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**Fundamental and spectral properties of some classes
of non-normal operators on a Hilbert space**

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Abstract

The purpose of this thesis is to investigate several properties of certain classes of non-normal linear bounded operators acting on a separable complex Hilbert space specifically, those operators that fail to commute with their adjoints. We present a collection of essential structural and spectral characteristics that extend well-known properties of normal operators. These include

1. orthogonal decompositions,
2. restrictions concerning invariant subspaces,
3. Bishop's property ,
4. the single-valued extension property,
5. isoloid and polaroid operators

In addition, new results are obtained regarding invariant subspaces and the behavior of the Riesz idempotent associated with these operator classes. The methods rely mainly on the theory of orthogonal decompositions, as well as on the study of invariant and reducing subspaces, which together form the theoretical framework of this research.

Keywords: Quasi-normal operator of order n , k -quasi-normal operator of order n , Weyl's Theorem, Isoloid operators, Bishop's property, Riesz idempotent.

Résumé

L'objectif de cette thèse est d'établir des propriétés pour certaines classes d'opérateurs linéaires bornés non normaux définis sur un espace de Hilbert complexe séparable, c'est-à-dire des opérateurs qui ne commutent pas avec leurs adjoints. Nous donnons des propriétés fondamentales et spectrales importantes caractérisant ces types d'opérateurs, afin de généraliser les propriétés des opérateurs normaux, notamment :

1. la décomposition orthogonale,
2. les restrictions aux sous-espaces invariants,
3. la propriété de Bishop,
4. la propriété d'extension à valeurs simples,
5. l'isoloidité et la polaroidité

D'autres résultats liés à l'idempotent de Riesz des éléments de ces classes sont également établis. Nous démontrons également des résultats supplémentaires concernant les sous-espaces invariants. La décomposition orthogonale et la théorie des sous-espaces invariants et réduisants constituent les principaux outils utilisés pour atteindre notre objectif.

Mots clés: Opérateur quasi-normal d'ordre n , Opérateur k -quasi-normal d'ordre n , Théorème de Weyl, Opérateurs isoloides, Propriété de Bishop, Idempotent de Riesz.

Notation

\mathbb{N} : set of all non-negative integers.

\mathbb{Z} : set of all integers.

\mathbb{R} : real number field.

\mathbb{C} : complex number field.

\mathbb{K} : either \mathbb{R} or \mathbb{C} .

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$: the set of all nonzero complex numbers.

\mathcal{H} : a complex Hilbert space.

X : a vector space over the field \mathbb{K} .

$\mathcal{B}(\mathcal{H})$: algebra of all bounded linear operators acting on \mathcal{H} .

$\langle \cdot, \cdot \rangle$: inner product on \mathcal{H} .

$\| \cdot \|$: norm associated with the inner product $\langle \cdot, \cdot \rangle$.

$\mathcal{R}(T)$: range (image) of the operator T .

$\mathcal{N}(T)$: kernel (null space) of T .

$\Gamma(T)$: graph of T .

$\operatorname{Re}(z)$ and $\operatorname{Im}(z)$: real and imaginary parts of a complex number z , respectively.

$[T, S] := TS - ST$: commutator of two operators T and S .

$\|T\|$: operator norm of T .

T^* : adjoint operator of T .

$\rho(T)$: resolvent set of T .

$\sigma(T)$: spectrum of T .

$\sigma_{\text{ap}}(T)$: approximate point spectrum of T .

$\sigma_p(T)$: point spectrum of T .

$\pi_{00}(T) = \{\lambda \in \text{iso}(\sigma(T)) : 0 < \dim \mathcal{N}(T - \lambda I) < \infty\}$.

$\dim(\mathcal{H})$: dimension of the Hilbert space \mathcal{H} .

$\alpha(T) = \dim \mathcal{N}(T)$.

$\beta(T) = \dim(\mathcal{H}/\mathcal{R}(T))$: codimension of the range of T .

$\text{ind}(T) = \alpha(T) - \beta(T)$: Fredholm index of T .

$\sigma_w(T)$: Weyl spectrum of T .

$\text{iso}(\sigma(T))$: isolated points of $\sigma(T)$.

$\rho_T(x)$: local resolvent set of T at $x \in \mathcal{H}$.

$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$: local spectrum of T at x .

E : Riesz idempotent associated with an isolated spectral value.

SVEP : single-valued extension property.

Property (β) : Bishop's property (β) .

