R}^d$. Consequently, by taking $\eta = \varepsilon$ in (1.4), and using the spectral decomposition (1.1) we obtain,

$$B : M_2(\varepsilon) = \frac{1}{2}B : A(\varepsilon, \varepsilon) = \frac{1}{2}B : A\left(\varepsilon, \sum_{i=1}^{d} \lambda_i X_i \otimes X_i\right) = \frac{1}{2}\sum_{i=1}^{d} \lambda_i B : A\left(\varepsilon, X_i \otimes X_i\right) = 0$$

■

Before closing this section, we introduce a definition that classifies the different types of minors based on the number of diagonal elements of the whole matrix contained in the sub-matrix.

Definition 1.1 *Let ε be a matrix in $\mathcal{S}^{d\times d}$, and p, m two integers, such that $m \leq p$. A $p\times p$ minor of ε is said m-diagonal if it contains exactly m diagonal elements of ε. If $m = p$, a $p\times p$ minor is p-diagonal if it is principal. A minor is 0-diagonal, if it is totally outside the diagonal. The picture bellow shows the different type of minors*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} & a_{56} \\ a_{16} & a_{26} & a_{36} & a_{46} & a_{56} & a_{66} \end{pmatrix}$$

The red minor is 3-diagonal, the blue one is 0-diagonal while the green is a 2-diagonal.

1.0.2 S-Tensor product

In all the sequel we associate the space of skew-symmetric $d \times d$ matrices with the space $\mathbb{R}^{\binom{d}{2}}$, such that the vector components are ordered with respect to the lexicographic order, this means that any vector $X = (x_1, x_2, \ldots, x_{\binom{d}{2}})$ is identified as the skew-symmetric matrix, denoted by $Skew(X)$ and having entries $\widetilde{x}_{ij}$ defined as follows:

$$\widetilde{x}_{ij} = x_{\sigma(i,j)}, \text{ such that } \sigma(i,j) = \left(\sum_{k=1}^{i}(d-k)\right) - (d-j), \text{ for } 1 \le i < j \le d \tag{1.5}$$

Specifically, for the case where $d = 4$ and $X = (x_1, x_2, \ldots, x_6)$, we have

$$Skew(X) = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix}$$

Definition 1.2 *For X any vector in $\mathbb{R}^{\binom{d}{2}}$, we call S-tensor product of X by it self the matrix of all two by two minors of $Skew(X)$, this matrix is denoted by $X \boxtimes X$, in other words*

$$X \boxtimes X = M_2(Skew(X)).$$

Thanks to Lemma 1.1, the matrix $X \boxtimes X$ is symmetric.

Remark 1.1 *It is evident that for $d = 2$ or $d = 3$, any vector $X \in \mathbb{R}^{\binom{d}{2}}$ satisfies $X \boxtimes X = X \otimes X$. However if $d \geq 4$, then in general $X \boxtimes X \neq X \otimes X$, as a counter-example we take the vector $X = (x_1, x_2, \ldots, x_{\binom{d}{2}})$ such that $x_2 = x_{d+1} = 1$ and $x_i = 0$ for $i \neq 2$ or $i \neq d+1$ then $Skew(X) = (\widetilde{x}_{ij})$, where $\widetilde{x}_{13} = \widetilde{x}_{24} = 1, \widetilde{x}_{31} = \widetilde{x}_{42} = -1$ and all the other entries are zero. By strain forward calculation we get $M_2(Skew(X))_{IJ} = M_2(Skew(X))_{JI} = 1, M_2(Skew(X))_{II} = M_2\left(Skew\left(X\right)\right)_{JJ} = 0$, where $I = (1,2)$ and $J = (3,4)$ which is sufficient to conclude that $M_2(Skew(X))$ and so that $X \boxtimes X$ is not positive semi-definite, hence it can not be equal to $X \otimes X$ which is clearly positive semi-definite.*

1.0.3 S-Positive semi-definite matrices

We will now define a new concept of positivity on symmetric matrices of order $\binom{d}{2}$ which will be used in the proof of Theorem3.5.

Definition 1.3 *A symmetric matrix A in $\mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ is called S-positive semi-definite (S-SDP) if*

$$A : X \boxtimes X \geq 0, \quad \textit{for all } X \in \mathbb{R}^{\binom{d}{2}}. \tag{1.6}$$

Additionally, if the inequality in (1.6) is strict, then A is called S-positive definite (S-DP). The set of all symmetric S-positive semi-definite (S-positive definite) matrices is denoted by $\mathcal{S}_{s^+}^{\binom{d}{2}\times\binom{d}{2}}$ $\left(\mathcal{S}_{s^{++}}^{\binom{d}{2}\times\binom{d}{2}}\right)$. *Conversely, if the symmetric matrix* $-A$ *is S-positive semi-definite (S-positive definite), then the symmetric matrix A is referred to as S-negative semi-definite (S-negative definite), respectively. The set of all symmetric S-negative semi-definite (S-negative definite) matrices is denoted by* $\mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$ $\left(\mathcal{S}_{s^{--}}^{\binom{d}{2}\times\binom{d}{2}}\right)$.

Remark 1.2

1. *For* $d=2$ *and* $d=3$, *the condition that* $A\in\mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ *is S-positive semi-definite is equivalent to the condition that A is positive semi-definite.*

2. *For the case* $d=4$, *there exist S-positive semi-definite matrices which are not positive semi-definite and vis versa. In fact the matrix* $A=(a_{ij})\in\mathcal{S}^{6\times 6}$ *where*

$$a_{ij}=\frac{1}{4}\ \textit{if}\ i+j=7,\ \ a_{ij}=0\quad \textit{if}\ (i,j)\neq(2,2),(i,j)\neq(5,5)\quad \textit{or}\quad i+j\neq 7$$

and $a_{22}=a_{55}=1$ *is clearly not positive semi-definite. By straightforward computation we get* $A:X\boxtimes X=x_2^2+x_5^2+x_2x_5\geq 0$. *Hence, according to (1.6), the matrix A is S-positive semi-definite. However, the symmetric matrix* $B=(b_{ij})\in\mathcal{S}^{6\times 6}$ *defined by:*

$$b_{ii}=1, b_{16}=b_{61}=-\frac{3}{4}, b_{25}=b_{52}=b_{34}=b_{43}=\frac{3}{4},$$

and $b_{ij}=0$ *for all other entries is positive semi-definite. Moreover*

$$B:X\boxtimes X=\sum_{i=1}^{6}x_i^2+3x_3x_4$$

which is not always positive, since by taking $x_1=x_2=x_5=x_6=0$ *and* $x_3=-1, x_4=1$, *we get* $B:X\boxtimes X=-1$. *Hence B is not S-positive semi-definite.*

We will now present the following auxiliary result, which proves that if the quadratic form $A:X\boxtimes X$ vanishes for all vectors $X\in\mathbb{R}^{\binom{d}{2}}$, then the matrix A is zero. This property is useful in demonstrating that symmetric poly-affine functions are affine.

Lemma 1.3 *Let* $A\in\mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, *if the condition*

$$A:X\boxtimes X=0\quad \textit{for all}\ X\in\mathbb{R}^{\binom{d}{2}}, \tag{1.7}$$

holds, then A must be the zero matrix.

Proof. Let consider $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ a matrix satisfying (1.7). Note that since the correspondence σ defined in (1.5) is one to one between the index sets $\{(i,j) : 1 \leq i < j \leq d\}$ and $\{1, 2, \ldots, \binom{d}{2}\}$, the entries of the matrix A can be noted as A_{IJ} such that $I, J \in \{(i,j) : 1 \leq i < j \leq d\}$. The proof is divided to three main steps.

i) We will show, first that all diagonal entries of A vanish. For a fixed index $I = (i,j)$, let $X = \big(x_{\sigma(k,l)}\big)$ be the matrix with $x_{\sigma(i,j)} = 1$, and $x_{\sigma(k,l)} = 0$, for $(k,l) \neq (i,j)$. By using (1.7), we get $A_{II} = 0$.

ii) In this case, we consider entries A_{IJ} such that $I = (i,k), J = (j,l)$ and $\{i,k\} \cap \{j,l\}$ has exactly one element in common, Without loss of generality, we can restrict our focus to $I = (i,j)$ and $J = (j,l)$. In this case, we choose $X = \big(x_{\sigma(p,q)}\big)$ by setting $x_{\sigma(i,j)}$ and $x_{\sigma(j,l)}$ to 1, and $x_{\sigma(p,q)} = 0$ otherwise. The only non vanishing elements of the matrix $C = M_2(Skew(X))$ satisfy, $C_{IJ} = C_{JI} = C_{II} = C_{JJ} = 1$. Hence (1.7), together with the first step, leads to $A_{IJ} = 0$.

iii) Finally, we will demonstrate that $A_{IJ} = 0$ for all pair of indices (I, J) where $I = (i,k)$ and $J = (j,l)$ with the sets $\{i,k\}$ and $\{j,l\}$ being disjoint. Since any tuple (p,q) satisfies $p < q$, we can, without loss of generality, suppose that $i = \min\{i,j,k,l\}$. Furthermore, there are only three possible orderings for the indices i,j,k,l : $i < j < k < l$, $i < j < l < k$ and $i < k < j < l$. We assume that $i < j < k < l$. The remaining cases are treated analogously. Consider the pairs

$$I_1 = (i,j),\ I_2 = I = (i,k),\ I_3 = (i,l)$$
$$J_1 = (k,l),\ J_2 = J = (j,l),\ J_3 = (j,k).$$

We choose a vector $X = \big(x_{\sigma(p,q)}\big)$, where $x_{\sigma(p,q)} = 0$ for all $(p,q) \neq (i,j), (k,l)$ and $x_{\sigma(i,j)} = x_{\sigma(k,l)} = 1$. The only non null sub-matrix of $Skew(X)$ exhibits the following form

$$\begin{array}{c|cccc} & i & j & k & l \\ \hline i & 0 & 1 & 0 & 0 \\ j & -1 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 1 \\ l & 0 & 0 & -1 & 0 \end{array}$$

Consequently, the matrix $C = M_2(Skew(X))$ has only six non zero entries. As the diagonal elements of A vanish, we will only consider non diagonal minors.

$$C_{IJ} = C_{JI} = 1 \text{ and } C_{I_3J_3} = C_{J_3I_3} = -1$$

Therefore, by the symmetry of A, along with (1.7), and the fact that $A_{II} = A_{JJ} = 0$, we obtain,

$$A_{IJ} - A_{I_3J_3} = 0. \tag{1.8}$$

Next, consider a second choice for X, defined as $X = (x_{\sigma(p,q)})$, where $x_{\sigma(p,q)} = 0$ for all $(p,q) \neq (i,k),(j,l)$ and $x_{\sigma(i,k)} = x_{\sigma(j,l)} = 1$. Then, the no zero sub-matrix of $Skew(X)$ has the following form:

$$\begin{array}{c c} & \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{array}{c} i \\ j \\ k \\ l \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

The non diagonal, non zero elements of C are the following,

$$C_{I_1J_1} = C_{J_1I_1} = C_{I_3J_3} = C_{J_3I_3} = 1.$$

By using the same reasoning as above, we get,

$$A_{I_1J_1} + A_{I_3J_3} = 0. \tag{1.9}$$

As a third choice for the vector X, consider $X = (x_{\sigma(p,q)})$, where $x_{\sigma(p,q)} = 0$ for all $(p,q) \notin \{(i,l),(j,k)\}$ and $x_{\sigma(i,l)} = x_{\sigma(j,k)} = 1$. Then, the non zero sub-matrix of $Skew(X)$ has the following form:

$$\begin{array}{c c} & \begin{array}{cccc} i & j & k & l \end{array} \\ \begin{array}{c} i \\ j \\ k \\ l \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The non zero elements of C except the diagonal elements are,

$$C_{IJ} = C_{JI} = 1 \text{ and } C_{I_1J_1} = C_{J_1I_1} = -1.$$

Similarly as above, we obtain,

$$A_{IJ} - A_{I_1J_1} = 0. \tag{1.10}$$

Consequently, from (1.8),(1.9) and (1.10) we conclude that $A_{IJ} = A_{I_1J_1} = A_{I_3J_3} = 0$, which is the desired result.

■

1.0.4 Quadratic Forms

By imitating [11], we give now a result on convexity of a special quadratic form which will be used in the proof of Theorem3.5.

Lemma 1.4 *Let* $q_A : \mathbb{R}^{d\times d} \to \mathbb{R}$, *be the quadratic form defined as follow:*

$$q_A(F) = A : M_2(F^a), \quad A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}.$$

Then, the equivalence of the following statements holds:

(i) *The function* q_A *is convex,*

(ii) $q_A(F) \geq 0$ *for all* $F \in \mathbb{R}^{d\times d}$,

(iii) A *is an S-positive semi-definite matrix.*

Furthermore, there exists a matrix A *that is neither S-positive semi-definite nor positive semi-definite, and yet* q_A *is rank-one convex.*

Remark 1.3 *In contrast to the* 3d *case see [11], the equivalence between rank-one convexity and convexity of* q_A *in the general case is currently an unresolved issue.*

Proof. The convexity of quadratic forms directly leads to the equivalence between (i) and (ii). Equivalence of (ii) and (iii) is a consequence of S-positive semi-definiteness.

We will give a counterexample of a matrix $A \in \mathcal{S}^{6\times 6}$, that is neither S-positive semi-definite nor positive semi-definite, for which the quadratic form q_A is rank one convex. Consider the block matrix

$$A = \left(\begin{array}{c|c} B & D \\ \hline D^T & C \end{array}\right),$$

where $B = e_1 \otimes e_1,\ D = e_1 \otimes e_3 - e_2 \otimes e_2 + e_3 \otimes e_1$ and $C = 0$, with (e_1, e_2, e_3) being the vectors of the canonical basis in $\mathbb{R}^3$. By a straightforward computation, we find for any vectors $a = (a_i), b = (b_i)$ in $\mathbb{R}^4,\ i = 1, 2, \ldots, 4$ that

$$M_2((a\otimes b)^a) = \frac{1}{4}\begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} u \\ v \end{pmatrix},$$

such that $v = (a_2b_3-a_3b_2, a_2b_4-a_4b_2, a_3b_4-a_4b_3)$, $u = a_1\hat{b}-b_1\hat{a}$ with $\hat{b} = (b_2, b_3, b_4)$ and $\hat{a} = (a_2, a_3, a_4)$. We notice that $Dv = \hat{a} \times \hat{b}$ represents the cross product of the vectors under consideration, Consequently, due to the orthogonality of u and Dv together with the positive semi-definiteness of B, we obtain

$$\begin{aligned} q_A(a\otimes b) &= A : M_2((a\otimes b)^a) = \tfrac{1}{4}\begin{pmatrix} u \\ v \end{pmatrix}^T \left(\begin{array}{c|c} B & D \\ \hline D^T & C \end{array}\right)\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \tfrac{1}{4}(u^TBu + v^TCv + 2u^TDv) = \tfrac{1}{4}(u^TBu) \geq 0, \end{aligned}$$

which means that q_A is rank one convex. In other hands, the matrix A is clearly non positive semi-definite, lets show that in fact it is not S-positive semi-definite. According to (1.5), we have for $X = (x_1, x_2, \dots, x_6)^T \in \mathbb{R}^6$,

$$A : X \boxtimes X = A : M_2(Skew(X)) = x_1^2 - 4x_3x_4 + 4x_2x_5 - 4x_1x_6. \qquad (1.11)$$

The right hand side in (1.11) is not a positive quadratic form, consequently the matrix A is not S-positive semi-definite. ■

1.0.5 Subdifferential of Convex Functions

Definition 1.4 *1. Consider a convex function $f : \mathbb{R}^d \to \mathbb{R}$. The subdifferential of f at $x \in \mathbb{R}^d$ is defined as follow:*

$$\partial f(x) = \{s \in \mathbb{R}^d : f(y) \geq f(x) + < s, y - x > \ \text{for all } y \in \mathbb{R}^d\}.$$

The set $\partial f(x)$ is compact and convex. For differentiable functions, $\partial f(x)$ is reduced to the set $\{\nabla f(x)\}$.

The following lemma provide sufficient conditions for a convex function to have non S-positive semi-definite sub-differentials . The proof is not done here, it is a direct adaptation of lemma 3.4 in [11] by taking S-tensor product instead of tensor product and S-semi-negative definite matrices instead of negative semi-definite ones.

Lemma 1.5 *Let $g : \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}} \to \mathbb{R}$ be a convex function. Then, the following implications hold:*

(i) If the subdifferential $\partial g(\varepsilon, \eta)$ is a subset of $\mathcal{S}^{d\times d} \times \mathcal{S}_{s-}^{\binom{d}{2}\times\binom{d}{2}}$ for any $(\varepsilon, \eta) \in \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, then the partial subdifferential $\partial_2 g(\varepsilon, \eta)$ is a subset of $\mathcal{S}_{s-}^{\binom{d}{2}\times\binom{d}{2}}$ for any $(\varepsilon, \eta) \in \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$;

(ii) The existence of an $\hat{\varepsilon} \in \mathcal{S}^{d\times d}$ and a constant $C > 0$ satisfying

$$g(\hat{\varepsilon}, X \boxtimes X) \leq C \ \text{for any } X \in \mathbb{R}^{\binom{d}{2}}, \qquad (1.12)$$

implies that the partial subdifferential $\partial_2 g(\varepsilon, \eta)$ is a subset of $\mathcal{S}_{s-}^{\binom{d}{2}\times\binom{d}{2}}$ for any $(\varepsilon, \eta) \in \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$.

Note that the constant C in (1.12) which is supposed positive in [11] can be assumed to be arbitrary in $\mathbb{R}$ without affecting the proof.

Chapter 2

Polyconvexity: Theory And Applications

In this chapter, we focus on the main properties and different classes of polyconvex functions, as well as discuss other important examples relevant to the application of the concept in nonlinear elasticity.

2.1 Definitions and Proprieties

The concept underlying polyconvexity was first introduced by Morrey [52, 53] as a sufficient condition for quasiconvexity. However, the term ”polyconvexity” itself was coined by Ball in his 1977 paper [7], where he investigated the concept within the framework of nonlinear elasticity. Ball characterized the properties of polyconvex functions in dimensions two and three, while the extension to arbitrary dimensions was later provided by Ball, Currie, and Olver [8], as well as by Dacorogna in his works [17, 19].

Definition 2.1 *A function $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ is termed polyconvex if it possesses the following structure:*

$$f(\xi) = g\left(\xi, M_2(\xi), \dots, M_r(\xi)\right),$$

where $g : \mathbb{R}^{N(n,m)} \to \mathbb{R} \cup \{+\infty\}$ is a convex function, $M_k(\xi)$ represents the matrix of all k-th order minors of ξ, $r = \min\{n, m\}$, and

$$N(n,m) = \sum_{i=1}^{r} \binom{m}{i}\binom{n}{i}.$$

Remark 2.1 *The definition of a polyconvex function is grounded in the work of Yan [71], Šilhavỳ [64, 65]. For the special cases when $m = n = 2$ or 3, we* *adhere to the definition provided by Ball [7], as it aligns with physical criteria. A function f is called polyconvex if it exhibits the following structure:*

$$\begin{aligned} f(\xi) &= g(\xi, \det \xi), && \text{for all } \xi \in \mathbb{R}^{2\times 2}, \\ f(\xi) &= g(\xi, \operatorname{cof} \xi, \det \xi), && \text{for all } \xi \in \mathbb{R}^{3\times 3}, \end{aligned}$$

where g is a convex function, $\operatorname{cof} \xi$ denotes the cofactor matrix of ξ, and $\det \xi$ represents to the determinant of ξ.

Remark 2.2 *A particular case of polyconvex functions is the so-called additive polyconvex functions where the expression is a sum of convex functions of minors of each order, namely in the 3×3 case a function $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ defined as*

$$f(\xi) = f_1(\xi) + f_2(\operatorname{cof} \xi) + f_3(\det \xi).$$

such that $f_1, f_2 : \mathbb{R}^{3\times 3} \to \mathbb{R}$ and $f_3 : \mathbb{R} \to \mathbb{R}$. If f_i, $i = 1, 2, 3$, are convex in their respective arguments, is polyconvex.

An example of a non-convex function that is polyconvex in $\mathbb{R}^{2\times 2}$ is illustrated by the function $\det \xi$. In the case of functions allowed to take the value infinity, an example is provided in [56, Remark 10.26].

Example 2.1 *Consider the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ specified by*

$$f(\xi) = \begin{cases} \frac{1}{\det \xi}, & \det \xi > 0, \\ \infty, & \det \xi \leq 0. \end{cases}$$

This function serves as an example of a polyconvex function that is not convex. A more detailed illustration of this example can be found in [35], where they consider the convex function $g : \mathbb{R}^5 \to \mathbb{R} \cup \{\infty\}$ defined by

$$g(\xi, \delta) = \begin{cases} \frac{1}{\delta}, & \delta > 0, \\ \infty, & \delta \leq 0. \end{cases}$$

Since $f(\xi) = g(\xi, \det \xi)$, this implies that f is polyconvex. The non-convexity is demonstrated by the two matrices $\xi_1, \xi_2 \in \mathbb{R}^{2\times 2}$ given by

$$\xi_1 = \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.$$

With the choice $t = \frac{1}{2}$, it follows that the inequality

$$f\left(\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\right) \leq \frac{1}{2} f(\xi_1) + \frac{1}{2} f(\xi_2)$$

is not satisfied.

Remark 2.3 *There are specific definitions for finite-valued polyconvex functions defined on the sets* $\mathbb{R}_+^{d\times d}$ *and* $SL(d)$*, as discussed in [51, 27].*

i) A function $f : \mathbb{R}_+^{d\times d} \to \mathbb{R}$ *is said polyconvex if the function*

$$\tilde{f} : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}, \quad \tilde{f}(\xi) = \begin{cases} f(\xi) & \xi \in \mathbb{R}_+^{d\times d} \\ \infty & \xi \notin \mathbb{R}_+^{d\times d} \end{cases}$$

is polyconvex in the sense described by Ball [7].

ii) A function $f : SL(d) \to \mathbb{R}$ *is regarded as polyconvex if the function*

$$\tilde{f} : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}, \quad \tilde{f}(\xi) = \begin{cases} f(\xi) & \xi \in SL(d) \\ \infty & \xi \notin SL(d) \end{cases}$$

is polyconvex according to Ball's definition [7].