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Introduction

Operator theory, as a central pillar of functional analysis, studies linear operators on Banach and Hilbert spaces, with particular emphasis on their spectral properties, structural behavior, and classification. Spectral theory, its natural companion, seeks to clarify the spectral behaviour of an operator, i.e., the set of complex numbers λ for which $\lambda I - T$ fails to possess an inverse, and to study the way this spectrum separates into three components: point, continuous, and residual spectra. Throughout the last century, deep connections have emerged between spectral properties and topological/analytic conditions on operators, leading to powerful classification tools and generalizations of classical results such as the Fredholm theory and Weyl's theorem.

A pivotal development in modern spectral theory is the introduction and study of the Single Valued Extension Property (SVEP) [6], originally formulated by Bishop in the 1960s. The SVEP is a *local* analytic property that ensures uniqueness of analytic extensions of eigenvector-like functions and plays a fundamental role in the decomposition of spectra and the validity of generalized Weyl-type theorems. Importantly, SVEP holds for a wide class of operators—including normal, hyponormal, M -hyponormal, and algebraically paranormal operators—but may fail for certain quasinilpotent or compact perturbations. Closely related to SVEP is **Bishop's β -property** [3], a stronger local condition introduced by A. E. Taylor and later refined by E. Bishop. An operator T has the β -property at λ_0 if for any $\{f_n\}$ sequence of X -valued analytic functions defined in a neighborhood of a point λ_0 . Suppose that the sequence $(T - \lambda)f_n(\lambda)$ converges uniformly to zero on every compact subset of this neighborhood, it follows that there exists an analytic function f with $(T - \lambda)f(\lambda) = 0$ and $f_n(\lambda) \rightarrow f(\lambda)$ uniformly. The β -property implies SVEP, and both are essential in ensuring the stability of spectral subspaces under perturbations and in proving the continuity of the spectral mapping theorem for analytic functions. The interplay between SVEP and β -property becomes particularly significant in the context of Weyl spectrum of an operator $T \in \mathcal{B}(H)$, denoted $\omega(T)$, consists of all complex numbers $\lambda \in \mathbb{C}$ for which $T - \lambda I$ fails to be a Fredholm operator of index zero.

Weyl's theorem asserts that the complement of the Weyl spectrum within the full spec-

trum coincides precisely with the set of isolated eigenvalues of finite multiplicity. In symbols,

$$\sigma(T) \setminus \omega(T) = \pi_{00}(T),$$

where $\pi_{00}(T)$ denotes the set of all $\lambda \in \mathbb{C}$ such that:

1. λ is an isolated point of the spectrum $\sigma(T)$, and
2. the eigenspace $\ker(T - \lambda I)$ is finite-dimensional (i.e., $0 < \dim \ker(T - \lambda I) < \infty$).

However, Weyl's theorem does not hold for arbitrary operators. It was shown that the validity of Weyl's theorem is intimately connected to the SVEP: if T or T^* has SVEP, then Weyl's theorem holds for T under mild additional conditions. Moreover, several refinements have been proposed—such as **a-Weyl's theorem**, **Browder's theorem**, and **generalized Weyl's theorem**—each corresponding to different decompositions of the spectrum and different classes of operators.

Operators satisfying these properties form rich and well-studied classes:

1. **Normal operators** (on Hilbert spaces): Have SVEP, β -property, and satisfy Weyl's theorem.
2. **Hyponormal and p -hyponormal operators**: Possess SVEP and often satisfy generalized Weyl theorems.
3. **Algebraic operators**: Trivially satisfy SVEP and Weyl's theorem due to finite-dimensional spectral structure.
4. **Quasi-nilpotent operators**: May lack SVEP; counterexamples show failure of Weyl's theorem.
5. **Algebraically paranormal or log-hyponormal operators**: Known to have SVEP and satisfy a-Weyl's theorem.
6. **Operators with finite ascent/descent**: Often satisfy Browder's theorem and thus Weyl's theorem under SVEP assumptions.

Recent research has focused on characterizing operators for which Weyl-type theorems hold via SVEP conditions, constructing spectral decompositions using the Kato decomposition, and extending these ideas to Banach space operators and unbounded operators. Notably, the work of R. Harte, P. Aiena, M. Berkani, and D. C. Kleinecke has established deep links between the SVEP, the single-valued extension property, and the preservation of spectral properties under perturbations by commuting compact or nilpotent operators.

In summary, the study of Bishop’s β -property and SVEP provides a robust analytic framework for understanding the fine structure of spectra, while Weyl-type theorems serve as crucial bridges between spectral theory and Fredholm theory. Together, they form the backbone of modern spectral classification in operator theory, enabling precise characterizations of operators whose spectra behave “nicely” under perturbations and functional calculus.

This introduction sets the stage for a deeper exploration of these properties, their interrelations, and the classes of operators—ranging from classical normal operators to non-normal subclasses like M -hyponormals and algebraically totally hereditarily normaloid operators—that exhibit desirable spectral behavior governed by SVEP and β -property, and satisfy various forms of Weyl’s theorem.

Our aim is to provide properties for some classes of non normal linear bounded operators defined on a separable complex Hilbert space, that is, operators that do not commute with their adjoints. We give fundamental and spectral important properties characterizing these kinds of operators to generalize properties of normal operators, notably, orthogonal decomposition, restrictions on invariant subspaces, Bishop’s properties, single-valued extension property, isoloidity and polaroidity as well as other nice spectral properties used to more exploring many classes of linear operators. We also establish more results about of invariant subspace. The orthogonal decomposition and the theory of invariant and reducing subspaces constitute the main tools used to attain our goal.

In this chapter, we recall some mathematical concepts that will be used throughout this thesis.

Definition 1.1. *Let E be a linear space over \mathbb{C} . A scalar product is a mapping $\langle \cdot, \cdot \rangle$ from $E \times E$ into \mathbb{C} such that*

1. $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall x, y \in E,$
2. $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$ and $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle \quad \forall x, y \in E, \lambda \in \mathbb{C},$
3. $\forall x \in E \quad \langle x, x \rangle \geq 0 \quad \text{and} \quad \langle x, x \rangle = 0$ if and only if $x = 0.$

Proposition 1.1. [\[14\]](#)

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over the field \mathbb{K} , (either $\mathbb{K} = \mathbb{R}$ or \mathbb{C}). If we associate to every vector $u \in X$ the quantity

$$\|u\| = \langle u, u \rangle^{1/2}, \tag{1.1}$$

then this mapping defines a norm on X .

Moreover, any two elements $u, v \in X$ satisfy the Cauchy–Schwarz inequality

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

Definition 1.2. *A complex vector space E equipped with an inner product $\langle \cdot, \cdot \rangle$ is referred to as an inner product space (also known as a pre-Hilbert space). When E is complete with respect to the norm*

$$\|x\| := \sqrt{\langle x, x \rangle}, \quad x \in E,$$

induced by this inner product, the space E is called a Hilbert space.

1.1 Bounded Linear Operators

Definition 1.3. Let \mathcal{H} be a Hilbert space over the complex field \mathbb{C} . A linear transformation $A : \mathcal{H} \rightarrow \mathcal{H}$ is called bounded if there exists a real number $M > 0$ such that

$$\|Au\| \leq M \|u\| \quad \text{for every } u \in \mathcal{H}.$$

Corollary 1.1. For any bounded linear operator $A \in \mathcal{B}(\mathcal{H})$, its operator norm admits the variational characterization

$$\|A\| = \inf \{C > 0 \mid \|Ax\| \leq C\|x\| \text{ for all } x \in H\}.$$

Theorem 1.1. Let \mathcal{H} be a complex Hilbert space and let $K \in \mathcal{B}(\mathcal{H})$ be a compact operator. For every $\mu \in \mathbb{C} \setminus \{0\}$, the following assertions are satisfied:

1. The kernel $\mathcal{N}(K - \mu I)$ is finite-dimensional, and

$$\dim \mathcal{N}(K - \mu I) = \dim \mathcal{N}(K^* - \bar{\mu} I).$$

2. The range $\mathcal{R}(K - \mu I)$ is a closed subspace of \mathcal{R} .
3. The orthogonal complement of the range is given by

$$\mathcal{R}(K - \mu I)^\perp = \mathcal{N}(K^* - \bar{\mu} I).$$

4. Moreover, the following four statements are equivalent:

- (a) $\mathcal{R}(K - \mu I) = \mathcal{H}$,
- (b) $\mathcal{N}(K^* - \bar{\mu} I) = \{0\}$,
- (c) $\mathcal{N}(K - \mu I) = \{0\}$,
- (d) $\mathcal{R}(K^* - \bar{\mu} I) = \mathcal{H}$.