The following lemma characterizes $SL(2)$-polyconvexity, as referenced in [27].

Lemma 2.1 *Let* $f : SL(2) \to \mathbb{R}$*. A necessary and sufficient condition for* f *to be polyconvex is the existence of a convex function* $\tilde{f} : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ *where*

$$f(\xi) = \tilde{f}(\xi) \quad \text{for any } \xi \in SL(2).$$

Note that the function g is not unique, as illustrated by the example in Dacorogna [19, p. 158]. Indeed, let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be defined as

$$f(\xi) = (\xi_{11} + \xi_{22})^2 + (\xi_{12} - \xi_{21})^2,$$

and consider $g, h : \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ be two convex functions defined by

$$g(\xi, x) = |\xi|^2 + 2x, \quad h(\xi, x) = (\xi_{11} + \xi_{22})^2 + (\xi_{12} - \xi_{21})^2.$$

Although g and h are different functions, they both satisfy $f(\xi) = g(\xi, \det(\xi)) = h(\xi, \det(\xi))$.

In the following theorem, we recall the characterization of polyconvex functions in any dimension, for more details, the reader is refered to [19]. This characterization was established for the 2d and 3d cases by Ball [7].

Theorem 2.1

- *Let* $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$*, and consider the following notations, for any integer* J

$$T_J = \left\{ \alpha = (\alpha_1, \dots, \alpha_J),\ \alpha_i \geq 0,\ \sum_{i=1}^{J} \alpha_i = 1 \right\}.$$

and $M(\xi) = (\xi, M_2(\xi), \dots, M_r(\xi))$*, where* $r = \min\{n, m\}$*, and denote* $N = N(n, m)$*. Then, the polyconvexity of* f *is equivalent to the following statement:*

i) The following two properties hold:

a) A convex function $h : \mathbb{R}^{N(n,m)} \to \mathbb{R} \cup \{+\infty\}$ *exists, that satisfies*

$$f(\xi) \geq h(M(\xi)) \text{ for each } \xi \in \mathbb{R}^{m\times n},$$

b) for all $\xi_i \in \mathbb{R}^{m\times n}$ *and* $\alpha \in T_{N+1}$ *where*

$$\sum_{i=1}^{N+1} \alpha_i M(\xi_i) = M\left(\sum_{i=1}^{N+1} \alpha_i \xi_i\right),$$

is satisfied, hence

$$f\left(\sum_{i=1}^{N+1} \alpha_i \xi_i\right) \leq \sum_{i=1}^{N+1} \alpha_i f(\xi_i).$$

- *Supposing statement i) holds, and* $g : \mathbb{R}^{N(n,m)} \to \mathbb{R} \cup \{+\infty\}$ *is considered as follows*

$$g(\eta) = \inf\left\{\sum_{i=1}^{N+1} \alpha_i f(\xi_i) \ : \alpha \in T_{N+1},\ \eta = \sum_{i=1}^{N+1} \alpha_i M(\xi_i)\right\},$$

hence, g is convex, and

$$f(\xi) = g(M(\xi)) \text{ for every } \xi \in \mathbb{R}^{m\times n}.$$

Furthermore, for each $\eta \in \mathbb{R}^{N(n,m)}$,

$$g(\eta) = \sup\left\{\phi(\eta),\ \phi : \mathbb{R}^{N(n,m)} \to \mathbb{R} \cup \{+\infty\} \text{ convex and } f(\xi) = \phi(M(\xi)),\ \forall \xi \in \mathbb{R}^{m\times n}\right\}.$$

- *In the finite case, i.e.,* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$*. Hence, f is polyconvex if and only if the following requirement is fulfilled,*

i') for each $\xi \in \mathbb{R}^{m\times n}$*, there exists* $\gamma = \gamma(\xi) = (\gamma_1(\xi), \gamma_2(\xi), \ldots, \gamma_r(\xi)) \in \mathbb{R}^{N(n,m)}$ *for which*

$$f(\eta) \geq f(\xi) + \gamma_1(\xi) : (\eta - \xi) + \sum_{i=2}^{r} \gamma_i(\xi) : (M_i(\eta) - M_i(\xi))$$

for every $\eta \in \mathbb{R}^{m\times n}$.

- *Under the assumption of i'), the function*

$$g(A_1, A_2, \ldots, A_r) = \sup_{\xi \in \mathbb{R}^{m\times n}} \left\{\gamma_1(\xi) : (A_1 - M_1(\xi)) + \sum_{i=2}^{r} \gamma_i(\xi) : (A_i - M_i(\xi)) + f(\xi)\right\}$$

is convex and finite-valued, satisfying

$$f(\xi) = g(M(\xi)) \text{ for every } \xi \in \mathbb{R}^{m\times n}.$$

The following result provides a necessary condition for polyconvexity in the case of functions satisfying growth conditions, see [71, Lemma 5.1] and [19, Corollary 5.9].

Lemma 2.2 *Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be a polyconvex function. Then, the following statements hold:*

i) For $m = n$, the existence of a constant $c > 0$ and $1 \leq q < n$ satisfying

$$0 \leq f(\xi) < c(1 + |\xi|^q) \quad \text{for every } \xi \in \mathbb{R}^{n\times n},$$

implies the existence of an integer $k \in [1, n-1]$, along with a convex function $g(\eta_1, \eta_2, \ldots, \eta_k)$ that satisfies

$$f(\xi) = g(\xi, M_2(\xi), \ldots, M_k(\xi)) \quad \text{for every } \xi \in \mathbb{R}^{n\times n},$$

with

$$\frac{\partial g}{\partial \eta_k}(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_k) \neq 0,$$

for some $(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_k)$ within the domain of g.

ii) The existence of a constant $c \geq 0$ and $0 \leq q < 2$ satisfying

$$f(\xi) \leq c(1 + |\xi|^q) \quad \text{for every } \ \xi \in \mathbb{R}^{m\times n},$$

implies that the function f is convex.

iii) There exists a non-negative constant $\alpha \geq 0$ such that the inequality

$$f(\xi) \geq -\alpha(1 + |\xi|^r) \quad \text{holds for all } \xi \in \mathbb{R}^{m\times n},$$

where $r = \min\{m, n\}$.

Remark 2.4 *In statement i) of the above lemma, the author in [71] mentions that if g is not differentiable at the point $(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_k)$, its derivative with respect to η_k may be substituted by its sub-differential at $(\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_k)$.*

The following lemma provides a sufficient condition for polyconvexity, see in [28, 46].

Lemma 2.3 *Let $g : [1, \infty) \to \mathbb{R}$, be a non-decreasing convex function, and $f : \mathbb{R}^{2\times 2}_{+} \to [1, \infty)$ be a polyconvex function. Then the composition $g \circ f$ is polyconvex.*

We finish this section by outlining some examples, see [40, Lemma 3.4], [24, Exercise 1.21], [38, Propositions 10.2 and 10.4], and [30].

Examples 2.1 *The following functions are polyconvex:*

1. *For* $f : \mathbb{R}^{2\times m} \to \mathbb{R}$, *defined as*
$$f(\xi) = \begin{cases} 1 + |\xi|^2, & \sqrt{|\xi|^2 + 2\gamma} \geq 1, \\ 2\sqrt{|\xi|^2 + 2\gamma} - 2\gamma, & \sqrt{|\xi|^2 + 2\gamma} \leq 1, \end{cases}$$
such that $\gamma = \sqrt{\sum_{1\leq i<j\leq m} (\xi_{1i}\xi_{2j} - \xi_{1j}\xi_{2i})^2}$.

2. *For* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$, *given by*
$$f(\xi) = |\xi|^4 - 2|\xi|^2 \det \xi.$$

3. *For* $f : \mathbb{R}^{d\times d} \to \mathbb{R}$, *given by*
$$f(\xi) = (\det \xi - 1)^2.$$

4. *For* $f_{d,q} : \mathbb{R}^{d\times d} \to \mathbb{R}$, *given by*
$$f_{d,q}(\xi) = (|\xi|^d - d^{\frac{d}{2}} \det \xi)^q, \quad q \geq 1.$$

5. *For* $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$, *given by*
$$f(\xi) = \frac{1}{2}\big(||\xi||^2 + \det \xi \ log(||\xi||^2)\big).$$

6. *For* $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$, *given by*
$$f(\xi) = \frac{1}{2}\big(\frac{||\xi||^2}{\det \xi} + log\frac{||\xi||^2}{\det \xi} - log \det \xi\big).$$

2.2 k-Polyconvexity

In this section, we explore a particular class of polyconvex functions known as k-polyconvex functions. This class of functions is initially defined in [8] and further examined in [64].

Definition 2.2 *A function* $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$ *is termed k-polyconvex if it possesses the following structure:*
$$f(\xi) = g(\xi, M_2(\xi), \ldots, M_k(\xi)),$$
where $g : \mathbb{R}^{N(k,n,m)} \to \mathbb{R}\cup\{+\infty\}$ *is a convex function,* $N(k, n, m) = \sum_{i=1}^{k} \binom{m}{i}\binom{n}{i}$ *and* $1 \leq k \leq r$.

The theorem below outlines the properties of k-polyconvexity. For more details, see [8, Theorem 5.2, p. 159]; see also [64].

Theorem 2.2 *Let* $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$*. The following statements provide equivalent characterizations of the* k*-polyconvexity of* f *:*

i) $a \in \mathbb{R}$ *and* $A_i \in \mathbb{R}^{\binom{m}{i}\times\binom{n}{i}}$ *exist where*

$$f(\xi) \geq a + \sum_{i=1}^{k} A_i : M_i(\xi), \quad \textit{for all } \xi \in \mathbb{R}^{m\times n},$$

and if $\xi, \xi_i \in \mathbb{R}^{m\times n}$ *and* $\alpha \in T_p$ *satisfy*

$$M_q(\xi) = \sum_{i=1}^{p} \alpha_i M_q(\xi_i), \quad q = 1, \ldots, k,$$

then

$$f(\xi) \leq \sum_{i=1}^{p} \alpha_i f(\xi_i).$$

ii) If $f : \mathbb{R}^{m\times n} \to \mathbb{R}$*, then for each* $\xi \in \mathbb{R}^{m\times n}$*,* $B_i(\xi) \in \mathbb{R}^{\binom{m}{i}\times\binom{n}{i}}$ *exist such that*

$$f(\eta) \geq f(\xi) + \sum_{i=1}^{k} B_i(\xi) : (M_i(\eta) - M_i(\xi)), \quad \textit{for all } \eta \in \mathbb{R}^{m\times n}.$$

Remark 2.5 *As noted in [64], if* f *takes only finite (and/or non-negative) values, then* g*, as in condition (ii), can also be chosen to take only finite (and/or non-negative) values. Consequently, by Carathéodory's theorem, we can select the non-negative integer* p *such that* $p \leq N(k, n, m) + 1$*; see [8].*

2.3 Polyconvexity for C^1 functions: Necessary and Sufficient Conditions

Based on [32], we have the characterization of C^1 polyconvexity in two dimensions:

Theorem 2.3 *Let* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ *define a* C^1 *function. an equivalent condition for the polyconvexity of* f *is that for all* $\varepsilon \in \mathbb{R}^{2\times 2}$*, there is a real number* $b(\varepsilon)$ *satisfying*

$$\inf_{\eta} F(\eta, \varepsilon, b(\varepsilon)) = F(\varepsilon, \varepsilon, b(\varepsilon)) = 0.$$

and the function $F : \mathbb{R}^{2\times 2} \times \mathbb{R}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *is expressed as*

$$F(\eta, \varepsilon, b) = f(\eta) - f(\varepsilon) - Df(\varepsilon) : (\eta - \varepsilon) - b \det(\eta - \varepsilon).$$

Remark 2.6 *For $f \in C^2(\mathbb{R}^{2\times 2})$, an equivalent condition for the polyconvexity of f is that for every $\varepsilon \in \mathbb{R}^{2\times 2}$, the set*

$$G(\varepsilon) = \{b,\ \varepsilon \text{ is the global minimizer of } g(\cdot, \varepsilon, b)\}$$

is nonempty; see [32].

Additionally, Aubert [5] provides another necessary and sufficient condition for polyconvexity of functions of class C^1 in dimensions 2 and 3. Let's study the cases $d = 2$ and $d = 3$ and present the following sets:

$$\mathbb{R}^{d\times d}_{+} = \{\varepsilon \in \mathbb{R}^{d\times d}, det\varepsilon > 0\},\ \mathbb{R}^{d\times d}_{-} = \{\varepsilon \in \mathbb{R}^{d\times d}, det\varepsilon < 0\},$$

especially, in the case $d = 3$, we define

$U^{ij} = \{\varepsilon \in \mathbb{R}^{3\times 3}, \varepsilon_{ip} = \varepsilon_{qj} = 0, \quad \text{for each } p, q = 1, 2, 3\},$
$V_{+} = \{(i, j),\ i, j = 1, 2, 3,\ (cof\varepsilon)_{ij} > 0\}\,,\ V_{-} = \{(i, j),\ i, j = 1, 2, 3,\ (cof\varepsilon)_{ij} < 0\}\,,$
$U^{ij}_{+} = \{\varepsilon \in U^{ij}, (cof\varepsilon)_{ij} > 0\},\ U^{ij}_{-} = \{\varepsilon \in U^{ij}, (cof\varepsilon)_{ij} < 0\}.$

Where $cof\varepsilon$ is the cofactor matrix of ε and $det\varepsilon$ its determinant.

Theorem 2.4 *Let $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ define a C^1 function. Hence, the polyconvexity of f is equivalent to:*

i) If $d = 2$, the next inequality is satisfied:

$$F_1(\eta) \leq G_1(\eta), \quad \text{for all } \eta \in \mathbb{R}^{2\times 2},$$

with

$$F_1(\eta) = \sup_{\varepsilon\in\mathbb{R}^{2\times 2}_{-}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta)}{\det\varepsilon},\ G_1(\eta) = \inf_{\varepsilon\in\mathbb{R}^{2\times 2}_{+}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta)}{det\varepsilon}.$$

ii) If $d = 3$, the next inequalities are satisfied

$$F_{ij}(\eta) \leq G_{ij}(\eta),\ F(\eta) \leq G(\eta)$$

for every $\eta \in \mathbb{R}^{3\times 3}$, such that

$$F_{ij}(\eta) = \sup_{\varepsilon\in U^{ij}_{-}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta)}{(cof\varepsilon)_{ij}},\ G_{ij}(\eta) = \inf_{\varepsilon\in U^{ij}_{+}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta)}{(cof\varepsilon)_{ij}}$$

$$F(\eta) = \sup_{\varepsilon\in\mathbb{R}^{3\times 3}_{-}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta) - \sum\limits_{V_{+}(\varepsilon)} F_{ij}(\eta)(cof\varepsilon)_{ij} - \sum\limits_{V_{-}(\varepsilon)} G_{ij}(\eta)(cof\varepsilon)_{ij}}{\det\varepsilon},$$

$$G(\eta) = \inf_{\varepsilon\in\mathbb{R}^{3\times 3}_{+}} \frac{f(\eta+\varepsilon) - f(\eta) - \varepsilon : Df(\eta) - \sum\limits_{V_{+}(\varepsilon)} F_{ij}(\eta)(cof\varepsilon)_{ij} - \sum\limits_{V_{-}(\varepsilon)} G_{ij}(\eta)(cof\varepsilon)_{ij}}{\det\varepsilon}.$$

Here, $Df(\eta)$ denotes the gradient of the function f at η.

Corollary 2.1 *Let $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be a polyconvex function that it of class C^2. Hence, for every $\eta \in \mathbb{R}^{2\times 2}$, the functions $F_2(\eta)$ and $G_2(\eta)$ are finite, and satisfy*

$$G_2(\eta) \geq F_2(\eta),$$

with

$$G_2(\eta) = \inf_{\varepsilon \in \mathbb{R}^{2\times 2}_{+}} \frac{\varepsilon^T D^2 f(\eta) \varepsilon}{\det \varepsilon}, \quad F_2(\eta) = \sup_{\varepsilon \in \mathbb{R}^{2\times 2}_{-}} \frac{\varepsilon^T D^2 f(\eta) \varepsilon}{\det \varepsilon}.$$

Additionally, for each $\beta \in [F_2(\eta), G_2(\eta)]$ and all $A \in \mathbb{R}^{2\times 2}$,

$$A^T D^2 f(\eta) A \geq \beta \det A.$$

Here, $D^2 f(\eta)$ denotes the Hessian matrix of f at η. This result follows from the work of Aubert in [5].

Remark 2.7 *By using Theorem 2.4, Aubert [5] reduced the number of matrices required to test the polyconvexity of a given function defined on $\mathbb{R}^{2\times 2}$ from six to three. Specifically, the polyconvexity of the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is equivalent to the fact that, for every $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^{2\times 2}$ and $\alpha \in T_3$ verifying*

$$\det(\xi_1 - \xi_3)\det(\xi_2 - \xi_3) < 0 \quad \textit{and} \quad \sum_{i=1}^{3} \alpha_i \det(\xi_i) = \det\left(\sum_{i=1}^{3} \alpha_i \xi_i\right),$$

we obtain

$$f\left(\sum_{i=1}^{3} \alpha_i \xi_i\right) \leq \sum_{i=1}^{3} \alpha_i f(\xi_i).$$

2.4 Quasiconvex & Rank One Convex Functions

An overview of concepts related to polyconvexity, including quasiconvexity and rank-one convexity, will be presented in this section. Sufficient and necessary conditions for polyconvexity will be discussed in relation to these concepts. In particular, we will explore cases where these three notions of semiconvexity are equivalent.

Historically, quasiconvexity was introduced by Morrey [52] to show that the functional

$$I(u) = \int_{\Omega} f(x, u(x), \nabla u(x))\, dx,$$

is weak lower semi-continuous, where $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is the energy density function, $u : \mathbb{R}^n \to \mathbb{R}^m$ is the deformation, and $\Omega \subset \mathbb{R}^n$ is an open set.

Definition 2.3 *The local integrable function* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, *is termed*

i) quasiconvex function if

$$\int_{\Omega} f(\xi + \nabla\varphi(x))dx \geq m(\Omega)f(\xi)$$

for each open bounded set $\Omega \subset \mathbb{R}^n$, *each* $\xi \in \mathbb{R}^{m\times n}$ *and any* $\varphi \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$, *with* $m(\Omega)$ *denote Lebesgue measure.*

ii) quasiaffine if f *and* $-f$ *are quasiconvex*

Since verifying quasiconvexity directly can be challenging, Morrey [52, 53] introduced polyconvexity and rank-one convexity as sufficient and necessary conditions, respectively, to simplify the process.

Definition 2.4 *A function* $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$, *is called rank-one convex if it satisfies the following inequality:*

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda)f(\eta)$$

for every $\lambda \in [0,1]$ *and all* $\xi, \eta \in \mathbb{R}^{m\times n}$ *where,* $rank(\xi - \eta) \leq 1$.

Note that the notion of quasiconvexity is defined only for finite-valued functions, and it is equivalent to weak lower semicontinuity, as shown in [17]. However, for functions allowed to take the value infinity, an extension of the quasiconvexity definition is provided in [9].

Additionally, we notice that polyconvexity and quasiconvexity are non-local properties, while rank-one convexity is local (see [42, 43]; see also [29]).

Definition 2.5 *A property* H *is said local if a local operator* h *exists, wherein for every smooth function* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, *then* f *possesses the property* H *equivalently to* $h(f) = 0$.

Additionally, if $f = g$ *in the neighborhood of* A, *it implies that* $h(f) = h(g)$ *in the neighborhood of* A, *then the operator* h *is local.*

In accordance with Kristensen [42, 43], Guerra [29], polyconvexity and quasiconvexity are generally non-local properties,

Theorem 2.5 *Consider* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. *Then:*

i) For $n \geq 2$ *and* $m \geq 2$, *polyconvexity does not exhibit local properties;*

ii) For $n \geq 2$ *and* $m \geq 3$, *quasiconvexity does not qualify as a local property;*

iii) Rank-one convexity possesses the property of locality.

Remark 2.8 *The local operator for rank-one convexity is given by:*

$$h(f)(\xi) = \inf\left\{\sum_{i,j=1}^{m}\sum_{p,q=1}^{n}\frac{\partial^2 f(\xi)}{\partial\xi_{ip}\partial\xi_{jq}}a_i a_j b_p b_q \mid a \in \mathbb{R}^m, b \in \mathbb{R}^n\right\}.$$

The following theorems outline on the relationship between the various semi-convexity notions. For more details, see [17, p. 40].

Morrey [52, 53] showed the following,

Theorem 2.6 *Consider a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. It follow that,*

- *i) f is convex implies f is polyconvex, which implies f is quasiconvex, and this implies f is rank-one convex.*
- *ii) For $f \in C^2(\mathbb{R}^{m\times n})$, the necessary and sufficient condition for f to be rank one convex is*

  $$\sum_{i,j=1}^{m}\sum_{p,q=1}^{n}\frac{\partial^2 f(\xi)}{\partial\xi_{ip}\partial\xi_{jq}}a_i a_j b_p b_q \geq 0.$$

  For all $\xi \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$.
- *iii) If $m = n + 1$, then $M_n(\xi) \in \mathbb{R}^{n+1}$ for $\xi \in \mathbb{R}^{m\times n}$. Let $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ be such that,*

  $$f(\xi) = \varphi(M_n(\xi)),$$

  often referred as the minimal surface problem. Then, The properties of polyconvexity, quasiconvexity, and rank-one convexity for f are equivalent to the property of convexity for φ.
- *iv) For functions $f : \mathbb{R}^{m\times n} \to \mathbb{R} \cup \{+\infty\}$, convexity implies polyconvexity, which in turn implies rank-one convexity.*
- *v) For $m = 1$ or $n = 1$, all semiconvexity concepts are equivalent to the convexity of the function f.*
- *vi) Convexity, polyconvexity, quasiconvexity, or rank-one convexity of f implies that f is locally Lipschitz*

For the case of polyaffine, quasiaffine, and rank-one affine functions (i.e., functions f and $-f$ that are polyconvex, quasiconvex, and rank-one convex), we have the following characterization (see [17, 19]).

Theorem 2.7 *Consider $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, the following equivalences holds:*

i) *f is polyaffine.*

ii) *f is quasiaffine.*

iii) *f is rank one affine.*

iv) *for each $1 \leq k \leq r$, there exist $\eta_k \in \mathbb{R}^{\binom{m}{k} \times \binom{n}{k}}$, and $a \in \mathbb{R}$ so that*

$$f(\xi) = a + \sum_{i=1}^{r} \eta_i : M_i(\xi)$$

where $r = \min\{n, m\}$.

v) *f is a function that is continuously differentiable, and*

$$f(\xi + x \otimes y) = f(\xi) + Df(\xi) : x \otimes y,$$

for every $\xi \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, with $Df(\xi) = \left(\frac{\partial f(\xi)}{\partial \xi_{ij}}\right)_{1 \leq i,j \leq m,n}$

2.4.1 Equivalence Case

In the following, we present several results that illustrate the equivalence of different semiconvexity notions under certain conditions.

For functions depending on quasi-affine functions, we have the equivalence of the different notions of convexity, as stated in [18, 16]; see also [17, 19].

Theorem 2.8 *Consider $f, \psi : \mathbb{R}^{m \times n} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$, which satisfy*

$$f(\xi) = \varphi(\psi(\xi)).$$

If ψ is quasiaffine, then the polyconvexity, quasiconvexity, rank-one convexity of f, are equivalent to the convexity of φ.

The validity of this equivalence is maintained for functions depending on the Euclidean norm. Refer to [16, 17] for more details.

Theorem 2.9 *Consider $f : \mathbb{R}^{m \times n} \to \mathbb{R}$. and $\varphi : \mathbb{R}^+ \to \mathbb{R}$, verifying*

$$f(\xi) = \varphi(|\xi|)$$

Then, the conditions of convexity, polyconvexity, quasiconvexity, rank-one convexity of f, as well as the convexity of φ with $\varphi(0) = \min\{\varphi(x),\ x \geq 0\}$, are all equivalent.

The subsequent result is attributed to [9].

Theorem 2.10 *Let $m = n,\ 1 \leq \alpha < 2n$, and let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$, verifying*

$$f(\xi) = |\xi|^{\alpha} + \varphi(det\xi).$$

Then, the polyconvexity, quasiconvexity, rank-one convexity of the function f, along with the convexity of φ, are all equivalent.

Charrier [13] provides the following example.

Theorem 2.11 *Consider $m = n,\ 1 \leq i \leq n-1,\ \alpha > 0$ and*

$$f(\xi) = \begin{cases} \left(\frac{|M_i(\xi)|^{n/i}}{det\xi}\right)^{\alpha} & \textit{if } det\xi > 0 \\ +\infty & \textit{if } det\xi \leq 0. \end{cases}$$

Then, the polyconvexity and rank-one convexity of f, along with the condition $\alpha \geq \frac{i}{n-i}$, are all equivalent.

The next example is due to Ball and Murat [9], see also [21].

Example 2.2 *Consider $\beta \geq \frac{1}{4},\ \alpha \geq 0$ and the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ given by*

$$f_{\alpha}(\xi) = |\xi|^{4\beta} - 2^{2\beta-1}\alpha((\det\xi)^2)^{\beta}.$$

Then, the following result hold, see [20]:

1. *For $\frac{1}{4} < \beta < 1$, the function f_{α} is convex, polyconvex, quasiconvex, and rank-one convex if and only if $\alpha = 0$.*
2. *For $\beta = 1$, the function f_{α} is convex, polyconvex, quasiconvex, and rank-one convex if and only if $\alpha \leq 1$.*
3. *For $\beta > 1$, the function f_{α} is rank one convex if and only if $\alpha \leq \alpha_r$, where*

$$\alpha_r = \begin{cases} \frac{2(2\beta+(2\beta-1)x)}{(2\beta-1)(1-x)(1-x^2)^{\beta-1}} & \textit{if} \ \ 1 < \beta < 1 + \frac{1}{\sqrt{2}}, \\ \frac{\frac{2(2\beta-1)}{4\beta-1}}{\left(1-\frac{\beta}{(2\beta-1)^2}\right)^{2(\beta-1)}} & \textit{if} \ \ \beta \geq 1 + \frac{1}{\sqrt{2}}, \end{cases}$$

such that

$$x = \frac{-(4\beta^2-1) + \sqrt{(4\beta^2-1)^2 - 8(2\beta-1)(\beta-1)(4\beta-1)}}{4(2\beta-1)(\beta-1)}$$

2.4.2 Quadratic Case

In the case of quadratic forms, quasiconvexity is equivalent to rank-one convexity. Furthermore, in the 2×2 case, these notions are also equivalent to polyconvexity. This equivalence has been established in [69, 2, 55, 36, 67, 57] and [45], as well as, [17].

Theorem 2.12 *Consider* $A \in \mathcal{S}^{(m\times n)\times(m\times n)}$ *and let*

$$f(\xi) = A\xi.\xi$$

where "." represent the scalar product in $\mathbb{R}^{mn}$ *and the matrix* ξ *is considered as a vector of* mn *element. Then, the quadratic function* f *is quasiconvex if and only if* f *is rank-one convex.*

Moreover, in the case where $m = 2$ *or* $n = 2$, *the conditions that* f *is polyconvex, quasiconvex, and rank-one convex are all equivalent.*

In general, if $m, n \geq 3$, *the rank-one convexity of* f *is not a sufficient condition for its polyconvexity of.*

For the case where $m = n = 3$, Serre, see [57], provided a specific counterexample of a quadratic form that is rank-one convex but not polyconvex.

Example 2.3 *Consider* $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$, *defined as*

$$\begin{aligned} f(\xi) =& (\xi_{12} - \xi_{31} + \xi_{13})^2 + (\xi_{21} - \xi_{31} - \xi_{13})^2 \\ &+ (\xi_{11} - \xi_{32} - \xi_{23})^2 + (\xi_{22})^2 + (\xi_{33})^2. \end{aligned}$$

Then, the function $g : \mathbb{R}^{3\times 3} \to \mathbb{R}$ *given by*

$$g(\xi) = f(\xi) - \epsilon|\xi|^2,$$

with $\epsilon > 0$, *is not polyconvex, whereas it is rank-one convex.*

2.4.3 Counterexample

Sverak [66] provides a counterexample that addresses Morrey's problem [52]. This long-standing open problem in the calculus of variations asks whether a rank-one convex function that is not quasiconvex exists. Sverak's counterexample answers this question for the case $n \geq 2$ and $m \geq 3$. However, the planar case remains an open problem.

Example 2.4 *For* $\beta, \alpha > 0$, *let* $f_{\alpha,\beta} : \mathbb{R}^{3\times 2} \to \mathbb{R}$ *be defined as*

$$f_{\alpha,\beta}(\xi) = g(P(\xi)) + \beta|\xi - P(\xi)|^2 + \alpha(|\xi|^2 + |\xi|^4).$$

such that

$$L = \left\{ \xi \in \mathbb{R}^{3\times 2},\ \xi = \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} ; x, y, z \in \mathbb{R} \right\},$$

$$P : \mathbb{R}^{3\times 2} \to L, \quad P(\xi) = \begin{pmatrix} \xi_{11} & 0 \\ 0 & \xi_{22} \\ (\xi_{31}+\xi_{32})/2 & (\xi_{31}+\xi_{32})/2 \end{pmatrix}$$

$$g : L \to \mathbb{R}, \quad g \begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix} = -xyz.$$

Thus,

i) For all $\beta > 0$ and for all $\alpha > 0$ sufficiently small, the function $f_{\alpha,\beta}$ is not quasiconvex.

ii) For all $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$, such that $f_{\alpha,\beta}$ is rank one convex.

The first properties i) was proved by using the periodic function $\phi \in W^{1,\infty}_{per}((0,1)^2; \mathbb{R}^3)$ given by

$$\phi(x_1, x_2) = \frac{1}{2\pi} \begin{pmatrix} sin 2\pi x_1 \\ sin 2\pi x_2 \\ sin 2\pi (x_1 + x_2) \end{pmatrix}.$$

Such that $W^{1,\infty}_{per}((0,1)^n; \mathbb{R}^m)$ is the set of periodic functions $\varphi \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m)$.

Remark that $D\phi \in L$ and $P(D\phi) = D\phi$, then we get

$$\int\limits_{(0,1)^2} g(D\phi)dx = -\int\limits_0^1 \int\limits_0^1 (cos 2\pi x_1)^2 (cos 2\pi x_2)^2 dx_1 dx_2 < 0.$$

Consequently, for each $\alpha > 0$ that is sufficiently small and for each $\beta > 0$,

$$\int\limits_{(0,1)^2} f_{\alpha,\beta}(D\phi)dx < 0 = f_{\alpha,\beta}(0).$$

According to the following lemma from [19],

Lemma 2.4 *Consider a continuous function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$. A necessary and sufficient condition for the quasiconvexity of f is that the following inequality holds:*

$$\int\limits_{(0,1)^n} f(\xi + D\varphi(x))dx \geq f(\xi).$$

for all $\xi \in \mathbb{R}^{m\times n}$ and for all $\varphi \in W^{1,\infty}_{per}((0,1)^n; \mathbb{R}^m)$.

Using this lemma, we can conclude that the function $f_{\alpha,\beta}$ *is not quasiconvex.*

In the second properties ii), it is equivalent to show the Legendre-Hadamard condition, i.e.,

$$L_f(\xi,\eta) = \frac{d^2}{dt^2}(f_{\alpha,\beta}(\xi + t\eta))\bigg|_{t=0} \geq 0,$$

for every $\xi, \eta \in \mathbb{R}^{3\times 2}$ *such that* $\mathrm{rank}\{\eta\} = 1$.

Sverak [66], generalize his example to arbitrary dimension $n \geq 2$ *and* $m \geq 3$ *by definining* $\widetilde{f} : \mathbb{R}^{m\times n} \to \mathbb{R}$*, as*

$$\widetilde{f}(\xi) = f_{\alpha,\beta}(\lambda(\xi)).$$

such that $\lambda : \mathbb{R}^{m\times n} \to \mathbb{R}^{3\times 2}$ *defined as*

$$\lambda(\xi) = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \\ \xi_{31} & \xi_{32} \end{pmatrix}.$$

Then, it follows that

- *i)* $\widetilde{f}$ *is rank one convex, as a consequence of the rank-one convexity of* $f_{\alpha,\beta}$.
- *ii)* $\widetilde{f}$ *is not quasiconvex. This can be shown by considering the periodic function*

$$\widetilde{\phi}(x_1, ..., x_n) = (\phi_1(x_1,x_2), \phi_2(x_1,x_2), \phi_3(x_1,x_2), 0, ..., 0).$$

We observe that $\widetilde{\phi} \in W^{1,\infty}_{per}((0,1)^n; \mathbb{R}^m)$ *and*

$$\int_{(0,1)^n} \widetilde{f}(\nabla\widetilde{\phi})dx < 0 = \widetilde{f}(0).$$

Therefore, by Lemma 2.4, $\widetilde{f}$ *is not quasiconvex.*

According to the characterizations provided in [5], and the characterization of rank-one convex functions of class C^2 from [39, 6], Aubert [4, 5] provides an isotropic[1] function in the two-dimensional case ($d = 2$) to demonstrate that rank one convexity does not imply polyconvexity.

Example 2.5 *Let* $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ *be given by*

$$f(\xi) = \begin{cases} \frac{1}{3}|\xi|^4 - \frac{1}{6}(\det\xi)^2 - \frac{2}{3}|\xi|^2\det\xi & \text{if } \det\xi > 0, \\ \infty & \text{else,} \end{cases}$$

---

[1] A function $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is isotropic if $f(Q\xi R) = f(\xi)$ for all $\xi \in \mathbb{R}^{d\times d}$ and $Q, R \in SO(d)$.

where $|\xi|$ *represents the Frobenius norm of* ξ*. Equivalently, for* $\det \xi > 0$*,* $f(\xi)$ *can be expressed in terms of the singular values* λ_1, λ_2 *of* ξ *as:*

$$f(\xi) = \frac{1}{3}(\lambda_1^4 + \lambda_2^4) - \frac{2}{3}\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2) + \frac{1}{2}\lambda_1^2\lambda_2^2.$$

Thus, f *is rank-one convex but not polyconvex.*

Next, we present the example of Dacorogna, Alibert, and Marcellini [3, 23], as given in its general version in [21], for the case $n = m = 2$.

Example 2.6 *Let* $\beta \geq 1$*,* $\alpha \in \mathbb{R}$*, and consider* $f_\alpha : \mathbb{R}^{2\times 2} \to \mathbb{R}$*, given by*

$$f_\alpha(\xi) = |\xi|^{2\beta}(|\xi|^2 - \alpha \det \xi).$$

Then, it follows that,

i) For $\beta = 1$*, the convexity, polyconvexity, quasiconvexity, and rank-one convexity of* f_α *are equivalent to* $|\alpha| \leq \alpha_c$*,* $|\alpha| \leq \alpha_p$*,* $|\alpha| \leq \alpha_q$*, and* $|\alpha| \leq \alpha_r$*, respectively, where* $\alpha_c = \frac{4}{3}\sqrt{2}$*,* $\alpha_p = 2$*,* $\alpha_q = 2 + \delta$ *with* $\delta > 0$*, and* $\alpha_r = \frac{4}{\sqrt{3}}$*.*

ii) For $\beta > 1$*, the rank one convexity of* f_α *is equivalent to* $|\alpha| \leq \alpha_r$*, as stated in [20], where*

$$\alpha_r = \begin{cases} \alpha_1 & \text{if } 1 \leq \beta < \frac{9+5\sqrt{5}}{4}, \\ \alpha_2 & \text{if } \beta \geq \frac{9+5\sqrt{5}}{4}. \end{cases}$$

with

$$\alpha_1 = \left(1 + \frac{1}{\beta}\right) \min_{x>0} \left\{ \frac{x^4 + 2(\beta+1)x^2 + 2\beta + 1}{3x^3 + (2\beta+1)x} \right\}$$

$$\alpha_2 = 1 + \sqrt{1 - \frac{1}{2\beta} - \frac{1}{2\beta^2}}$$

Note that, the question of whether $2 + \delta = \frac{4}{\sqrt{3}}$ *is still open. However, if* $2 + \delta < \frac{4}{\sqrt{3}}$*, then a rank-one convex function that is not quasiconvex exists.*

We conclude this section by presenting a result given in [61].

Proposition 2.1 *Let* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ *defined as*

$$f(\xi) = \begin{cases} f_1(\xi) & \text{if } \xi \in \mathbb{M}_1 \\ f_2(\xi) & \text{if } \xi \in \mathbb{M}_2 \end{cases}$$

such that, $f_1, f_2 : \mathbb{R}^{m\times n} \to \mathbb{R}$ defined two function satisfy along with the two closed sets $\mathbb{M}_1, \mathbb{M}_2 \subset \mathbb{R}^{m\times n}$

$$\begin{cases} f_1 = f_2 & \text{over } \mathbb{M}_1 \cap \mathbb{M}_2, \\ f_1 \geq f_2 & \text{over } \mathbb{M}_1. \end{cases} \quad \text{and} \quad \begin{cases} \mathbb{M}_1 \cup \mathbb{M}_2 = \mathbb{R}^{m\times n}, \\ \mathbb{M}_1 - \mathbb{M}_2 \text{ is an open set.} \end{cases}$$

Then

- *i) there exist f_1, f_2 tow polyconvex function such that f is not;*
- *ii) The rank one convexity of f_1 and f_2 implies the rank one convexity of f;*
- *iii) The convexity of f is ensured if f_1 and f_2 are both convex.*

Note that the first statement is demonstrated by considering a counterexample of a non-polyconvex function that is rank-one convex. Specifically, consider the functions $f_1, f_2 : \mathbb{R}^{2\times 2} \to \mathbb{R}$, defined by the singular values $\lambda_1(\xi) \geq \lambda_2(\xi)$ of ξ as

$$f_1(\xi) = \lambda_1(\xi)\lambda_2(\xi), \qquad f_2(\xi) = \lambda_1(\xi) + \lambda_2(\xi) - 1,$$

both satisfying the hypotheses of the lemma with

$$\mathbb{M}_1 = \{\xi \in \mathbb{R}^{2\times 2} : \ \lambda_1(\xi) \leq 1\}, \ \mathbb{M}_2 = \{\xi \in \mathbb{R}^{2\times 2} : \ \lambda_1(\xi) \geq 1\}.$$

Therefore, the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$, given as

$$f(\xi) = \begin{cases} \lambda_1(\xi)\lambda_2(\xi) & \text{if } \ \lambda_1(\xi) \leq 1, \\ \lambda_1(\xi) + \lambda_2(\xi) - 1 & \text{if } \ \lambda_1(\xi) \geq 1, \end{cases}$$

is not a polyconvex function but it is rank one convex, see [61, Theorem 5].

2.5 Invariant and Fully Invariant Polyconvex Functions

In this section, we focus on functions that are invariant under orientation. We begin with some preliminary results, following [60, 24], we provide the following definitions.

Definition 2.6 *Let $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$ and $g : \mathbb{R}^{d} \to \mathbb{R} \cup \{\infty\}$.*

- *i) f is said to be invariant if,*

$$f(Q\xi R) = f(\xi) \quad \text{for all } \xi \in \mathbb{R}^{d\times d} \text{ and every } Q, R \in SO(d).$$

ii) *f is said to be fully invariant if,*

$$f(Q\xi R) = f(\xi) \quad \text{for all } \xi \in \mathbb{R}^{d\times d} \text{ and every } Q, R \in O(d).$$

iii) *g is symmetric if for all $d \times d$ permutation matrices P and all $x \in \mathbb{R}^d$,*

$$g(Px) = g(x).$$

iv) *g is even if for every $x \in \mathbb{R}^d$,*

$$g(\epsilon x) = g(x) \quad \text{for every } \epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d \text{ where } \prod_{i=1}^{d} \epsilon_i = 1.$$

v) *g is fully even if for every $x \in \mathbb{R}^d$,*

$$g(\epsilon x) = g(x) \quad \text{for every } \epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{-1, 1\}^d.$$

In nonlinear elasticity, invariant functions are also called isotropic functions.

Let $\lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_d(\xi))$ denote the vector of the singular values of the matrix $\xi \in \mathbb{R}^{d\times d}$, where $\lambda_i(\xi)$ are the eigenvalues of $(\xi^T\xi)^{1/2}$ ordered as follows:

$$\lambda_1(\xi) \geq \cdots \geq \lambda_d(\xi) \geq 0.$$

Consider the vector $\mu(\xi) = (\mu_1(\xi), \dots, \mu_d(\xi))$ representing the signed singular values of the matrix $\xi \in \mathbb{R}^{d\times d}$ where the components of $\mu(\xi)$ are defined as follows:

$$\mu_d(\xi) = \lambda_d(\xi) \operatorname{sign}(\det \xi) \quad \text{and} \quad \mu_i(\xi) = \lambda_i(\xi), \ i = 1, \dots, d-1.$$

The following proposition can be derived from the singular value decomposition. For more details see [30, Lemma 3.1 and Remark 3.6] and [60, 59, 62, Proposition 2.1], see also [19, Proposition 5.31].

Proposition 2.2 *Let $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$,*

i) *The necessary and sufficient condition for f to be invariant is that*

$$f(\xi) = g(\mu_1(\xi), \dots, \mu_d(\xi)) \tag{2.1}$$

where $g : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is the unique even and symmetric function.

ii) *The necessary and sufficient condition for f to be fully invariant is that*

$$f(\xi) = g(\lambda_1(\xi), \dots, \lambda_d(\xi))$$

such that $g : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ represents the unique symmetric and fully even function.

Remark 2.9 *It is noteworthy that the above proposition is slightly different from the one in [19, Proposition 5.31]. However, while this reference uses an increasing order for the singular values and includes the definition:*

$$\mu_1(\xi) = \lambda_1(\xi)\,\mathit{sign}(\det \xi) \quad \mathit{and} \quad \mu_i(\xi) = \lambda_i(\xi),\ i = 2, \ldots, d,$$

in our case, we use a decreasing order, and the signed singular values are defined as:

$$\mu_d(\xi) = \lambda_d(\xi)\,\mathit{sign}(\det \xi) \quad \mathit{and} \quad \mu_i(\xi) = \lambda_i(\xi),\ i = 1, \ldots, d-1.$$

Consider the set

$$\mathbb{G}^d = \left\{x \in \mathbb{R}^d : x_1 \geq \cdots \geq x_{d-1} \geq |x_d|\right\}.$$

The following example is due to [61] and is provided in a general version in [24], which represent an example of a fully invariant function that is rank-one convex but not polyconvex.

Theorem 2.13 *Let* $1 \leq k < \infty$, $d \geq 2$, *and let* f *be a fully invariant function on* $\mathbb{R}^{d\times d}$ *given as*

$$f(\mathrm{diag}(x)) = \begin{cases} x_1^k + |x_2|^k - 1 & \mathit{if} \quad x_1 \geq 1, \\ |x_1 x_2|^k & \mathit{if} \quad x_1 \leq 1, \end{cases}$$

for every $x \in \mathbb{G}^d$, *where diag*(x) *is the diagonal matrix whose diagonal entries are the components of the vector* x, *it follows that:*

i) f *is rank-one convex;*

ii) The function f *is not polyconvex for* $1 \leq k < 2$, *and it is polyconvex otherwise.*

2.5.1 Invariant Function Characterization

We now provide a characterization of invariant polyconvex functions, following [63, Theorem 4.1]; see also [24].

Theorem 2.14 *Given an invariant function* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}\cup\{\infty\}$ *and its associated signed singular value map* $g : \mathbb{R}^2 \to \mathbb{R}\cup\{\infty\}$ *(as defined in (2.1)). Therefore, the following conditions are equivalent.*

i) f *exhibits polyconvexity;*

ii) A convex function $\widetilde{g} : \mathbb{R}^3 \to \mathbb{R}\cup\{\infty\}$ *exists such that* $\widetilde{g}(\cdot, z)$ *is even and symmetric for every* $z \in \mathbb{R}$, *and*

$$g(x) = \widetilde{g}(x_1, x_2, x_1 x_2) \ \mathit{for\ all}\ x \in \mathbb{G}^2;$$

iii) There exists a convex function $\widehat{g} : \mathbb{R}^{+2} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ *such that* $\widehat{g}(\cdot, z)$ *is non-decreasing for all* $z \in \mathbb{R}$ *and*

$$g(x) = \widehat{g}(x_1 + x_2, x_1 - x_2, x_1 x_2) \text{ for all } x \in \mathbb{G}^2;$$

iv) Suppose $x, y^i \in \mathbb{G}^2$ *and* $\alpha^i \geq 0$ *for* $i = 1, \ldots, q$ *with* $\sum_{i=1}^q \alpha^i = 1$, $x_1 x_2 = \sum_{i=1}^q \alpha^i y_1^i y_2^i$, *and*

$$x_1 + \epsilon x_2 \leq \sum_{i=1}^{q} \alpha^i y_1^i + \epsilon y_2^i \text{ for every } \epsilon \in \{-1, 1\}.$$

Then, it follows that

$$g(x) \leq \sum_{i=1}^{q} \alpha^i g(y^i).$$

In [22], it was proved that the polyconvexity of the function $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$, in the case of invariant functions, coincides with its restriction on the diagonal matrices. Note that the function restricted to the diagonal matrices $\mathbb{R}_D^{2\times 2}$ is said to be polyconvex if there exists a function $g : \mathbb{R}^3 \to \mathbb{R}$, such that for all

$$\xi = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathbb{R}_D^{2\times 2}, \ f(\xi) = g(a, b, ab).$$

Consider a matrix $\xi \in \mathbb{R}^{2\times 2}$, given by $\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$. Define the associated matrix $\tilde{\xi}$ as $\tilde{\xi} = \begin{pmatrix} \xi_{22} & -\xi_{21} \\ -\xi_{12} & \xi_{11} \end{pmatrix}$. Denote by $\xi^+ = \frac{1}{2}(\xi + \tilde{\xi})$ and $\xi^- = \frac{1}{2}(\xi - \tilde{\xi})$.

Theorem 2.15 *Let* $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ *define an invariant function. Then, the following equivalence hold:*

i) f *is a polyconvex function;*

ii) f *is polyconvex on diagonal matrices;*

iii) For any $\eta \in \mathbb{R}_D^{2\times 2}$, *there are* $\alpha(\eta), \beta(\eta), \gamma(\eta) \in \mathbb{R}$ *that satisfy*

$$f(\xi) \geq f(\eta) + \begin{pmatrix} \alpha(\eta) & 0 \\ 0 & \beta(\eta) \end{pmatrix} : (\xi - \eta) + \gamma(\eta)(\det(\xi) - \det(\eta))$$

for all $\xi \in \mathbb{R}_D^{2\times 2}$; *specifically, if*

$$h(x, y, \delta) \equiv \sup_{a,b \in \mathbb{R}} \left\{ f \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \alpha(a, b)(x - a) + \beta(a, b)(y - b) + \gamma(a, b)(\delta - ab) \right\},$$

it follows that h is convex and

$$f(\xi) = h\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}, \det(\xi)\right),$$

$$f\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = h(x, y, xy)$$

for all $\xi \in \mathbb{R}^{2\times 2}$*;*

iv) For every $\xi_i \in \mathbb{R}^{2\times 2}_D$*, and every* $\beta_i \geq 0$ *such that* $\sum_{i=1}^4 \beta_i = 1$*, satisfying*

$$\sum_{i=1}^{4} \beta_i \det(\xi_i) = \det\left(\sum_{i=1}^{4} \beta_i \xi_i\right),$$

it follows that

$$\sum_{i=1}^{4} \beta_i f(\xi_i) \geq f\left(\sum_{i=1}^{4} \beta_i \xi_i\right).$$

Specifically, if $g : \mathbb{R}^3 \to \mathbb{R}$ *is given as*

$$g(a, b, \delta) \equiv \inf\left\{\sum_{i=1}^{4} \beta_i f\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}, \sum_{i=1}^{4} \beta_i (a_i, b_i, a_i b_i) = (a, b, \delta)\right\},$$

it follows that g is convex and

$$f(\xi) = g\left(\frac{|\xi^+| + |\xi^-|}{\sqrt{2}}, \frac{|\xi^+| - |\xi^-|}{\sqrt{2}}, \det(\xi)\right),$$

$$f\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = g(a, b, ab)$$

for all $\xi \in \mathbb{R}^{2\times 2}$*.*

As a result to the above characterization and Theorem 2.3, the author in [32] provide a characterization of C^1 polyconvex functions that are invariant in the case $d = 2$.

Theorem 2.16 *Consider an invariant function* $f \in C^1(\mathbb{R}^{2\times 2})$*. Then, a necessary and sufficient condition for f to be polyconvexity is that, for any* $\eta \in \mathbb{R}^{2\times 2}_D$*, there exists* $\delta(\eta) \in \mathbb{R}$ *satisfying*

$$\inf_{\xi \in \mathbb{R}^{2\times 2}_D} g(\xi, \eta, \delta(\eta)) = g(\eta; \eta, \delta(\eta)),$$

where, for $\xi \in \mathbb{R}^{2\times 2}_D$*,*

$$g(\xi, \eta, \delta(\eta)) = f(\xi) - f(\eta) - f_{\xi_{11}}(\eta)(\xi_{11} - \eta_{11}) - f_{\xi_{22}}(\eta)(\xi_{22} - \eta_{22}) - \delta(\xi_{11} - \eta_{11})(\xi_{22} - \eta_{22})$$

The following proposition provides a necessary condition to the polyconvexity for invariant functions, see [63, Proposition 4.2].

Proposition 2.3 *Let* $f : \mathbb{R}^{2\times 2} \to \mathbb{R} \cup \{\infty\}$ *be polyconvex. Then its associated signed singular value map* $g : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$ *(see 2.1 for a precise definition) satisfies the following inequality:*

For every $x, y, z \in \mathbb{G}^2$ *and* $\alpha \in [0,1]$ *that satisfies*

$$\begin{cases} x_1 + \epsilon x_2 \le (1-\alpha)(y_1 + \epsilon y_2) + \alpha(z_1 + \epsilon z_2), \\ x_1 x_2 = (1-\alpha) y_1 y_2 + \alpha z_1 z_2, \end{cases}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } (y_1 - z_1)(y_2 - z_2) \ge 0, \\ -1 & \text{if } (y_1 - z_1)(y_2 - z_2) < 0, \end{cases}$$

It follows that $g(x) \le (1-\alpha) g(y) + \alpha g(z)$.

Consider the sets $V_d = \{x \in \mathbb{R}^{++d},\ x_1 \ge \cdots \ge x_d > 0\}$,

$E(x) = \{\xi \in \mathbb{R}^{d\times d},\ \det \xi > 0 \text{ and } \ x = \lambda(\xi)\} = \{R_1(diag(x))R_2,\ R_1, R_2 \in SO(d)\}$,

and $\mathbb{PA}(\mathbb{R}^{d\times d}) = \{\varphi_\beta \ : \ \beta \in \mathbb{R}^{\widetilde{\tau}(d)}\}$ such that $\varphi_\beta(\xi) = \beta : \widetilde{M}(\xi)$ where $\widetilde{M}(\xi) \in \mathbb{R}^{\widetilde{\tau}(d)}$ represent the vector of all minors of the matrix $\xi \in \mathbb{R}^{d\times d}_+$ that include 1 as a minor of order 0 and $\widetilde{\tau}(d) = \sum_{i=0}^{d} \binom{d}{i}\binom{d}{i}$.

In [51], the author give a necessary and sufficient condition on the associated function $g : V_d \to \mathbb{R} \cup \{\infty\}$ to obtain a polyconvex function $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$, such that

$$f(\xi) = g \circ \lambda(\xi) = \begin{cases} g(\lambda(\xi)) & \text{if } \det \xi \ge 0 \\ \infty & \text{if } \det \xi < 0 \end{cases} \tag{2.2}$$

where it is assumed that $g : V_d \to \mathbb{R} \cup \{\infty\}$ is lower semi-continuous.

Note that, if f is polyconvex, the function g is called singular-value polyconvex.

Definition 2.7 *A lower semi-continuous function* $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$, *is said to be polyconvex, if*

$$f(\xi) = \sup\{\varphi(\xi) \ : \ \varphi \in \mathbb{PA}(\mathbb{R}^{d\times d}),\ \varphi \le f\}.$$

Theorem 2.17 **([51])** *Let* $f : \mathbb{R}^{d\times d} \to \mathbb{R} \cup \{\infty\}$, *be defined as in 2.2. Then, the following assertions are equivalents* *i)* *g is singular-value polyconvex;*

ii) $g(x) = \sup\{\phi(\alpha, x) : \phi(\alpha, .) \leq g\}$ *for every* $x \in V_d$

where

$$\phi(\alpha, x) = \max\{\alpha : \widetilde{M}(\xi),\ \xi \in E(x)\}$$

with $\alpha \in \mathbb{R}^{\widetilde{\tau}(d)}$

Corollary 2.2 *i) For every* $\alpha \in \mathbb{R}^{\widetilde{\tau}(d)}$ *the function* $\phi(\alpha, x) : V_d \to \mathbb{R}$ *is singular-value polyconvex.*

ii) If $g : V_d \to \mathbb{R} \cup \{\infty\}$ *is singular-value polyconvex, then for all* $y \in V_d$ *with* $g(y) < \infty$*, there exists* $\alpha \in \mathbb{R}^{\widetilde{\tau}(d)}$ *that satisfies*

$$g(x) \geq g(y) + \phi(\alpha, x) - \phi(\alpha, y).$$

Consider

$$D(x) = \{\xi \in E(x),\ \xi \text{ is diagonal matrix}\} = \{diag(Sx),\ S \in S_d\}$$

where

$$S_d = \left\{ Pdiag(\epsilon) \in O(d), P \in Perm(d), \epsilon \in \{-1, 1\}^d, \prod_{i=1}^{d} \epsilon_i = 1 \right\}$$

and define

$$\psi(\alpha, x) = \max_{\xi \in D(x)} \{\alpha : \widetilde{M}(\xi)\} = \max_{S \in S_d} \{\alpha : \widetilde{M}(diag(Sx))\} \leq \phi(\alpha, x).$$

Proposition 2.4 *If the function g as stated in 2.2 is singular-value polyconvex, then for any* $x \in V_d$*, it follows that*

$$g(x) = \sup\{\psi(\alpha^*, x),\ \psi(\alpha^*, x) \leq g\},$$

where $\alpha^* \in \mathbb{R}^{\widetilde{\tau}(d)}$ *is a vector whose components, which do not correspond to the determinants of sub-matrices symmetric to the diagonal of* ξ*, are equal to* 0*, for instance for* $d = 2$ *and* $\widetilde{M}(\xi) = (1, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}, \det \xi)$ *than* $\alpha^* = (\alpha_1, , \alpha_2, 0, 0, \alpha_5, \alpha_6)$.

Note that, by using the following property $\psi(\alpha^*, x) = \phi(\alpha^*, x)$ for all $x \in V_3$ see [51, Proposition 3.5], the author show that the above necessary condition is also a sufficient one in the case $d \leq 3$.

Theorem 2.18 *Consider* $d = 2, 3$*. Then,* *i) for every* α^* *the function* $\psi(\alpha^*, .)$ *is singular-value polyconvex;*

ii) The necessary and sufficient condition for the singular-value polyconvexity of a function $g : V_d \to \mathbb{R} \cup \{\infty\}$ *as defined in 2.2 is that, for all* $x \in V_d$*, it follows that*

$$g(x) = \sup\{\psi(\alpha^*, x),\ \psi(\alpha^*, .) \leq g\};$$

iii) For the finite case, the necessary and sufficient condition for the singular-value polyconvexity of a function $g : V_d \to \mathbb{R}$ *as defined in 2.2 is that, for all* $x, y \in V_d$*, there exists* α^* *such that*

$$g(x) \geq g(y) + \psi(\alpha^*, x) - \psi(\alpha^*, y).$$

For differentiable functions, Šilhavý [60] provides a characterization of polyconvex functions in dimension 2, and Mielke [51] extends this characterization to dimension 3.

Theorem 2.19 *Let* $f : \mathbb{R}^{d\times d} \to \mathbb{R}$*, be a differentiable function defined by its representation* $g \in C^1(V_d, \mathbb{R})$ *as in 2.2, then the following conditions are equivalent*

i) g *is singular-value polyconvex;*

ii) In the case $d = 2$*, for all* $x, y \in V_2$ *there exist* $b \in [-\frac{\partial_1 g - \partial_2 g}{y_1 - y_2}, \frac{\partial_1 g + \partial_2 g}{y_1 + y_2}]$ *such that*

$$g(x) \geq g(y) + Dg(y).(x - y) + b(x_1 - y_1)(x_2 - y_2)$$

iii) In the case $d = 3$*, for all* $x, y \in V_3$ *there exist* $a \in \mathbb{R}^3$ *and* $b \in \mathbb{R}$ *satisfying*

$$\begin{aligned} g(x) \geq g(y) + \max_{S \in S_3} \big((Dg(y) - A(a)y - b\bar{y}).(Sx) + a.(S\bar{x})\big) - Dg(y).y \\ + a.\bar{y} + b(x_1x_2x_3 + 2y_1y_2y_3) \end{aligned}$$

such that $\bar{z} = (z_2z_3, z_1z_3, z_1z_2)$ *for all* $z \in \mathbb{R}^3$*, and* $Dg(y)$ *denotes the gradient vector of* g *at* y *and the matrix* $A(a)$ *is defined as*

$$A(a) = \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{pmatrix}, \quad \textit{for every} \quad a \in \mathbb{R}^3.$$

The next example was obtained in connection with elasto-plasticity see [49, 50]

Example 2.7 *Let* $f:\mathbb{R}^{2\times 2}\to\mathbb{R}\cup\{\infty\}$ *and* $g:V_2\to\mathbb{R}\cup\{\infty\}$, *for which*

$$f(\xi)=g(\lambda(\xi)),$$

and g *is given by*

$$g(x)=g_1(x)+g_2(x_1x_2)$$

such that

$$g_1(x)=\begin{cases}\frac{2}{p}\sqrt{x_1^p(x_2^p+b)} & \textit{for}\;\; x_1\geq (x_2^p+b)^{\frac{1}{p}},\\ \frac{1}{p}(x_1^p+x_2^p+b) & \textit{for}\;\; x_1\in[x_2,(x_2^p+b)^{\frac{1}{p}}],\end{cases}$$

$p\geq 2$, *and* $g_2:\mathbb{R}^{++}\to\mathbb{R}\cup\{\infty\}$ *is a lower semi-continuous, convex function, then as a consequence of the Theorem 2.19, [51] showed that* g *generates a polyconvex function for all* $p\geq 2$.

We will now deal with functions that describe incompressible materials and we assume that $\det\xi=1$ where $\xi\in\mathbb{R}^{d\times d}$ is the deformation gradient, hence we focus on the function

$$f(\xi)=g(\lambda(\xi)).$$

The function g is determined by the function $h:\mathcal{K}^d\to\mathbb{R}\cup\{\infty\}$ as follows

$$g(x)=\begin{cases}h(x) & \text{for}\;\; \prod_{i=1}^d x_i=1,\\ \infty & \text{otherwise}\end{cases}\tag{2.3}$$

and $\mathcal{K}^d=\left\{x\in V_d:\ \prod_{i=1}^d x_i=1\right\}$. Note that, the author in [51] provides a characterization of the set $\mathcal{K}^d$ in the cases $d=2$ and $d=3$ as follows

$$\mathcal{K}^2=\left\{\left(x,\frac{1}{x}\right)^T\in V_2:\ x\in[1,\infty[\right\},$$
$$\mathcal{K}^3=\left\{\left(x_1,\frac{x_2}{x_1},\frac{1}{x_2}\right)^T\in V_3:\ x_1,x_2\in[1,\infty[,\sqrt{x_1}\leq x_2\leq x_1^2\right\}.$$

Therefore, he established the following result:

Theorem 2.20 *Let* $f:\mathbb{R}^{d\times d}\to\mathbb{R}\cup\{\infty\}$ *defined by its representation* $g:V_d\to\mathbb{R}\cup\{\infty\}$ *where* g *is defined by* $h:\mathcal{K}^d\to\mathbb{R}\cup\{\infty\}$ *as in (2.3), then, in the case* $d=2$, *consider* $\widetilde{h}:[1,\infty[\to\mathbb{R}\cup\{\infty\}$ *introduce the functions*

$$h:\mathcal{K}^d\to\mathbb{R}\cup\{\infty\}$$
$$\left(x,\frac{1}{x}\right)^T\to\widetilde{h}(x).$$

Then the following conditions are equivalent, *i)* *g is a singular-value polyconvex function;*

ii) $\widetilde{h}(x) = \sup\left\{\varphi_{\alpha,\beta}(x) = \alpha + \beta\left(x - \frac{1}{x}\right),\ \varphi_{b,\beta} \leq \widetilde{h},\ \alpha \in \mathbb{R},\ \beta \geq 0\right\};$

iii) $\widetilde{h} \circ p : [0,\infty[\to \mathbb{R} \cup \{\infty\}$ *is a non-decreasing, convex function, where p denotes the inverse of the function $x \to y = x - \frac{1}{x}$, which is defined by*

$$\begin{aligned} p :& [0,\infty[\to [1,\infty[\\ y \to& \frac{y}{2} + \sqrt{1 + \frac{y^2}{4}} \end{aligned}$$

For $d = 3$, the following equivalence holds:

i) *g is singular-value polyconvex,*

ii) $\widetilde{h}(x) = \sup\{\varphi(\alpha, a, b, x) : \varphi(\alpha, a, b, .) \leq \widetilde{h}\}$, *such that $\widetilde{h} : L \to \mathbb{R} \cup \{\infty\}$ represents the function h by*

$$\begin{aligned} h :& \mathcal{K}^3 \to \mathbb{R} \cup \{\infty\}, \\ \left(x_1, \frac{x_2}{x_1}, \frac{1}{x_2}\right)^T \to& \widetilde{h}(x) \end{aligned}$$

and

$$\varphi(\alpha, a, b, x) = \alpha + \max_{S \in S_3}\left(Sa.\bar{x} + Sb.\widetilde{x}\right),$$

where $\alpha \in \mathbb{R},\ a, b \in \mathbb{R}^3,\ \bar{x} = \left(x_1, \frac{x_2}{x_1}, \frac{1}{x_2}\right)^T,\ \widetilde{x} = \left(\frac{1}{x_1}, \frac{x_1}{x_2}, x_2\right)^T$ and $x \in L = \left\{[1,\infty)^2 :\ \sqrt{x_1} \leq x_2 \leq x_1^2\right\}$.

2.5.2 Characterization of fully invariant functions

The characterization of fully invariant functions is presented, providing a sufficient condition for obtaining a polyconvex function.

Consider the sets,

$$\begin{aligned} \mathbb{R}^d_+ &= \{x \in \mathbb{R}^d :\ x_i \geq 0,\ i = 1, \ldots, d\}, \\ K^d_+ &= \{x \in \mathbb{R}^d :\ 0 \leq x_1 \leq \cdots \leq x_d\}. \end{aligned}$$

Based on the notations in [19, Section 5.39, p. 202], we have, for $\xi \in \mathbb{R}^{\binom{d}{s}\times\binom{d}{s}}$ where $1 \leq s \leq d-1$, denote the singular value of ξ by $\Lambda^s(\xi) \in K_+^{\binom{d}{s}}$

For every $x \in K_+^d$ and $2 \leq s \leq d$, we let $M_s x \in K_+^{\binom{d}{s}}$ be the vector in $\mathbb{R}^{\binom{d}{s}}$ composed of each product $x_{i_1} \dots x_{i_s}$, arranged in increasing order where $i_1 < \dots < i_s$.

Note that for every $1 \leq s \leq d$ and every $\xi \in \mathbb{R}^{d\times d}$

$$\Lambda^s(M_s(\xi)) = M_s \Lambda^1(\xi).$$

Theorem 2.21 *Let $0 \leq \lambda_1(\xi) \leq \dots \leq \lambda_d(\xi)$ be the singular value of $\xi \in \mathbb{R}^{d\times d}$, and consider the functions $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ and $g : \mathbb{R}_+^d \to \mathbb{R}$, where*

$$f(\xi) = g(\lambda_1(\xi), \dots, \lambda_d(\xi)).$$

- *Suppose the existence of a function*

  $$\begin{aligned} G :&\mathbb{R}_+^d \times \mathbb{R}_+^{\binom{d}{2}} \times \dots \times \mathbb{R}_+^{\binom{d}{d-1}} \times \mathbb{R}_+ \to \mathbb{R} \\ G(z) &= G(z^1, z^2, \dots, z^{d-1}, z^d) \end{aligned}$$

  which is convex, non-decreasing in every variable, and symmetric in each variable separately i.e., for every permutation P_i of $\binom{d}{i}$ elements

  $$G(P_1\Lambda^1, P_2\Lambda^2, \dots, P_{d-1}\Lambda^{d-1}, P_d\Lambda^d) = G(\Lambda^1, \Lambda^2, \dots, \Lambda^{d-1}, \Lambda^d)$$

  and for which

  $$g(x) = G(x, M_2 x, \dots, M_{d-1}x, M_d x).$$

  Then f is polyconvex.

As a consequence of the above theorem, the function

$$f_k(\xi) = \prod_{i=k}^{d} \lambda_i(\xi),$$

is polyconvex for all $k = 1, \dots, d$. See [19].

2.5.3 Objective, Isotropic & Isochoric Functions

Definition 2.8 *Let $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$, f is said to be*

i) objective-isotropic if,

$$f(Q\xi R) = f(\xi), \quad \text{for every } \xi \in \mathbb{R}_+^{2\times 2} \quad \text{and every } Q,\ R \in SO(2).$$

ii) isochoric if,

$$f(\alpha\xi) = f(\xi), \quad \textit{for all } \alpha \in \mathbb{R}^{++}.$$

Note that, the extended definition of isochoric functions to the hole matrix space $\mathbb{R}^{2\times 2}$, is given by,

$$\begin{cases} f(\xi) & \det \xi > 0, \\ \infty & \text{otherwise.} \end{cases}$$

Refer to [31, 46].

Proposition 2.5 ([46]) *Let $f : \mathbb{R}^{2\times 2}_{+} \to \mathbb{R}$. Then, a necessary and sufficient condition for f to be objective, isochoric, and isotropic is*

$$f(Q\xi R) = f(\xi), \qquad \textit{for all } Q, R \in CSO(2)$$

with $CSO(2) = \{\alpha Q : Q \in SO(2),\ \alpha \in \mathbb{R}^{++}\}$.

The following lemma is an auxiliary result used to prove the equivalence between rank-one convexity and polyconvexity for isochoric functions, as demonstrated in [46].

Lemma 2.5 *The function $f : \mathbb{R}^{2\times 2}_{+} \to [1, \infty)$ such that*

$$f(\xi) = \frac{||\xi||^2_{\text{op}}}{\det \xi},$$

is polyconvex on $\mathbb{R}^{2\times 2}_{+}$. In addition, the function f can be expressed as

$$f(\xi) = h(\lambda_1(\xi), \lambda_2(\xi)) = \frac{\max\{\lambda_1^2(\xi), \lambda_2^2(\xi)\}}{\lambda_1(\xi)\lambda_2(\xi)}$$

in which $\lambda_i(\xi)$ stands for the singular values of the matrix ξ, and $||\xi||_{\text{op}} = \max\{\lambda_1(\xi), \lambda_2(\xi)\}$ denotes the spectral norm of $\xi \in \mathbb{R}^{2\times 2}_{+}$.

The following proposition, derived as a consequence of Lemmas 2.3 and 2.5, is considered in [46] the main tool for demonstrating that an isochoric rank-one convex function is polyconvex.

Proposition 2.6 *Let $f : \mathbb{R}^{2\times 2}_{+} \to \mathbb{R}$. If a non-decreasing convex function $g : [1, \infty) \to \mathbb{R}$ exists such that*

$$f = g \circ h,$$

with $h(\xi) = \frac{||\xi||^2_{\text{op}}}{\det \xi}$, then f is polyconvex.

Furthermore, characterization of the above notions in terms of singular values representation is given in [46, Lemma 3.1],

Lemma 2.6 *For an objective, isochoric, and isotropic function* $f : \mathbb{R}^{2\times 2}_{+} \to \mathbb{R}$*, a unique function* $g : \mathbb{R}^{++} \to \mathbb{R}$ *exists, satisfying* $g(x) = g(\frac{1}{x})$*, where*

$$f(\xi) = g\left(\frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right) \qquad \textit{for all } \xi \in \mathbb{R}^{2\times 2}_{+},$$

such that $\lambda_1(\xi), \lambda_2(\xi) \in \mathbb{R}^{++}$ *represent the singular values of the matrix* ξ*. If* f *is only objective-isotropic, then a unique symmetric function* $g : \mathbb{R}^{++} \times \mathbb{R}^{++} \to \mathbb{R}$ *exists such that*

$$f(\xi) = g(\lambda_1(\xi), \lambda_2(\xi)) \qquad \textit{for all } \xi \in \mathbb{R}^{2\times 2}_{+}.$$

The equivalence between rank-one convex functions and polyconvex functions, assumed to be objective, isochoric, and isotropic, is established in [46].

Theorem 2.22 *Let* $h : \mathbb{R}^{++} \to \mathbb{R}$*,* $g : \mathbb{R}^{++} \times \mathbb{R}^{++} \to \mathbb{R}$ *be the unique singular value representation function that define the objective, isochoric, and isotropic function* $f : \mathbb{R}^{2\times 2}_{+} \to \mathbb{R}$*, where*

$$f(\xi) = g(\lambda_1(\xi), \lambda_2(\xi)) = h\left(\frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right) = h\left(\frac{\lambda_2(\xi)}{\lambda_1(\xi)}\right) \qquad \textit{for all } \xi \in \mathbb{R}^{2\times 2}_{+}.$$

Then, the following conditions are equivalent:

- *i)* f *is polyconvex;*
- *ii)* f *is rank one convex;*
- *iii)* g *is separately convex;* [2]
- *iv)* h *is convex on* $\mathbb{R}^{++}$;
- *v)* h *is non decreasing convex function on* $[1, \infty)$.

In the case of a continuously differentiable condition on h, the following corollary is derived from the above theorem; see [46].

Corollary 2.3 *Let* $f : \mathbb{R}^{2\times 2}_{+} \to \mathbb{R}$ *be an objective, isochoric, and isotropic function, and let* $h : \mathbb{R}^{++} \to \mathbb{R}$ *be its unique singular values representation such that*

$$f(\xi) = h\left(\frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right) \qquad \textit{for all } \xi \in \mathbb{R}^{2\times 2}_{+}.$$

Hence, if $h \in C^1(\mathbb{R}^{++})$*, the polyconvexity of* f *is equivalent to the convexity of* h *on* $[1, \infty)$.

---

[2] i.e., the functions $a \to g(a, b)$ and $b \to g(a, b)$ are convex; see [19].

In the case of C^2 functions, we have the following:

Theorem 2.23 *Consider $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$, be defined as an objective, isochoric and isotropic function of class C^2, and let $g, h : [0, \infty) \to \mathbb{R}$ be the unique functions such that*

$$f(\xi) = g(|\operatorname{dev}_2 \log U|^2) = h\left(\log^2 \frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right) \qquad \text{for all } \xi \in \mathbb{R}_+^{2\times 2}.$$

If $g, h \in C^2([0, \infty))$, then the following conditions are equivalent,

- *i) f is polyconvex,*
- *ii) f is rank one convex,*
- *iii) $2xh^{''}(x) + (1 - \sqrt{x})h^{'}(x) \geq 0$ for every $x \in (0, \infty)$,*
- *iv) $2yg^{''}(y) + (1 - \sqrt{2y})g^{'}(y) \geq 0$ for every $y \in (0, \infty)$.*

Where, $\operatorname{dev}_2 \xi = \xi - \frac{1}{2}\operatorname{tr}(\xi).\mathbb{I}$, log represent the principal matrix logarithm, and $U = \sqrt{\xi^T \xi}$.

Corollary 2.4 *Under the assumptions of theorem 2.23, if f is polyconvex (which is rank one convex), then $h^{'}(x) \geq 0$ for all $x > 0$.*

In the class of distortion function, We also have following propostion, see [46].

Proposition 2.7 *Consider $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$, be defined as an objective, isochoric and isotropic function and let $g : [1, \infty) \to \mathbb{R}$ be the unique determined function such that*

$$f(\xi) = g\left(\frac{|\xi|^2}{2\det \xi}\right) \qquad \text{for all } \xi \in \mathbb{R}_+^{2\times 2}.$$

If $g \in C^2([0, \infty))$, then the following conditions are equivalent,

- *i) f is polyconvex;*
- *ii) f is rank one convex;*
- *iii) $(x^2 - 1)(x + \sqrt{x^2 - 1})g^{''}(x) + g^{'}(x) \geq 0$ for all $x \in (1, \infty)$.*

The next following example was ed by [46].

Corollary 2.5 *i) The isochoric Hencky energy $|\operatorname{dev}_2 \log U|^2 = \frac{1}{2}\log^2\left(\frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right)$ is non-polyconvex and non-rank one convex function on $\mathbb{R}_+^{2\times 2}$.*

ii) The exponentiated isochoric Hencky energy function

$$\exp(k|\operatorname{dev}_2 \log U|^2) = \exp\left(k\left|\log \frac{U}{\det U^{\frac{1}{2}}}\right|^2\right) = \exp\left(\frac{k}{2}\log^2 \frac{\lambda_1(\xi)}{\lambda_2(\xi)}\right)$$

is rank one convex function (and hence polyconvex) on $\mathbb{R}_+^{2\times 2}$ *if and only if* $k \geq \frac{1}{4}$.

In which $U = \sqrt{\xi^T \xi}$, *and* $\lambda_1(\xi), \lambda_2(\xi)$ *represent the singular values of* $\xi \in \mathbb{R}_+^{2\times 2}$.

The following result provided the equivalence between polyconvexity and rank-one convexity for an objective-isotropic function defined on the set $SL(2)$, as stated in [27].

Theorem 2.24 *Consider* $f : SL(2) \to \mathbb{R}$ *be defined as an objective and isotropic function, hence the next assertions are equivalent,*

i) f *is polyconvex,*

ii) f *is rank one convex,*

iii) the function $g : \mathbb{R} \to \mathbb{R}, \quad g(x) = f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)$ *is convex,*

iv) the function $h : [0, \infty) \to \mathbb{R}$ *such that* $f(\xi) = h(\sqrt{|\xi|^2 - 2}) = h\left(\lambda_{\max}(\xi) - \frac{1}{\lambda_{\max}(\xi)}\right)$ *is convex and non-decreasing.*

where $\lambda_{\max}(\xi)$ *denotes the largest singular value of* ξ.

In the case of differentiable functions, equivalence between rank one convexity and polyconvexity is treated in [27].

Proposition 2.8 *Let* $f : SL(2) \to \mathbb{R}$ *be a differentiable, objective and isotropic function. The following are equivalent,*

i) f *is polyconvex,*

ii) f *is rank one convex,*

iii) The function $h : [0, \infty) \to \mathbb{R}$ *with* $f(\xi) = h\left(\lambda_{\max}(\xi) - \frac{1}{\lambda_{\max}(\xi)}\right)$ *is non decreasing and convex.*

Note that the function $f : SL(2) \to \mathbb{R}$, $f(\xi) = \lambda_{\max}(\xi) - \frac{1}{\lambda_{\max}(\xi)}$ is polyconvex on $SL(2)$, see [27].

Proposition 2.9 *([27, Propostion 4.2])*

Consider $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$ *defined as an objective, isochoric, and isotropic function. Let* $g : SL(2) \to \mathbb{R}$ *characterize the function* f *as stated in [46], defined as*

$$f(\xi) = g\left(\frac{\xi}{(\det \xi)^{\frac{1}{2}}}\right).$$

Then the following statement hold,

i) The rank-one convexity (equivalently, polyconvexity) of f *on* $\mathbb{R}_+^{2\times 2}$ *is a sufficient condition for the rank-one convexity (equivalently, polyconvexity) of* g *on* $SL(2)$*,*

ii) The reverse implication does not hold in general, as shown with the counterexample [27], Let $f : \mathbb{R}_+^{2\times 2} \to \mathbb{R}$ *given by*

$$f(\xi) = \left|\sqrt{\frac{\lambda_1(\xi)}{\lambda_2(\xi)}} - \sqrt{\frac{\lambda_2(\xi)}{\lambda_1(\xi)}}\right| \quad \xi \in \mathbb{R}_+^{2\times 2},$$

with $\lambda_1(\xi), \lambda_2(\xi) \in \mathbb{R}^{++}$ *represent the singular values of* $\xi \in \mathbb{R}_+^{2\times 2}$*. Then* f *is not a rank-one convex function on* $\mathbb{R}_+^{2\times 2}$*, whereas it is objective, isochoric and isotropic on* $\mathbb{R}_+^{2\times 2}$*, and its restriction on* $SL(2)$ *is polyconvex and rank one convex on* $SL(2)$*, see [27, Remark 4.1].*

As an example of isochoric function that is polyconvex; see [31].

Lemma 2.7 *The isochoric function* $f : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$*, given as*

$$f(\xi) = \begin{cases} \frac{|\xi|^2}{(\det \xi)^{\frac{2}{3}}}, & \det \xi > 0, \\ \infty & \det \xi \leq 0, \end{cases}$$

is polyconvex function.

Numerous other functions in the general case of the isochoric class, which are shown in [31] to be polyconvex, are presented in the following proposition.

Proposition 2.10 *Let* $\xi \in \mathbb{R}^{3\times 3}$*, and* $p, q \geq 1$*. The following functions, in the case* $\det \xi > 0$*, are defined by:*

i) $\xi \to \left(\frac{|\xi|^{2q}}{(\det \xi)^{\frac{2q}{3}}} - 3^q\right)^p.$

ii) $\xi \to \left(\frac{|\operatorname{adj} \xi|^{3q}}{(\det \xi)^{2q}} - (3\sqrt{3})^q\right)^p.$

iii) $\xi \to \exp\left(\left(\frac{|\xi|^{2q}}{(\det \xi)^{\frac{2q}{3}}} - 3^q\right)^p\right) - 1.$

iv) $\xi \to \exp\left(\left(\frac{|\operatorname{adj} \xi|^{3q}}{(\det \xi)^{2q}} - (3\sqrt{3})^q\right)^p\right) - 1.$

In the case $\det \xi \leq 0$, *each of the functions tend to infinity. Then, for all* $p, q \geq 1$, *these functions are polyconvex.*

However, following functions are not polyconvex; see [31, Lemma 2.4].

Proposition 2.11 *For all* $k, l \geq 1$, *let the functions* $f_1, f_2 : \mathbb{R}^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ *defined in the case* $\det \xi > 0$, *as*

$$f_1(\xi) = \left(\frac{|\xi|^2}{(\det \xi)^{\frac{2}{3}}} - 3\right)^k \left(\frac{|\operatorname{adj} \xi|^3}{(\det \xi)^2} - 3\sqrt{3}\right)^l$$

$$f_2(\xi) = \left(\frac{|\operatorname{adj} \xi|^2}{(\det \xi)^{\frac{4}{3}}} - 3\right)^k$$

and $f_1, f_2 \longrightarrow \infty$ *as* $\det \xi \leq 0$. *Then the two functions* f_1, f_2 *are not polyconvex.*

For

2.6 Applications

In this section, we present a set of stored energy functions that describe the behavior of materials, such as Saint Venant-Kirchhoff, Ogden's materials, and Mooney-Rivlin materials. Following [15], we drive the next different stored energy functions.

2.6.1 Ogden Material

The stored energy function of Ogden material is represent in the following form,

$$f(\xi) = h(\det \xi) + \sum_{i=1}^{k} a_i \operatorname{tr}(\xi^T \xi)^{\frac{\alpha_i}{2}} + \sum_{j=1}^{l} b_i \operatorname{tr}(\operatorname{cof}(\xi^T \xi))^{\frac{\beta_i}{2}}.$$

where

i) $h : (0, \infty) \to \mathbb{R}$ is a convex function satisfied $\lim_{x\to 0^+} h(x) = +\infty$.

ii) $\alpha_i, \beta_j \geq 1,\ a_i, b_j > 0$ for every $1 \leq i, j \leq k, l$

By using the formula,

$$\begin{aligned}
&\operatorname{tr}(\xi^T\xi)^{\frac{\alpha}{2}} = \lambda_1^\alpha + \lambda_2^\alpha + \lambda_3^\alpha,\\
&\operatorname{tr}(\operatorname{cof}(\xi^T\xi))^{\frac{\beta}{2}} = (\lambda_2\lambda_3)^\beta + (\lambda_1\lambda_3)^\beta + (\lambda_2\lambda_1)^\beta,
\end{aligned}$$

with $\lambda_i = \lambda_i(\xi)$ represent the singular values of the matrix ξ, hence, the stored energy of Ogden material, becomes as in the following theorem

Theorem 2.25 *Let* $f : \mathbb{R}_+^{3\times 3} \to \mathbb{R}$, *be defined as*

$$f(\xi) = h(\det \xi) + \sum_{i=1}^{k} a_i(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i}) + \sum_{j=1}^{l} b_i((\lambda_2\lambda_3)^{\beta_i} + (\lambda_1\lambda_3)^{\beta_i} + (\lambda_2\lambda_1)^{\beta_i}),$$

where f *satisfied the two proposal i) and ii) without the condition* $\lim_{x\to 0^+} h(x) = +\infty$ *in the above assumption. Then the function* f *is polyconvex.*

2.6.2 Saint Venant-Kirchhoff material

The stored energy function of the St Venant-Krichhoff material is presented in the following form

$$f(\xi) = \mu \operatorname{tr} E^2 + \frac{\lambda}{2}(\operatorname{tr} E)^2,\ \mathbb{I} + 2E = \xi^T\xi.$$

It has an equivalent form, given as

$$f(\xi) = -\left(\frac{3\lambda + 2\mu}{4}\right)\operatorname{tr}(\xi^T\xi) + \left(\frac{\lambda + 2\mu}{8}\right)\operatorname{tr}(\xi^T\xi)^2 + \frac{\lambda}{4}\operatorname{tr}\operatorname{cof}(\xi^T\xi) + \left(\frac{9\lambda + 6\mu}{8}\right),$$

such that λ and μ represent two positive constants called Lamé's coefficients. It is consider as a particular class of Ogden material.

The non-polyconvexity on the set $\mathbb{R}_+^{3\times 3}$ [3] of the stored energy function of the St Venant-Krichhoff material is proved by the counterexample, that is presented in [54].

Theorem 2.26 *Let* $f : \mathbb{R}_+^{3\times 3} \to \mathbb{R}$ *be given by*

$$f(\xi) = a \operatorname{tr}(\xi^T\xi) + b \operatorname{tr}(\xi^T\xi)^2 + c \operatorname{tr}(\operatorname{cof}(\xi^T\xi)),$$

where the coefficients $b, c > 0$. *Then, if* $a < 0$, *it follows that the function* f *is not polyconvex.*

---

[3] The polyconvexity on the set $\mathbb{R}_+^{3\times 3}$ is defined by the existence of a convex function $g : \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R}^{++} \to \mathbb{R}$ such that $f(\xi) = g(\xi, \operatorname{cof}\xi, \det \xi)$ for every $\xi \in \mathbb{R}_+^{3\times 3}$, refer to [7, 54]

Note that the author in [54] provides a direct proof by considering the two matrices $\xi_1, \xi_2 \in \mathbb{R}^{3\times 3}_{+}$ and considering the positive number α for which

$$\xi_1 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 3\alpha \end{pmatrix}.$$

Then, under the polyconvexity assumption of f, it follows that

$$f\left(\frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\right) \leq \frac{1}{2}f(\xi_1) + \frac{1}{2}f(\xi_2).$$

Hence, straightforward computations lead to the fact that the last inequality is not valid for some α.

Chapter 3

Symmetric Polyconvexity: Characterization in Higher Dimensions

The purpose of this chapter is to describe the notion of symmetric polyconvexity within higher dimensions. In [11] the authors have obtained a complete characterization in the 2d and 3d cases, respectively. Symmetric polyconvexity is crucial in establishing the existence of minimizers for certain nonlinear elasticity problems within the calculus of variations, using the direct methods in calculus of variations, which are also known as the methods of weak lower semicontinuity.

This notion is a special case of polyconvexity, defined exclusively for symmetric matrices, so that the function combined with the symmetric part of the matrix is polyconvex.

Definition 3.1 *Consider $f : \mathcal{S}^{d\times d} \to \mathbb{R}$, we said that f is symmetric polyconvex, if the function $\widetilde{f} : \mathbb{R}^{d\times d} \to \mathbb{R}$, such that*

$$\widetilde{f}(\xi) = f(\xi^s)$$

is polyconvex.

Analogous definitions can be given for symmetric quasiconvex and symmetric rank-one convex functions.

3.1 Symmetric Polyconvexity in 2 and 3 Dimensions

We present in this section the characterization of the symmetric polyconvex function in 2 and 3 dimensions included in [11], where they prove the following theorem; see [11, Theorem 4.1 and 5.1]

Theorem 3.1 *Consider $f : \mathcal{S}^{d\times d} \to \mathbb{R}$. Then, it follows that,*

i) If $d = 2$, then the necessary and sufficient condition for the function f to be symmetric polyconvex is the existence of a convex function $g : \mathcal{S}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$, where, g has a non-increasing behavior with respect to the second variable, and

$$f(\xi) = g(\xi, det(\xi)) \text{ for every } \xi \in \mathcal{S}^{2\times 2}.$$

ii) If $d = 3$, then the necessary and sufficient condition for the symmetric polyconvexity of the function f is the existence of a convex function $g : \mathcal{S}^{3\times 3} \times \mathcal{S}^{3\times 3} \to \mathbb{R}$, where $\partial_2 g(\varepsilon, \eta) \subset \mathcal{S}^{3\times 3}_-$ for all $\epsilon, \eta \in \mathcal{S}^{3\times 3}$, and

$$f(\xi) = g(\xi, cof(\xi)) \text{ for every } \xi \in \mathcal{S}^{3\times 3}.$$

A significant result derived from this theorem is presented as follows:

Theorem 3.2 *Consider $f : \mathcal{S}^{3\times 3} \to \mathbb{R}$, the following hold:*

i) If $f(\xi) = h(cof\xi)$ for all $\xi \in \mathcal{S}^{3\times 3}$, where $h : \mathcal{S}^{3\times 3} \to \mathbb{R}$ is a continuous function, then the symmetric polyconvexity of f is equivalent to the convexity of the function h with $\partial_2 h(\varepsilon) \subset \mathcal{S}^{3\times 3}_-$ for every $\epsilon \in \mathcal{S}^{3\times 3}$.

ii) If $f(\xi) = h(det\xi)$ for all $\xi \in \mathcal{S}^{3\times 3}$, where $h : \mathbb{R} \to \mathbb{R}$, then the symmetric polyconvexity of f is equivalent to h being a constant function.

In the case of quadratic functions, Boussaid and al., [11], gave the characterization of symmetric polyconvex quadratic form in 2d and 3d cases, see [11, Proposition 4.5 and Proposition 5.6].

Theorem 3.3 *Let $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ define a quadratic form. Then, the following conditions are equivalent:*

i) f is a symmetric polyconvex function;

ii) In $d = 2$, there exists $\alpha \geq 0$ satisfying

$$f(\xi) + \alpha \det \xi \geq 0 \text{ for every } \xi \in \mathcal{S}^{2\times 2},$$

with $\det \xi$ representing the determinant of ξ.

iii) In $d = 3$, a positive semi-definite matrix $A \in \mathcal{S}^{3\times 3}$ exists, satisfying

$$f(\xi) + A : \operatorname{cof} \xi \geq 0 \text{ for every } \xi \in \mathcal{S}^{3\times 3},$$

where $\operatorname{cof} \xi$ denotes the cofactor matrix of $\xi \in \mathcal{S}^{3\times 3}$.

We conclude this section by presenting the characterization for $d = 2$ and $d = 3$ of the symmetric polyaffine function $f : \mathcal{S}^{d\times d} \to \mathbb{R}$, i.e., f and $-f$ are symmetric polyconvex; see [11].

Theorem 3.4 *Consider $f : \mathcal{S}^{d\times d} \to \mathbb{R}$. Hence, the symmetric polyaffinity of f is equivalent to its affinity.*

3.2 Symmetric Polyconvexity in Higher Dimensions

Our main result in this section is the following:

Theorem 3.5 (Characterization of symmetric polyconvexity in any dimension) *A function* $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ *is symmetric polyconvex if and only if, a convex function* $g : \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}} \to \mathbb{R}$ *exists, where* $\partial_2 g(\varepsilon,\eta) \subset \mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$ *and*

$$f(\varepsilon) = g(\varepsilon, M_2(\varepsilon)),\ \forall \varepsilon \in \mathcal{S}^{d\times d}. \tag{3.1}$$

Proof.

1. Sufficiency:

Let f and g be two functions as in Theorem3.5 such that (3.1) is satisfied. We want to show that the function $\widetilde{f}(F) = g(F^s, M_2(F^s))$ is polyconvex. According to [19], we will show first the existence of a convex function $k : \mathbb{R}^{d\times d} \times \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}} \times \cdots \times \mathbb{R} \to \mathbb{R}$ that satisfies $\widetilde{f}(F) \geq k(M(F))$, with $M(F)$ representing the vector containing all minors of the matrix F, see [19]. As the function g is convex and $\partial_2 g(\varepsilon,\eta) \subset \mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$, we can find two matrices $(B, \tilde{B})$ such that, $\tilde{B}$ is S-positive semi-definite and

$$\begin{aligned}
\widetilde{f}(F) = g(F^s, M_2(F^s)) \ &\geq g(0,0) + B : F^s - \tilde{B} : M_2(F^s)\\
&= g(0,0) + B : F^s - \tilde{B} : (M_2(F))^s + \tilde{B} : M_2(F^a)\\
&= k(M(F))
\end{aligned}$$

Owing to the convexity of the function $F \to \tilde{B} : M_2(F^a)$, see Lemma 1.4, we conclude that k is convex.

Let $F_i \in \mathbb{R}^{d\times d}$ be matrices and $\lambda_i \in [0,1]$ reals such that $\sum_{i=1}^{\tau(d)+1} \lambda_i = 1$, where $\tau(d) = \sum_{k=1}^{d} \binom{d}{k}^2$. Assume that the matrix $F = \sum_{i=1}^{\tau(d)+1} \lambda_i F_i$ satisfy $M(\sum_{i=1}^{\tau(d)+1} \lambda_i F_i) = \sum_{i=1}^{\tau(d)+1} \lambda_i M(F_i)$. We will show that $\widetilde{f}(\sum_{i=1}^{\tau(d)+1} \lambda_i F_i) \leq \sum_{i=1}^{\tau(d)+1} \lambda_i \widetilde{f}(F_i)$. Remark first, since g is convex and its partial sub-differential with respect to the second variable satisfies $\partial_2 g(\varepsilon,\eta) \subset \mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$, for any $\varepsilon \in \mathcal{S}^{d\times d}$ and $\eta \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, there exist an S-negative semi-definite matrix $\tilde{B}$ in $\partial_2 g(F^s, M_2(F^s))$ that satisfies

$$\begin{aligned}
g\left(F^s, M_2(F^s) + M_2(F^a) - \textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i M_2(F_i^a)\right) \ &\geq g(F^s, M_2(F^s))\\
&\quad + \tilde{B} : \left(M_2(F^a) - \textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i M_2(F_i^a)\right)\\
&\geq g(F^s, M_2(F^s))
\end{aligned} \tag{3.2}$$

Note that in the latter inequality, we utilized the concavity of the form $q_{\tilde{B}}$, (see Lemma 1.4).

Consequently, thanks to (1.2), (3.2) and the convexity of g, we get:

$$\begin{aligned}
\widetilde{f}(F) = g(F^s, M_2(F^s)) &\leq g\left(F^s, M_2(F^s) + M_2(F^a) - \textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i M_2(F_i^a)\right) \\
&= g(F^s, (M_2(F))^s - \textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i M_2(F_i^a)) \\
&= g(\textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i F_i^s, \sum_{i=1}^{\tau(d)+1} \lambda_i((M_2(F_i))^s - M_2(F_i^a))) \\
&= g(\textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i F_i^s, \sum_{i=1}^{\tau(d)+1} \lambda_i M_2(F_i^s)) \\
&\leq \textstyle\sum_{i=1}^{\tau(d)+1} \lambda_i g(F_i^s, M_2(F_i^s)) = \sum_{i=1}^{\tau(d)+1} \lambda_i \widetilde{f}(F_i),
\end{aligned}$$

which is the desired result.

1. Necessity:

We turn, now our attention to the second part of the theorem which is the most relevant here. We will show through the different steps that if a function f defined on the space of $d \times d$ symmetric matrices, is symmetric polyconvex, then there exist a convex function g defined on the space $\mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, such that $\partial_2 g(\varepsilon, \eta) \subset \mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$, for every $(\varepsilon, \eta) \in \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, and $f(\varepsilon) = g(\varepsilon, M_2(\varepsilon))$, for all $\varepsilon \in \mathcal{S}^{d\times d}$. Since f is symmetric polyconvex, the associated function $\widetilde{f}$ defined on $\mathbb{R}^{d\times d}$ is polyconvex. There exist then a convex function $\widetilde{g}$ defined on $\mathbb{R}^{d\times d} \times \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}} \times \cdots \times \mathbb{R}^{\binom{d}{d}\times\binom{d}{d}}$, such that for every matrix $F \in \mathbb{R}^{d\times d}$ we have $\widetilde{f}(F) = \widetilde{g}(M(F))$. We denote by g the restriction of $\widetilde{g}$ to the space of symmetric matrices in all the variables, except the last one which is real and correspond to the determinant. The function g is then clearly convex and $f(\varepsilon) = g(M(\varepsilon))$. We will show in the sequel that the function g is dependent only on the matrix and the second order minors and does not depend on higher order ones. We divide the proof into two main parts, in the first part we prove that the function g is independent on the minors of order greater or equal than 3. In the second part we focus on showing that the function partial sub-differential of g is such that $\partial_2 g(\varepsilon, \eta) \subset \mathcal{S}_{s^-}^{\binom{d}{2}\times\binom{d}{2}}$, for any $(\varepsilon, \eta) \in \mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$.

Part 1. Independence with respect to p-order minors

In this part we will show the independence of g on the minors of order greater or equal than 3. Our strategy is based on showing first the independence of g on all m-diagonal $p \times p$ minors, for $2 \leq m \leq p$. The cases $m = 0, 1$ are treated by induction on the order of the minor.

Step1 : m-diagonal minors, $2 \leq m \leq p$.

We start by showing that for any $3 \leq p \leq d$, the function g is constant in all $p \times p$ m-diagonal minors, for $2 \leq m \leq p$. To do so, we prove that g is independent on any arbitrary m-diagonal minor. Let $3 \leq p \leq d$ any integer, we denote by M_p the matrix of all $p \times p$ minors. For $I = (i_1, i_2, \dots, i_p)$ and $J = (j_1, j_2, \dots, j_p)$ two arbitrary p-tuples such that I and J have at least two common indices, the minor $(M_p)_{IJ}$ obtained by taking the lines i_k and the columns $j_k, k = 1, 2, \dots, p$, is then m-diagonal, we will show that the function g does not depend on the minor $(M_p)_{IJ}$. Suppose that $1 \leq i_1 < i_2 < \cdots < i_p \leq d$, and $1 \leq j_1 < j_2 < \cdots < j_p \leq d$. Without loss of generality, we can suppose that $i_1 = j_1$ and $i_2 = j_2$. The other possibilities are treated similarly by changing the matrix F_m below.

Next we will choose an appropriate $d \times d$ matrix which give us useful informations on the sub-gradients of g. Consider the matrix F_m defined as follows

$$(F_m)_{i_1 j_2} = \sqrt{t},\ (F_m)_{i_2 j_1} = -\sqrt{t}, (F_m)_{i_3 j_3} = \beta,\ \text{and for } 4 \leq k \leq p,\ (F_m)_{i_k j_k} = \alpha$$

Where t and α are positive reals and β is any real parameter. All the other remaining elements of F_m vanish. The only non-zero rows and columns of F_m can be represented as follows:

$$\begin{array}{c|ccccccc|}
 & j_1 & j_2 & j_3 & j_4 & \cdots & \cdots & j_p \\
i_1 & 0 & \sqrt{t} & 0 & 0 & 0 & \cdots & 0 \\
i_2 & -\sqrt{t} & 0 & 0 & 0 & 0 & \cdots & 0 \\
i_3 & 0 & 0 & \beta & 0 & 0 & & \vdots \\
i_4 & 0 & 0 & 0 & \alpha & 0 & & \vdots \\
\vdots & 0 & 0 & 0 & 0 & \alpha & & \\
\vdots & \vdots & \vdots & & & & \ddots & 0 \\
i_p & 0 & 0 & \cdots & \cdots & & 0 & \alpha
\end{array}$$

By a straightforward computation we get that the different minors of F_m satisfy:

$$\begin{cases}
F_m^s = \alpha A_0 + \beta A_1,\ A_0, A_1 \in \mathcal{S}^{d \times d} \\
(M_2(F_m))^s = tA_2 + \alpha^2 A_3 + \alpha\beta A_4,\quad A_2, A_3, A_4 \text{ are constants matrices in } \mathcal{S}^{\binom{d}{2} \times \binom{d}{2}}, \\
(M_k(F_m))^s = \alpha^k \tilde{A}_1 + \beta\alpha^{k-1}\tilde{A}_2 + \alpha^{k-1}\sqrt{t}\tilde{A}_3 + \beta\alpha^{k-2}\sqrt{t}\tilde{A}_4 + t\alpha^{k-2}\tilde{A}_5 + \beta\alpha^{k-3}t\tilde{A}_6, \\
\ 3 \leq k \leq p-3, \text{and } \tilde{A}_i, i = 1, 2, \dots, 6,\ \text{are constants matrices in } \mathcal{S}^{\binom{d}{k} \times \binom{d}{k}}, \\
(M_{p-2}(F_m))^s = \beta\alpha^{p-3}\bar{A}_1 + \alpha^{p-3}\sqrt{t}\bar{A}_2 + \beta\alpha^{p-4}\sqrt{t}\bar{A}_3 + t\alpha^{p-4}\bar{A}_4 + t\beta\alpha^{p-5}\bar{A}_5, \\
\bar{A}_i, i = 1, 2, \dots, 5,\ \text{are constants matrices in } \mathcal{S}^{\binom{d}{p-2} \times \binom{d}{p-2}}, \\
(M_{p-1}(F_m))^s = t\alpha^{p-3}B_1 + \beta\alpha^{p-3}\sqrt{t}B_2 + t\beta\alpha^{p-4}B_3 \\
B_1, B_2, B_3\ \text{ are constants matrices in } \mathcal{S}^{\binom{d}{p-1} \times \binom{d}{p-1}}, \\
(M_p(F_m))_{IJ} = t\beta\alpha^{p-3},\ \text{and } (M_p(F_m))_{KL} = 0 \text{ if } K \neq I \text{ or } L \neq J \\
M_k(F_m) = 0 \text{ for all } k \geq p+1.
\end{cases}$$

Due to the convexity of $\tilde{g}$, combined with the invariance property of f and the structure of the matrix F_m, we can write

$$\begin{aligned} f(F_m^s) &= \tfrac{1}{2}\tilde{f}(F_m) + \tfrac{1}{2}\tilde{f}(F_m^T) = \tfrac{1}{2}\tilde{g}(M(F_m)) + \tfrac{1}{2}\tilde{g}(M(F_m^T)) \\ &\geq \tilde{g}((M(F_m))^s) = g(F_m^s, (M_2(F_m))^s, (M_3(F_m))^s, \cdots, (M_p(F_m))^s, 0, \cdots, 0) \end{aligned}$$

In other hands, since the function g is convex, we can find for any real r, matrices $C_k \in \mathcal{S}^{\binom{d}{k}\times\binom{d}{k}}$, $k = 1, 2, \ldots, p-1$ and a real c_p which depends only on r such that

$$\begin{aligned} f(F_m^s) &\geq g(F_m^s, (M_2(F_m))^s, (M_3(F_m))^s, \cdots, (M_p(F_m))^s, 0, \cdots, 0) \\ &\geq g(0, \cdots, 0, r, 0, \cdots, 0) + C_1 : (\alpha A_0 + \beta A_1) + C_2 : (tA_2 + \alpha^2 A_3 + \alpha\beta A_4) \\ &+ \sum_{k=3}^{p-3} C_k : (\alpha^k \tilde{A}_1 + \beta\alpha^{k-1}\tilde{A}_2 + \alpha^{k-1}\sqrt{t}\tilde{A}_3 + \beta\alpha^{k-2}\sqrt{t}\tilde{A}_4 + t\alpha^{k-2}\tilde{A}_5 + \beta\alpha^{k-3}t\tilde{A}_6) \\ &+ C_{p-2} : (\beta\alpha^{p-3}\bar{A}_1 + \alpha^{p-3}\sqrt{t}\bar{A}_2 + \beta\alpha^{p-4}\sqrt{t}\bar{A}_3 + t\alpha^{p-4}\bar{A}_4 + t\beta\alpha^{p-5}\bar{A}_5) \\ &+ C_{p-1} : (t\alpha^{p-3}B_1 + \beta\alpha^{p-3}\sqrt{t}B_2 + t\beta\alpha^{p-4}B_3) + c_p(t\beta\alpha^{p-3} - r). \end{aligned} \tag{3.3}$$

We will show that the constant c_p is necessary zero. We divide the two sides of the above inequality by $t\alpha^{p-3}$ and tend first t and then α to $+\infty$ to get

$$(C_{p-1} : B_1) + \beta c_p \leq 0$$

Varying β on positive and negative values respectively, we get necessarily $c_p = 0$. By taking in (3.3) $t = \alpha = \beta = 0$, we conclude that

$$g(0, \ldots, 0, r, 0, \ldots, 0) \leq f(0) \tag{3.4}$$

Lemma 3.3 *iv)* in [11], allow us to establish that g is constant relatively to the $p \times p$ m-diagonal minor $(M_p)_{IJ}$, for all $m \in \{2, 3, \ldots, p\}$. Since I and J are arbitrary in the set of p-tuples which gives rise to m-diagonal minors, we conclude that g is independent with respect to all m-diagonal minors.

Step2 : m-diagonal for $m = 0,\ m = 1$

We move now to the cases $m = 0$ and $m = 1$ and demonstrate that g is independent on all the m-diagonal minors ($m = 0, 1$) of order greater or equal to 3. The strategy is based on showing first the independence on the 3×3 m-diagonal minors and then arguing by induction to show the independence on the general order minors.

The sequel of the proof is done by induction, we will show first the property for all 0 and 1-diagonal 3×3 minors, and then generalize it to $p \times p$ minors of the same kind.

1. **Independence on 1-diagonal and 0-diagonal minors of order 3**

Let us first show that g is constant on the 3×3 m- diagonal minors, for $m = 0, 1$. We will use the same technique as in the first step by choosing appropriate matrices; this choice is based on the number of diagonal elements of the $d \times d$ matrix appearing in the minor under study, which is either 1 or 0. We divide this part of the proof into two main sub-steps depending on the number of diagonal elements appearing in the minor.

i) Case of 1-diagonal Minors

We consider the case of 1-diagonal minors, which consists on minors intersecting the diagonal of the entire matrix in just one element. Let $I_1 = (i_1, i_2, i_3), 1 \leq i_1 < i_2 < i_3 \leq d$ and $J_1 = (j_1, j_2, j_3)$, $1 \leq j_1 < j_2 < j_3 \leq d$ two arbitrary 3-tuples, such that $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\}$ contain exactly one element. We will show that g does not depend on the minor $(M_3)_{I_1 J_1}$. There is no loss of generality if we suppose that $i_1 = j_1 < i_2 < j_2 < i_3 < j_3$. The other situation are treated similarly. Next we choose the matrices F_1, F_2 in $\mathcal{S}^{d \times d}$ whose entries are given by:

$$\begin{array}{l} (F_1)_{i_1 i_1} = \alpha, (F_1)_{i_2 j_3} = \sqrt{t}, (F_1)_{i_3 j_2} = \sqrt{t}, (F_1)_{j_3 i_2} = -\sqrt{t}, (F_1)_{j_2 i_3} = -\sqrt{t} \\ (F_2)_{i_1 j_2} = \sqrt{t}, (F_2)_{i_1 i_3} = \sqrt{t}, (F_2)_{i_2 j_3} = \alpha, (F_2)_{j_2 i_1} = -\sqrt{t}, (F_2)_{i_3 i_1} = -\sqrt{t} \end{array} \tag{3.5}$$

and all the other entries non appearing in (3.5) in both F_1 and F_2 are supposed to be null. We deal first with the matrix F_1. By deleting all the null rows and columns, the only sub-matrix of F_1 containing non zero elements in each row or column is the following

$$\begin{array}{c|ccccc} & i_1 & i_2 & j_2 & i_3 & j_3 \\ \hline i_1 & \alpha & 0 & 0 & 0 & 0 \\ i_2 & 0 & 0 & 0 & 0 & \sqrt{t} \\ j_2 & 0 & 0 & 0 & -\sqrt{t} & 0 \\ i_3 & 0 & 0 & \sqrt{t} & 0 & 0 \\ j_3 & 0 & -\sqrt{t} & 0 & 0 & 0 \end{array}$$

Note that, $F_1^s = \alpha e_{i_1} \otimes e_{i_1}$, e_{i_1} is the $i_1 th$ vector in the canonical basis in $\mathbb{R}^d$. In other hands, the sub-matrix contain only one non zero element in each row and each column, we deduce that the number of non zero minors of order k, $1 \leq k \leq 5$ is exactly $\binom{5}{k}$. For the two order minors, we have $(M_2(F_1))^s = tA$, where A is a constant matrix in $\mathcal{S}^{\binom{d}{2} \times \binom{d}{2}}$. Next, in order to compute the different 3-order minors of F_1, we consider all the possible 3-tuples, defined in the index set $\{i_1, i_2, j_2, i_3, j_3\}$

$$\begin{array}{c} I_1 = (i_1, i_2, i_3), I_2 = (i_1, i_2, j_2), I_3 = (i_1, i_2, j_3), I_4 = (i_1, j_2, i_3), I_5 = (i_1, j_2, j_3), \\ I_6 = (i_1, i_3, j_3), I_7 = (i_2, j_2, i_3), I_8 = (i_2, j_2, j_3), I_9 = (i_2, i_3, j_3), I_{10} = (j_2, i_3, j_3), \end{array}$$

note that $J_1 = I_5$. Due to the structure of the matrix F_1, for each 3-tuple, I_k, correspond only one other 3-tuple I_l such that $(M_3)_{I_kI_l} \neq 0,\ k, l = 1, \ldots, 10$. Consequently, the only non zero elements of M_3 are given as follow:

$$\begin{array}{ll}
(M_3(F_1))_{I_1I_5} = (M_3(F_1))_{I_5I_1} = -\alpha t, & (M_3(F_1))_{I_2I_6} = (M_3(F_1))_{I_6I_2} = \alpha t \\
(M_3(F_1))_{I_7I_{10}} = -(M_3(F_1))_{I_{10}I_7} = t\sqrt{t}, & (M_3(F_1))_{I_8I_9} = -(M_3(F_1))_{I_9I_8} = -t\sqrt{t}, \\
(M_3(F_1))_{I_3I_3} = (M_3(F_1))_{I_4I_4} = \alpha t &
\end{array}$$

Note that the only possible non zero 5-order minor is a diagonal one which is a 5-diagonal minor, and the non zero 4-order minors are those obtained by choosing rows and columns in the set $I = \{i_1, i_2, j_2, i_3, j_3\}$, then necessary at least two of the rows and columns are equal, this means that those minors are m-diagonal with $m \geq 2$, The same thing apply on the 3×3 minors, $(M_3(F_1))_{I_3I_3}, (M_3(F_1))_{I_4I_4}$ which are 3-diagonal and the minors $(M_3(F_1))_{I_7I_{10}}, (M_3(F_1))_{I_{10}I_7}, (M_3(F_1))_{I_8I_9}$ and $(M_3(F_1))_{I_9I_8}$ are 2-diagonal. Thanks to the first step, the function g is independent on all the above minors and we will consider only the dependence of g on the minors

$$(M_3(F_1))_{I_1I_5},\ (M_3(F_1))_{I_5I_1},\ (M_3(F_1))_{I_2I_6},\ (M_3(F_1))_{I_6I_2}.$$

Consequently, proceeding as in the first step, for any two arbitrary reals r_1, r_2 , there exist $c_{I_1I_5}, c_{I_2I_6} \in \mathbb{R}$ such that

$$\begin{aligned} f(\alpha A_0) \ &\geq g(0, 0, r, 0, \cdots, 0) + C_1 : \alpha A_0 + C_2 : tA \\ &+2c_{I_1I_5}(-\alpha t - r) + 2c_{I_2I_6}\alpha t. \end{aligned} \tag{3.6}$$

Where A_0 and A are constant matrices. and r appear both in the position (I_1I_5) and its symmetric position with respect to the diagonal.

Tending first t to infinity and then varying α to take both positive and negative values, we get

$$-c_{I_1I_5} + c_{I_2I_6} = 0 \tag{3.7}$$

We consider, now the matrix F_2. By deleting null rows and columns as we did for F_1, we obtain the following 5×5 sub-matrix:

$$\begin{array}{c|ccccc}
 & i_1 & i_2 & j_2 & i_3 & j_3 \\
\hline
i_1 & 0 & 0 & \sqrt{t} & \sqrt{t} & 0 \\
i_2 & 0 & 0 & 0 & 0 & \alpha \\
j_2 & -\sqrt{t} & 0 & 0 & 0 & 0 \\
i_3 & -\sqrt{t} & 0 & 0 & 0 & 0 \\
j_3 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Remark that the fifth and the fourth order minors of F_2 are equal to zero. The non zero 3-order minors are the following

$$(M_3(F_2))_{I_1I_5} = (M_3(F_2))_{I_2I_6} = (M_3(F_2))_{I_1I_6} = (M_3(F_2))_{I_2I_5} = -\alpha t.$$

The minors $(M_3(F_2))_{I_1I_6}$ and $(M_3(F_2))_{I_2I_5}$ are 2-diagonal, since $I_1 = \{i_1, i_2, i_3\}, I_6 = \{i_1, i_3, j_3\}$ and $I_2 = \{i_1, i_2, j_2\}, I_5 = \{i_1, j_2, j_3\}$ then we will not consider the dependence of g on it. The non zero second order minors are either t or $\alpha\sqrt{t}$, i.e. we can write $(M_2(F_2))^s = t\tilde{A}_1 + \alpha\sqrt{t}\tilde{A}_2$, where $\tilde{A}_1$ and $\tilde{A}_2$ are constant matrices in the corresponding spaces. Finally $(F_2)^s = \alpha\tilde{A}_0$, such that $\tilde{A}_0$ is a constant matrix. arguing as in the previous step, we conclude that

$$c_{I_1I_5} + c_{I_2I_6} = 0 \tag{3.8}$$

Combining (3.7) and (3.8) we conclude that $c_{I_1I_5} = 0$ and $c_{I_2I_6} = 0$. We return to equation (3.6) and take $\alpha = t = 0$, we get an inequality as in (3.4). We conclude then that g is independent on the minor $(M_3)_{I_1I_5}$. Since the choice of the minor is arbitrary among the 1-diagonal ones, we conclude that g is independent of all third-order minors of the 1-diagonal form.

ii) Case of 0-diagonal Minors

In this part of the proof, we will show that the function under consideration g is also independent on the 0-diagonal minors of order 3, namely those which does not intersect the diagonal of the matrix. This case is the most relevant here since we are enable to prove it with just one, two or three different choices of special matrices. It will be done with seven different choices such that each choice leads to a dependence equation between exactly four entries of the sub-differential matrix associated to the 3×3 minors variable. Consider any 0-diagonal minor of order 3, this is obtained by choosing three rows and three columns which does not intersect. Suppose that this minor correspond to the rows (i_1, i_2, i_3) and (j_1, j_2, j_3) such that $\{i_1, i_2, i_3\} \cap \{j_1, j_2, j_3\} = \emptyset$. Without loss of generality we can suppose that $1 \le i_1 < i_2 < i_3 < j_1 < j_2 < j_3 \le d$. Next we will express the different corresponding choices of the matrices.

$$(F_1)_{i_1j_3} = \sqrt{t}, (F_1)_{i_2j_2} = \sqrt{t}, (F_1)_{i_3j_1} = \alpha, (F_1)_{j_3i_1} = -\sqrt{t}, (F_1)_{j_2i_2} = -\sqrt{t},$$

and all other elements of F_1 vanish. The sub-matrix $\hat{F}_1$ of F_1 where the rows i_1, i_2, i_3 and the columns j_1, j_2, j_3 are involved can be represented as follow

$$\begin{array}{c} \\ i_1 \\ i_2 \\ i_3 \\ j_1 \\ j_2 \\ j_3 \end{array} \begin{array}{c} \begin{array}{cccccc} i_1 & i_2 & i_3 & j_1 & j_2 & j_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{t} \\ 0 & 0 & 0 & 0 & \sqrt{t} & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{t} & 0 & 0 & 0 & 0 \\ -\sqrt{t} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

The non zero minors of F_1 are exactly those of $\hat{F}_1$. We note that F_1 is a 5-rank matrix, which means that all the minors of order more than 6 vanish, moreover, the minors of order 5 and 4 are either zero or m-diagonal since the intersection of two subset (rows and columns) containing both four (or five) elements respectively among six elements is never empty. Before studying the 3×3 minors, we recall that any such minor correspond to a specific choice of 3 rows and 3 columns among the elements $\{i_1, i_2, i_3, j_1, j_2, j_3\}$. Next, we denote the 3-tuples I_k and J_k for $k = 1, \dots, 9$ as follows:

$$\begin{array}{ll} I_1 = (i_1, i_2, i_3), & J_1 = (j_1, j_2, j_3), \\ I_2 = (i_1, i_3, j_2), & J_2 = (i_2, j_1, j_3), \\ I_3 = (i_2, i_3, j_3), & J_3 = (i_1, j_1, j_2), \\ I_4 = (i_3, j_2, j_3), & J_4 = (i_1, i_2, j_1), \\ I_5 = (i_1, i_2, j_3), & J_5 = (i_3, j_1, j_2), \\ I_6 = (i_1, j_2, j_3), & J_6 = (i_2, i_3, j_1), \\ I_7 = (i_1, i_2, j_2), & J_7 = (i_3, j_1, j_3), \\ I_8 = (i_1, i_3, j_1), & J_8 = (i_2, j_2, j_3), \\ I_9 = (i_2, i_3, j_2), & J_9 = (i_1, j_1, j_3). \end{array}$$

For the non null 3×3 minors, we distinguish the two possible situations. If a 3-minor contain j_2 or j_3 as a row and a column, it is a 1-diagonal minor, their number is exactly 6. There is four 0-diagonal minors of order 3, namely

$$\begin{array}{ll} (M_3(F_1))_{I_1 J_1} = -\alpha t, & (M_3(F_1))_{I_2 J_2} = \alpha t, \\ (M_3(F_1))_{I_3 J_3} = \alpha t, & (M_3(F_1))_{I_4 J_4} = -\alpha t. \end{array}$$

The 2×2 minors are either zero or $\pm t$ or $\pm \alpha\sqrt{t}$. Arguing as in the previous step, for any real r, there exist matrices $C_1 \in \mathcal{S}^{d \times d}, C_2 \in \mathcal{S}^{\binom{d}{2} \times \binom{d}{2}}$ and $C_3 \in \mathcal{S}^{\binom{d}{3} \times \binom{d}{3}}$ such that

$$f(F_1^s) \geq g(0, \dots, 0, r, 0, \dots, 0) + C_1 : F_1^s + C_2 : (M_2(F_1))^s + C_3 : (M_3(F_1))^s - r(C_3)_{I_1 J_1} \tag{3.9}$$

By varying first t then α we get the following equation:

$$-(C_3)_{I_1 J_1} + (C_3)_{I_2 J_2} + (C_3)_{I_3 J_3} - (C_3)_{I_4 J_4} = 0. \tag{3.10}$$

Next, we consider matrices F_l by changing the position of α in the 3×3 sub-matrix of F_1 corresponding to the rows I_1 and columns J_1, keeping in mind to have exactly one non zero value in each row and each column. The specific choices are as follows:

$$\begin{array}{l} (F_2)_{i_1 j_1} = \alpha, (F_2)_{i_2 j_2} = (F_2)_{i_3 j_3} = \sqrt{t},\ (F_2)_{j_2 i_2} = (F_2)_{j_3 i_3} = -\sqrt{t}, \\ (F_3)_{i_1 j_3} = \alpha, (F_3)_{i_2 j_2} = (F_3)_{i_3 j_1} = \sqrt{t},\ (F_3)_{j_2 i_2} = (F_3)_{j_1 i_3} = -\sqrt{t}, \end{array}$$

$$\begin{aligned}
&(F_4)_{i_2j_2}=\alpha, (F_4)_{i_1j_3}=(F_4)_{i_3j_1}=\sqrt{t},\ (F_4)_{j_1i_3}=(F_4)_{j_3i_1}=-\sqrt{t},\\
&(F_5)_{i_1j_3}=\alpha, (F_5)_{i_2j_1}=(F_5)_{i_3j_2}=\sqrt{t},\ (F_5)_{j_1i_2}=(F_5)_{j_2i_3}=-\sqrt{t},\\
&(F_6)_{i_2j_1}=\alpha, (F_6)_{i_1j_2}=(F_6)_{i_3j_3}=\sqrt{t},\ (F_6)_{j_2i_1}=(F_6)_{j_3i_3}=-\sqrt{t},\\
&(F_7)_{i_2j_3}=\alpha, (F_7)_{i_1i_3}=(F_7)_{j_1j_2}=\sqrt{t}, (F_7)_{i_3i_1}=(F_7)_{j_2j_1}=-\sqrt{t}.
\end{aligned}$$

and all the other entries non appearing in the matrices are equal to zero. Note that all the k-minors for $k\geq 6$ of all the F_l are zero, and all the 4-order (5-order) minors of all the matrices F_l are m-diagonal, $m\geq 2$. By substituting F_l together with its minors in (3.9) and combining with (3.10) we get the following system

$$\begin{cases}
-(C_3)_{I_1J_1}+(C_3)_{I_2J_2}+(C_3)_{I_3J_3}-(C_3)_{I_4J_4}=0,\\
(C_3)_{I_1J_1}-(C_3)_{I_5J_5}-(C_3)_{I_2J_2}+(C_3)_{I_6J_6}=0,\\
-(C_3)_{I_1J_1}+(C_3)_{I_4J_4}+(C_3)_{I_2J_2}-(C_3)_{I_3J_3}=0,\\
-(C_3)_{I_1J_1}+(C_3)_{I_4J_4}+(C_3)_{I_3J_3}-(C_3)_{I_2J_2}=0,\\
(C_3)_{I_1J_1}+(C_3)_{I_7J_7}+(C_3)_{I_8J_8}+(C_3)_{I_3J_3}=0,\\
-(C_3)_{I_1J_1}+(C_3)_{I_5J_5}-(C_3)_{I_9J_9}+(C_3)_{I_8J_8}=0,\\
-(C_3)_{I_4J_4}+(C_3)_{I_7J_7}-(C_3)_{I_6J_6}+(C_3)_{I_9J_9}=0.
\end{cases}$$

Note that the equations above are arranged with respect to the matrices F_i.

By solving the system, we get $C_{I_1J_1}=0$, which means as in the previous steps that g is independent on the variable which correspond to the 3×3 minor $(M_3)_{I_1J_1}$. Since the minor is arbitrary, we deduce that g is not dependent on all the 0-diagonal minors of order 3.

2. **Independence on 0-diagonal and 1-diagonal minors of order p**

By induction, we will prove that the function g is independent on the 0-diagonal and 1-diagonal minors of order p, for all $p>3$. Let's assume that $3<p\leq d$ is any integer, suppose that g is independent on all k- order minors, $3\leq k\leq p-1$, we want to show that g is also independent on $p\times p$ minors. we start as in the previous step by considering 1-diagonal minors and then 0-diagonal ones.

a) Case of 1-diagonal minors of order p

We note first that any $p\times p$ minor such that $2p>d+1$ is an m-diagonal with $m\geq 2$. In fact, if a minor correspond to the rows I and columns J, such that $Card(I)=Card(J)=p$ and $2p\geq d+2$, we conclude, since $I\cup J\subset\{1,2,\ldots,d\}$, that

$$Card(I\cap J)=Card(I)+Card(J)-Card(I\cup J)\geq 2p-d\geq 2.$$

Hence I and J have in common at least two elements, thus the minor is an m-diagonal, with $m\geq 2$.

Let's assume that $2p \leq d+1$ and consider an arbitrary 1-diagonal minors of order p, by choosing the rows $I_1 = \{i_1, i_2, \ldots, i_p\}$ and the columns $I_2 = \{j_1, j_2, \ldots, j_p\}$. Without lose of generality we can suppose that

$$i_1 < i_2 < \cdots < i_{p-1} < j_1 < j_2 < \cdots < j_p = i_p$$

We consider the matrix F_1 defined as follow:

$$\begin{aligned}&(F_1)_{i_{p-2}j_p} = \sqrt{t},\ (F_1)_{j_p i_{p-2}} = -\sqrt{t},\ (F_1)_{i_{p-1}j_{p-1}} = \sqrt{t},\ (F_1)_{j_{p-1}i_{p-1}} = -\sqrt{t},\\ &(F_1)_{i_p j_{p-2}} = \beta,\ (F_1)_{i_k j_k} = \alpha,\ for\ 1 \leq k \leq p-3,\end{aligned}$$

where α, β are real parameters and t a positive real. All the other entries of F_1 are supposed to be equal to zero. Note that $\operatorname{rank} F_1 \leq p+1$, consequently all the k-order minors, for $k \geq p+2$ are equal to zero. Moreover

$$F_1^s = \alpha A_0 + \beta A_1 \text{ and } M_2(F_1)^s = tA_2 + \alpha\sqrt{t}A_3 + \beta\sqrt{t}A_4 + \alpha\beta A_5 + \alpha^2 A_6, \tag{3.11}$$

such that $A_0,\ A_1 \in \mathcal{S}^{d\times d}$ and $A_m,\ \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}, 2 \leq m \leq 6$ are constant matrices. The index of non zero rows and columns are in the set $\{i_1, i_2, \ldots, i_{p-1}, j_{p-1}, j_p\}$, $\{i_{p-2}, i_{p-1}, j_1, j_2, \ldots, j_{p-1}, j_p\}$ respectively. As a consequence, all the $(p+1)$-order minors are m-diagonal, with $m = 3, 4$, since a $(p+1)$-order minors is obtained by deleting one column between the $(p+2)$ columns $i_{p-2}, i_{p-1}, j_1, j_2, \ldots, j_{p-1}, j_p$ and we have

$$\{i_1, i_2, \ldots, i_{p-1}, j_{p-1}, j_p\} \cap \{i_{p-2}, i_{p-1}, j_1, j_2, \ldots, j_{p-1}, j_p\} = \{i_{p-2}, i_{p-1}, j_{p-1}, j_p\} \tag{3.12}$$

which means that the columns and the lines coincide in at least three elements for any deleted column. For p-order minors, we note that since the intersection in (3.12) contain four elements, the only possible 1-diagonal minors are those obtained by deleting one row and two columns from the intersection index set. All other minors are either zero or 4-diagonal.

– Assume that we delete the row j_p, we get a possible non zero p-minor only if we delete the two columns i_{p-2}, j_{p-2}. But the obtained minor is 2-diagonal since it contain the indices i_{p-1} and j_{p-1} which appears both in the rows and columns.

– Assume that we delete the row j_{p-1}, to obtain a possible non zero minor we should delete the column i_{p-1}. If in addition we delete the column j_p, the obtained minor vanish since it contain the two dependent columns i_{p-2}, j_{p-2}. But if we delete the column i_{p-2} we get the minor under study

$$(M_p(F_1))_{I_1 I_2} = -t\beta\alpha^{(p-3)}$$

– Assume that we delete the row i_{p-1}, to obtain a possible non zero minor we should delete the column j_{p-1}. Moreover, If we delete the column j_p, the obtained minor vanish since it contain the two dependent columns i_{p-2}, j_{p-2}. However, if the second column deleted is i_{p-2}, the obtained minor is

$$(M_p(F_1))_{I_3 I_4} = -t\beta(-\alpha)^{(p-3)},$$

where $I_3$and I_4 are the rows and columns respectively given by:

$$I_3 = (i_1, \cdots, i_{p-3}, i_{p-2}, j_{p-1}, i_p),\ I_4 = (i_{p-1}, j_1, \cdots, j_{p-3}, j_{p-2}, j_p)$$

– Assume that we delete the row i_{p-2}, to obtain a possible non zero minor we should delete the column j_p. If in addition, we delete the column i_{p-1}, or the column j_{p-1}, the obtained minor vanishes since it contains the two dependent columns i_{p-2}, j_{p-2}.

Let's fix a real r, by sub-differential property of convex functions, keeping in mind, due to the induction hypothesis, that the function g is independent on the k-minors for $3 \leq k \leq (p-1)$, there exist two real constants $c_{I_1 I_2}, c_{I_3 I_4}$, two constant matrices $C_1 \in \mathcal{S}^{d\times d}$ and $C_2 \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, such that,

$$\begin{array}{ll} f(\alpha A_0 + \beta A_1) & \geq g(0, \cdots, 0, r, 0, \cdots, 0) + C_1 : (\alpha A_0 + \beta A_1) \\ & +C_2 : (tA_2 + \alpha\sqrt{t}A_3 + \beta\sqrt{t}A_4 + \alpha\beta A_5 + \alpha^2 A_6) \\ & +c_{I_1 I_2}(-t\beta\alpha^{(p-3)} - r) - t\beta(-\alpha)^{(p-3)} c_{I_3 I_4}. \end{array} \tag{3.13}$$

The constant matrices A_i are given in (3.11). By tending t and α to infinity and varying β we get

$$c_{I_1 I_2} + (-1)^{p-3} c_{I_3 I_4} = 0 \tag{3.14}$$

To conclude that $c_{I_1 I_2} = 0$, we will get a second equation of the form $c_{I_1 I_2} - c_{I_3 I_4} = 0$ by a suitable choice of a second matrix F_2 which will be chosen as follow:

$$\begin{array}{r} (F_2)_{i_{p-1} j_p} = \sqrt{t},\ (F_2)_{j_p i_{p-1}} = -\sqrt{t},\ (F_2)_{i_p j_{p-1}} = \sqrt{t},\ (F_2)_{j_{p-1} i_p} = -\sqrt{t}, \\ (F_2)_{i_{p-2} j_{p-2}} = \beta,\ (F_2)_{i_k j_k} = \alpha,\ \mathit{for}\ 1 \leq k \leq p-3, \end{array}$$

As in the treatment done for F_1, we will show that we only consider the p-minors $(M_p(F_2))_{I_1 I_2}$ and $(M_p(F_2))_{I_3 I_4}$.

We note first that F_2 is of rank $(p+1)$, moreover the maximal rank submatrix of F_2 is the one obtained by choosing the rows, $i_1, i_2, \ldots, i_{p-1}, j_{p-1}, j_p$ and columns $i_{p-1}, j_1, j_2, \ldots, j_p$. We denote this matrix by $\hat{F}_2$, consequently all the k-order minors of F_2 vanish for $k \geq (p+2)$. In other hands, the rows and columns intersect exactly in three indices, namely

$$\{i_1, i_2, \ldots, i_{p-1}, j_{p-1}, j_p\} \cap \{i_{p-1}, j_1, j_2, \ldots, j_p\} = \{i_{p-1}, j_{p-1}, j_p\}.$$

Since the columns i_{p-1} and j_{p-1} are dependent, all $(p+1)$-order minors vanish. We now turn our attention to the p-order minors of F_2.

– If we delete the row j_p, the obtained sub matrix of $\hat{F}_2 \in \mathbb{R}^{(p+1)\times(p+1)}$ contain two null columns, this means that all resulting p-order minors of F_2 are equal to zero.

– If we delete the row j_{p-1} and the column j_p, the obtained minor vanish since it contain the dependent columns i_{p-1} and j_{p-1}. But if we delete the row j_{p-1} and the columns i_{p-1} the obtained minor is

$$(M_p(F_2))_{I_1 I_2} = -t\beta\alpha^{p-3} \tag{3.15}$$

– If we delete the row i_{p-1} and the columns j_p the obtained minor vanish since it contain the column i_{p-1} and j_{p-1}. But if we delete the row i_{p-1} and the column j_{p-1} the obtained minor is

$$(M_p(F_2))_{I_3 I_4} = t\beta(-\alpha)^{p-3} \tag{3.16}$$

For the minors of order one or two, we obtain a same formula as in (3.11) with different constant matrices A_i. Arguing as previously by taking in account (3.15) and (3.16) we get the equation

$$-c_{I_1 I_2} + (-1)^{p-3} c_{I_3 I_4} = 0 \tag{3.17}$$

Combining (3.14) and (3.17) we conclude that $c_{I_1 I_2} = 0$. We replace in (3.13) t, α and β by zero, we get for any $r \in \mathbb{R}$

$$g(0, \ldots, 0, r, 0, \ldots, 0) \le f(0)$$

which allow us to conclude that g is independent on the minor $(M_p)_{I_1 I_2}$. Since the choice of I_1 and I_2 is arbitrary, we deduce that g does not depend on all 0-diagonal minors of order p.

b) Case of 0-diagonal minors of order p

We turn now to 0-diagonal minors of order p, by recurrence hypothesis, the function g is independent on minors of order less than p, we will show in this step that g is independent on 0-diagonal minors of order p,. Consider any 0-diagonal minors of order p, corresponding to the rows $I_1 = (i_1, i_2, \cdots, i_p)$ and columns $J_1 = (j_1, j_2, \cdots, j_p)$, without loss of generality, we suppose that the p-tuple are ordered as follows:

$$1 \le i_1 < i_2 < \cdots < i_p < j_1 < j_2 < \cdots < j_p \le d.$$

Consider the $d \times d$ matrix defined bellow:

$$(F_1)_{i_{p-2} j_{p-2}} = \beta,\ (F_1)_{i_{p-1} j_{p-1}} = \sqrt{t},\ (F_1)_{j_{p-1} i_{p-1}} = -\sqrt{t},$$
$$(F_1)_{i_p j_p} = \sqrt{t},\ (F_1)_{j_p i_p} = -\sqrt{t},\ (F_1)_{i_k j_k} = \alpha,\ \mathit{for}\ 1 \le k \le p-3,$$

where t is a positive integer, α, β are real numbers and all other entries of F_1 are zero. We have $F_1^s = \alpha A_0 + \beta A_1$, where A_0, A_1 are $d \times d$ constant symmetric matrices, and $(M_2(F_1))^s = tA_2 + \alpha\sqrt{t}A_3 + \beta\sqrt{t}A_4 + \alpha\beta A_5 + \alpha^2 A_6$ with $A_m \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$, $2 \leq m \leq 6$ are constant matrices. The non-zero row and columns of F_1 correspond respectively to the sets $\{i_1, i_2, \cdots, i_{p-1}, i_p, j_{p-1}, j_p\}$ and $\{i_{p-1}, i_p, j_1, j_2, \cdots, j_{p-1}, j_p\}$. As a consequence, all the minors of order greater than $(p+2)$ are equal to zero. Moreover the two sets of rows and columns satisfies

$$\{i_1, i_2, \cdots, i_{p-1}, i_p, j_{p-1}, j_p\} \cap \{i_{p-1}, i_p, j_1, j_2, \cdots, j_{p-1}, j_p\} = \{i_{p-1}, i_p, j_{p-1}, j_p\}. \tag{3.18}$$

This means that all the minors of F_1 of order $(p+1)$ or $(p+2)$ are either zero or m-diagonal, with $m = 3$ or $m = 4$. Consequently, by taking in account the induction hypothesis, it suffice to treat only the p-order minors. To obtain a possible non-zero p-order minor, since all the rows and all the columns has only one non-zero element, we need to delete two elements from the $(p+2)$ rows (the deleted columns are automatically the corresponding ones). If one at least of the deleted rows are outside the intersection set (3.18), the obtained p-order minor is m-diagonal, with $m = 3$ or $m = 4$. Let's consider, then the case where we delete two rows from the intersection set (3.18), the number of such possibilities is exactly six, two of them leads to 2-diagonal minors and correspond to deleting the rows i_{p-1}, j_{p-1} or the rows i_p, j_p. The other four possible choices gives the following minors:

$$\begin{aligned} (M_p(F_1))_{I_1J_1} &= t\beta\alpha^{(p-3)}, & (M_p(F_1))_{I_2J_2} &= -(-1)^{(p-3)}t\beta\alpha^{(p-3)}, \\ (M_p(F_1))_{I_4J_4} &= t\beta\alpha^{(p-3)}, & (M_p(F_1))_{I_3J_3} &= -(-1)^{(p-3)}t\beta\alpha^{(p-3)}. \end{aligned}$$

Where

$$\begin{aligned} I_2 &= (i_1, i_2, \cdots, i_{p-1}, j_p), & J_2 &= (i_p, j_1, \cdots, j_{p-2}, j_{p-1}), \\ I_3 &= (i_1, \cdots, i_{p-2}, i_p, j_{p-1}), & J_3 &= (i_{p-1}, j_1, \cdots, j_{p-2}, j_p), \\ I_4 &= (i_1, \cdots, i_{p-2}, j_{p-1}, j_p), & J_4 &= (i_{p-1}, i_p, j_1, \cdots, j_{p-2}), \end{aligned}$$

As we did before, for any real r, there exist constants $C_{I_1J_1}, C_{I_2J_2}, C_{I_3J_3}, C_{I_4J_4} \in \mathbb{R}$ and $C_1 \in \mathcal{S}^{d\times d}$, $C_2 \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ such that,

$$\begin{aligned} f(\alpha A_0 + \beta A_1) \geq\ & g(0, \cdots, 0, r, 0, \cdots, 0) + C_1 : (\alpha A_0 + \beta A_1) \\ & + C_2 : (tA_2 + \alpha\sqrt{t}A_3 + \beta\sqrt{t}A_4 + \alpha\beta A_5 + \alpha^2 A_6) + C_{I_1J_1}(t\beta\alpha^{(p-3)} - r) \\ & -(-1)^{(p-3)}t\beta\alpha^{(p-3)}(C_{I_2J_2} + C_{I_3J_3}) + t\beta\alpha^{(p-3)}C_{I_4J_4}. \end{aligned} \tag{3.19}$$

Tending first t then α to infinity and varying β we get the following equation

$$C_{I_1J_1} - (-1)^{(p-3)}C_{I_2J_2} - (-1)^{(p-3)}C_{I_3J_3} + C_{I_4J_4} = 0. \tag{3.20}$$

The above equation does not imply that $C_{I_1J_1} = 0$. To obtain this, we will choose eight other matrices which give us eight other equations, solving the obtained system leads to the desired conclusion. The matrix choices are summarized as follow:

$$\begin{aligned}
&(F_2)_{i_{p-2}j_{p-1}} = \beta,\ (F_2)_{i_{p-1}j_p} = (F_2)_{i_pj_{p-2}} = \sqrt{t},\ (F_2)_{j_pi_{p-1}} = (F_2)_{j_{p-2}i_p} = -\sqrt{t},\\
&(F_3)_{i_{p-2}j_p} = \beta,\ (F_3)_{i_{p-1}j_{p-2}} = (F_3)_{i_pj_{p-1}} = \sqrt{t},\ (F_3)_{j_{p-2}i_{p-1}} = (F_3)_{j_{p-1}i_p} = -\sqrt{t},\\
&(F_4)_{i_{p-2}j_p} = \beta,\ (F_4)_{i_{p-1}j_{p-1}} = (F_4)_{i_pj_{p-2}} = \sqrt{t},\ (F_4)_{j_{p-1}i_{p-1}} = (F_4)_{j_{p-2}i_p} = -\sqrt{t},\\
&(F_5)_{i_{p-2}j_{p-2}} = \beta,\ (F_5)_{i_{p-1}i_p} = (F_5)_{j_{p-1}j_p} = \sqrt{t},\ (F_5)_{i_pi_{p-1}} = (F_5)_{j_pj_{p-1}} = -\sqrt{t},\\
&(F_6)_{i_{p-1}j_{p-2}} = \beta,\ (F_6)_{i_{p-2}i_p} = (F_6)_{j_{p-1}j_p} = \sqrt{t},\ (F_6)_{i_pi_{p-2}} = (F_6)_{j_pj_{p-1}} = -\sqrt{t},\\
&(F_7)_{i_pj_{p-2}} = \beta,\ (F_7)_{i_{p-2}i_{p-1}} = (F_7)_{j_{p-1}j_p} = \sqrt{t},\ (F_7)_{i_{p-1}i_{p-2}} = (F_7)_{j_pj_{p-1}} = -\sqrt{t},\\
&(F_8)_{i_{p-2}i_{p-1}} = \beta,\ (F_8)_{i_pj_p} = (F_8)_{j_{p-2}j_{p-1}} = \sqrt{t},\ (F_8)_{j_pi_p} = (F_8)_{j_{p-1}j_{p-2}} = -\sqrt{t},\\
&(F_9)_{i_{p-2}i_{p-1}} = \beta,\ (F_9)_{i_pj_{p-1}} = (F_9)_{j_{p-2}j_p} = \sqrt{t},\ (F_9)_{j_{p-1}i_p} = (F_9)_{j_pj_{p-2}} = -\sqrt{t},
\end{aligned}$$

such that for any $2 \leq l \leq 9$, we have $(F_l)_{i_kj_k} = \alpha,\ for\ 1 \leq k \leq p-3,$, and all other entries of F_l are equal to zero, t is a positive integer and α, β are real numbers.

By making the same treatment for the matrices $F_l,\ l = 2, \ldots, 9$ as we made for F_1 there exist real constants $C_{I_kJ_k},\ k = 5, \ldots, 12$ satisfying eight equations which combined with (3.20) leads to the following system:

$$\begin{cases}
C_{I_1J_1} - (-1)^{(p-3)}C_{I_2J_2} - (-1)^{(p-3)}C_{I_3J_3} + C_{I_4J_4} & = 0,\\
C_{I_1J_1} - (-1)^{(p-3)}C_{I_5J_5} + (-1)^{(p-3)}C_{I_6J_6} - C_{I_7J_7} & = 0,\\
C_{I_1J_1} + (-1)^{(p-3)}C_{I_8J_8} + (-1)^{(p-3)}C_{I_9J_9} + C_{I_{10}J_{10}} & = 0,\\
-C_{I_1J_1} + (-1)^{(p-3)}C_{I_5J_5} + (-1)^{(p-3)}C_{I_3J_3} - C_{I_{10}J_{10}} & = 0,\\
-(-1)^{(p-3)}C_{I_8J_8} + (-1)^{(p-3)}C_{I_2J_2} + (-1)^{(p-3)}C_{I_3J_3} - (-1)^{(p-3)}C_{I_6J_6} & = 0,\\
(-1)^{(p-3)}C_{I_8J_8} - (-1)^{(p-3)}C_{I_2J_2} + (-1)^{(p-3)}C_{I_{11}J_{11}} - (-1)^{(p-3)}C_{I_{12}J_{12}} & = 0,\\
(-1)^{(p-3)}C_{I_3J_3} - (-1)^{(p-3)}C_{I_6J_6} - (-1)^{(p-3)}C_{I_{11}J_{11}} + (-1)^{(p-3)}C_{I_{12}J_{12}} & = 0,\\
-(-1)^{(p-3)}C_{I_9J_9} + (-1)^{(p-3)}C_{I_3J_3} + C_{I_7J_7} - C_{I_4J_4} & = 0,\\
(-1)^{(p-3)}C_{I_9J_9} + (-1)^{(p-3)}C_{I_6J_6} + C_{I_{10}J_{10}} + C_{I_4J_4} & = 0.
\end{cases}$$

Note that equation in the line $l, l = 1, 2, \ldots, 9$ is obtained by exploiting the corresponding matrix F_l.

The p-tuples I_k, J_k are the rows and columns defined by:

$$I_5 = (i_1, \cdots, i_{p-1}, j_{p-2}), \qquad J_5 = (i_p, j_1, \cdots, j_{p-3}, j_{p-1}, j_p),$$

$$\begin{aligned}
&I_6 = (i_1, \cdots, i_{p-2}, i_p, j_p), && J_6 = (i_{p-1}, j_1, \cdots, j_{p-2}, j_{p-1}),\\
&I_7 = (i_1, \cdots, i_{p-2}, j_{p-2}, j_p), && J_7 = (i_{p-1}, i_p, j_1, \cdots, j_{p-3}, j_{p-1}),\\
&I_8 = (i_1, i_2, \cdots, i_{p-1}, j_{p-1}), && J_8 = (i_p, j_1, \cdots, j_{p-2}, j_p),\\
&I_9 = (i_1, \cdots, i_{p-2}, i_p, j_{p-2}), && J_9 = (i_{p-1}, j_1, \cdots, j_{p-3}, j_{p-1}, j_p),\\
&I_{10} = (i_1, \cdots, i_{p-2}, j_{p-2}, j_{p-1}), && J_{10} = (i_{p-1}, i_p, j_1, \cdots, j_{p-3}, j_p),\\
&I_{11} = (i_1, \cdots, i_{p-3}, i_{p-1}, i_p, j_{p-1}), && J_{11} = (i_{p-2}, j_1, \cdots, j_{p-2}, j_p),\\
&I_{12} = (i_1, \cdots, i_{p-3}, i_{p-1}, i_p, j_p), && J_{12} = (i_{p-2}, j_1, \cdots, j_{p-2}, j_{p-1}).
\end{aligned}$$

Solving the system, we get $C_{I_1J_1} = 0$. Taking $t = \beta = \alpha = 0$ in (3.19), we have

$$g(0, \cdots, 0, r, 0, \cdots, 0) \leq f(0)$$

Which means that g is independent on the 0-diagonal minor $(M_p)_{I_1J_1}$. Since I_1, J_1 are arbitrary, we conclude that g is independent on all 0-diagonal minors of order p. By induction we get the independence of g on all p-minors for $p = 3, \ldots, d$.

Part2. **Non S-positivity of ∂_2g**

We turn now to the second part of theorem 3.1 which consist on showing that the set of partial sub-differential of g with respect to the second variable contain only S-negative semi-definite matrices. We will adapt the proof in [11] to our case, the change will be on considering S-positive semi-definite matrices instead of positive semi-definite ones. The strategy consist on adapting Lemma 3.4*ii*) to our situation. Let $X \in \mathbb{R}^{\binom{d}{2}}$ be any vector, we will show that there exist a real constant C such that $g(0, X \boxtimes X) \leq C$. Let us denote F_X the matrix $Skew(X)$

$$\begin{aligned}
g(0, X \boxtimes X) &= \tilde{g}(F_X^s, (M_2(F_X))^s) \leq \tfrac{1}{2}\tilde{g}(F_X, M_2(F_X)) + \tfrac{1}{2}\tilde{g}(F_X^T, M_2(F_X)^T)\\
&= \tfrac{1}{2}\tilde{f}(F_X) + \tfrac{1}{2}\tilde{f}(F_X^T) = f((F_X)^s) = g(0,0)
\end{aligned}$$

Thanks to Lemma 1.5, we conclude that $\partial_2 g$ contain only S-negative semi-definite matrices. ■

3.3 Symmetric Polyconvex Quadratic Forms in Higher Dimensions

In this section we give a characterization of symmetric polyconvex quadratic forms, based on Theorem3.5. The result asserts that the necessary and sufficient condition for the symmetric polyconvexity of any quadratic form defined on the space of $d \times d$ symmetric matrices is that it can be written as the sum of a convex quadratic form and the scalar product of an S-negative semi-definite matrix by the matrix of 2-order minors. The proof is not done here, it is similar to that of Proposition 5.6 in [11].

Proposition 3.1 *Let $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ be a quadratic form. Then the following conditions are equivalent:*

(i) *f is a symmetric polyconvex function;*

(ii) *There exist a convex quadratic form $h : \mathcal{S}^{d\times d} \to \mathbb{R}$ and a matrix $A \in \mathcal{S}_{s+}^{\binom{d}{2}\times\binom{d}{2}}$ such that*

$$f(\varepsilon) = h(\varepsilon) - A : M_2(\varepsilon) \quad \textit{for all } \varepsilon \in \mathcal{S}^{d\times d};$$

(iii) *there exist a matrix $A \in \mathcal{S}_{s+}^{\binom{d}{2}\times\binom{d}{2}}$ such that*

$$f(\varepsilon) + A : M_2(\varepsilon) \geq 0 \quad \textit{for all } \varepsilon \in \mathcal{S}^{d\times d}.$$

Remark 3.1 *Our result on quadratic forms in higher dimensions can be seen as a refinement of the classical case where no conditions on the matrix A are required, see [19], Lemma* 5.27*. Furthermore, the result is a generalization of the representations of symmetric polyconvex quadratic forms from three dimensions to higher dimensions, see [11].*

In the particular case of quadratic functions of the form: $f(\varepsilon) = -A : M_2(\varepsilon)$ we have the following:

Corollary 3.1 *Let $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ be a quadratic form such that*

$$f(\varepsilon) = -A : M_2(\varepsilon), \forall \varepsilon \in \mathcal{S}^{d\times d} \tag{3.21}$$

where $A \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$. The following assertions are equivalents:

i) *The quadratic form f is symmetric polyconvex,*

ii) *There exists a matrix $B \in \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}}$ such that $A - B$ is S-positive semi-definite and $B : M_2(\varepsilon) = 0$ for any matrix $\varepsilon \in \mathcal{S}^{d\times d}$.*

Proof.

- *ii)* $\Longrightarrow$ *i)*. Consider A and B as in *ii)*, then, clearly $f(\varepsilon) = -(A - B) : M_2(\varepsilon)$, as a consequence of Proposition 3.1, f is a symmetric polyconvex quadratic form.

- *i)* $\Longrightarrow$ *ii)*. Assume that f given by (3.21) is symmetric polyconvex. Making use of (1.2) we get,

$$f(F^s) = -A : M_2(F^s) = -A : (M_2(F))^s + A : M_2(F^a). \tag{3.22}$$

The symmetric polyconvexity of f along with the poly-affinity of the form $F \to -A : (M_2(F))^s$, allow us to conclude that the form $F \to A : M_2(F^a)$ is polyconvex. According to [19, Lemma 5.27, p. 192], there exist $\beta \in \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}}$, such that

$$A : M_2(F^a) \geq \beta : M_2(F) \ \text{ for all } F \in \mathbb{R}^{d\times d}. \tag{3.23}$$

Taking first symmetric matrices, and then skew-symmetric matrices in (3.23), bearing in mind that $M_2(F^a)$ is a symmetric matrix, we get for all $\varepsilon \in \mathcal{S}^{d\times d}$ and $X \in \mathbb{R}^{\binom{d}{2}}$,

$$\beta^s : M_2(\varepsilon) \leq 0, \ \text{ and } \ (A - \beta^s) : X \boxtimes X \geq 0.$$

choosing $B = \beta^s$, and making use of Lemma 1.2 as well as Definition 1.3, we obtain $ii)$.

■

Remark 3.2 *i) Note that, the symmetric polyconvexity of f does not imply that A is S-positive semi-definite. Indeed, let*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have for all $X \in \mathbb{R}^6$,

$$A : X \boxtimes X = 4(-x_3x_4 + x_2x_5 - x_1x_6),$$

then A is not S-positive semi-definite, whereas f is symmetric polyconvex, since,

$$f(\varepsilon) = -A : M_2(\varepsilon) = 0 \ \forall \varepsilon \in \mathcal{S}^{4\times 4}.$$

ii) As a consequence of (1.3), if A is a positive semi-definite matrix, then the function f defined in (3.21) is symmetric rank-one convex. In fact, for $a, b \in \mathbb{R}^{d\times d}$, we have

$$f(a \odot b) = -A : M_2(a \odot b) = A : M_2((a \otimes b)^a) = A : \Lambda \otimes \Lambda \geq 0.$$

However, at this stage, it is unknown whether f is symmetric polyconvex.

Remark 3.3 *the counterexample provided in [11, Theorem 5.7] in the 3d case can be extended to higher dimensions while preserving the same structure of the quadratic form. Specifically, we define the quadratic form* $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ *as*

$$\begin{aligned} f(\varepsilon) &= (\varepsilon_{12}-\varepsilon_{13})^2 + (\varepsilon_{12}-\varepsilon_{23})^2 + (\varepsilon_{13}-\varepsilon_{23})^2 + \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 \\ &\quad - \eta(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 + 2(\varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2)), \end{aligned}$$

where η *is defined as in [11, Theorem 5.7, p. 443]. Clearly,* f *is symmetric rank-one convex but, it is not symmetric polyconvex since by restricting* f*, initially defined on* $\mathcal{S}^{d\times d}$*, to matrices of the form* $\varepsilon \in \mathcal{S}^{d\times d}$*,*

$$\varepsilon = \begin{pmatrix} \varepsilon' & 0 \\ 0 & 0 \end{pmatrix},$$

with $\varepsilon' \in \mathcal{S}^{3\times 3}$*, the resulting quadratic form reduces to the one already proven to be non-symmetric polyconvex in the three-dimensional case in [11]. Consequently,* f *is also not symmetric polyconvex in higher dimensions.*

It is also important to note that the quadratic form studied by [14, 34, 33] for classical polyconvexity is not suitable as a counterexample in the symmetric case. Specifically, [33] have demonstrated that the form

$$Q(F) = F_{11}^2 + F_{22}^2 + F_{33}^2 - 2(F_{11}F_{22} + F_{11}F_{33} + F_{22}F_{33}) + F_{12}^2 + F_{31}^2 + F_{23}^2 + \sum_{i=4}^{d} F_{ii}^2,$$

defined on $\mathbb{R}^{d\times d}$ *is rank-one convex but not polyconvex. However, in the symmetric context, the quadratic form*

$$q(\varepsilon) = \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2 - 2(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) + \varepsilon_{12}^2 + \varepsilon_{13}^2 + \varepsilon_{23}^2 + \sum_{i=4}^{d} \varepsilon_{ii}^2$$

is symmetric polyconvex, as a consequence of Proposition 3.1, since

$$q(\varepsilon) + A : M_2(\varepsilon) = \sum_{i=1}^{d} \varepsilon_{ii}^2 - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{11}\varepsilon_{33} + \varepsilon_{22}\varepsilon_{33}) \geq 0, \quad \forall \varepsilon \in \mathcal{S}^{d\times d},$$

where $A = (A_{IJ})$*,* $A_{II} = 1$ *if* $I = (1,2)$*,* $(1,3)$*, or* $(2,3)$*, and* $A_{IJ} = 0$ *otherwise, is clearly S-positive semi-definite.*

3.4 Symmetric poly-affine functions in higher dimensions

We turn now our attention to symmetric poly-affine functions, we will show, as in the $3d$ case, that this notion is equivalent to affinity. We recall that a function $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ is said symmetric poly-affine if f and $-f$ are symmetric polyconvex.

Proposition 3.2 *Any symmetric poly-affine function is affine, i.e. if $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ is symmetric poly-affine, there exists a matrix $\beta \in \mathcal{S}^{d\times d}$ and $b \in \mathbb{R}$ such that*

$$f(\varepsilon) = \beta : \varepsilon + b, \;\; \forall \varepsilon \in \mathcal{S}^{d\times d} \tag{3.24}$$

Proof. Let $f : \mathcal{S}^{d\times d} \to \mathbb{R}$ be a symmetric poly-affine function, the function $\tilde{f}$ is then poly-affine. According to [19, Theorem 5.20, p. 179], there exist $\alpha = (\alpha_1, \dots, \alpha_k, \dots, \alpha_d) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{\binom{d}{2}\times\binom{d}{2}} \times \dots \times \mathbb{R}$, $\alpha_k \in \mathbb{R}^{\binom{d}{k}\times\binom{d}{k}}$ such that,

$$\tilde{f}(F) = \tilde{f}(0) + \sum_{k=1}^{d} \alpha_k : M_k(F), \quad \text{for all } F \in \mathbb{R}^{d\times d}.$$

Lets denote by $\tilde{g}$ the function $\tilde{g}(V) = \tilde{f}(0) + \sum_{k=1}^{d} \alpha_k : V_k$, $\forall\, V = (V_1, V_2, \dots, V_d)$, where $V_k \in \mathbb{R}^{\binom{d}{k}\times\binom{d}{k}}$ $\forall k = 1, 2, \dots, d$. Consider g the restriction of $\tilde{g}$ to the space $\mathcal{S}^{d\times d} \times \mathcal{S}^{\binom{d}{2}\times\binom{d}{2}} \dots \times \mathcal{S}^{\binom{d}{d-1}\times\binom{d}{d-1}} \times \mathbb{R}$ and

$$f(\varepsilon) = g(M(\varepsilon)) = \tilde{f}(0) + \sum_{k=1}^{d} \alpha_k : M_k(\varepsilon).$$

Since the function g is affine, then both g and $-g$ are convex. Apply the procedure of deleting the minor variables as in the proof of theorem 3.5, we get that g is independent on minors of order bigger than 2 and

$$g(\varepsilon, M_2(\varepsilon), \dots, \det \varepsilon) = g(\varepsilon, M_2(\varepsilon), 0, \dots, 0) \tag{3.25}$$

Hence,

$$g(\varepsilon, M_2(\varepsilon), \dots, \det \varepsilon) = \alpha_0 + \alpha_1 : \varepsilon + \alpha_2 : M_2(\varepsilon)$$

and the sub-differential of g with respect to the second variable is S-positive semi-definite, that is $\alpha_2 : X \boxtimes X \geq 0$ for any vector $X \in \mathbb{R}^{\binom{d}{2}}$. Applying the same procedure to the function $-g$, we end up with

$$\alpha_2 : X \boxtimes X = 0 \text{ for any } X \in \mathbb{R}^{\binom{d}{2}}.$$

Using Lemma 1.3, we deduce that $\alpha_2 = 0$. Consequently, taking $\beta = \alpha_1^s$, we get $f(\varepsilon) = \beta : \varepsilon + b$, so f is affine. ■

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