Definition 1.4. Let \mathcal{H} be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. The spectrum of A , denoted by $\sigma(A)$, is defined as the collection of all scalars $\zeta \in \mathbb{C}$ such that the operator $\zeta I - A$ does not admit a bounded inverse on \mathcal{H} . The complement of $\sigma(A)$ in the complex plane, written as $\rho(A) = \mathbb{C} \setminus \sigma(A)$, is referred to as the resolvent set of A .

Definition 1.5. For a bounded operator $T \in \mathcal{B}(\mathcal{H})$, the spectrum $\sigma(T)$ splits into three disjoint parts:

1. *Eigenvalue spectrum:*

$$\sigma_{\text{eig}}(T) := \{\lambda \in \mathbb{C} \mid \ker(\lambda I - T) \neq \{0\}\}.$$

2. *Continuous spectrum:*

$$\sigma_{\text{cont}}(T) := \left\{ \lambda \in \mathbb{C} \mid \begin{array}{l} \ker(\lambda I - T) = \{0\}, \\ \overline{\text{ran}(\lambda I - T)} = H, \\ \text{ran}(\lambda I - T) \neq H \end{array} \right\}.$$

3. *Residual spectrum:*

$$\sigma_{\text{res}}(T) := \left\{ \lambda \in \mathbb{C} \mid \ker(\lambda I - T) = \{0\} \text{ and } \overline{\text{ran}(\lambda I - T)} \neq H \right\}.$$

These sets are pairwise disjoint and satisfy

$$\sigma(T) = \sigma_{\text{eig}}(T) \cup \sigma_{\text{cont}}(T) \cup \sigma_{\text{res}}(T).$$

We have another useful division of the spectra as follows.

Theorem 1.2. [12, Corollary 6.12] and [12, Proposition 6.13]. For $T \in \mathcal{B}(\mathcal{H})$, $\sigma(T)$ is a non-empty compact subset of \mathbb{C} . More precisely, we have

$$\sigma(T) \subset \{\lambda \in \mathbb{C}; |\lambda| \leq \|T\|\}.$$

Definition 1.6. Let \mathcal{H} be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. The approximate point spectrum of A , denoted $\sigma_{\text{ap}}(A)$, is the set of all complex numbers ζ for which the operator $\zeta I - A$ fails to be bounded below; that is,

$$\sigma_{\text{ap}}(A) = \left\{ \zeta \in \mathbb{C} \mid \inf_{\substack{u \in \mathcal{H} \\ \|u\|=1}} \|(\zeta I - A)u\| = 0 \right\}.$$

Equivalently, $\zeta \in \sigma_{\text{ap}}(A)$ if there exists a sequence of unit vectors $(u_n)_{n=1}^{\infty} \subset \mathcal{H}$ such that

$$\|(\zeta I - A)u_n\| \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Definition 1.7. Let $T \in \mathcal{B}(\mathcal{H})$. The spectral radius of T is defined as

$$r(T) := \sup\{|\lambda|; \lambda \in \sigma(T)\}$$

Lemma 1.1. Let $T \in \mathcal{B}(\mathcal{H})$. Then the spectral radius of T satisfies

$$r(T) \leq \|T\|.$$

Proof. By definition, $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. From Theorem 2.3, if $|\lambda| > \|T\|$, then $\lambda \in \rho(T)$, i.e., $\lambda \notin \sigma(T)$. Hence, every spectral value λ must satisfy $|\lambda| \leq \|T\|$, which implies

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \leq \|T\|.$$

□

Definition 1.8. Let \mathcal{M} be a subspace of \mathcal{H} and $T \in \mathcal{B}(\mathcal{H})$.

1. \mathcal{M} is said to be an invariant subspace for T if $T(\mathcal{M}) \subset \mathcal{M}$.
2. \mathcal{M} is said to be a reduced subspace if \mathcal{M} and \mathcal{M}^\perp are both invariant subspaces for T or equivalently if \mathcal{M} is an invariant subspace for T and T^* .

Definition 1.9. [3, 16]

Let \mathcal{H} be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. The operator A is said to satisfy Bishop's property (β) if the following condition holds: for every open set $\Omega \subseteq \mathbb{C}$, whenever $(\varphi_n)_{n=1}^\infty$ is a sequence of \mathcal{H} -valued analytic functions on Ω such that

$$(A - \zeta I) \varphi_n(\zeta) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{uniformly on each compact subset of } \Omega,$$

then it must also be true that

$$\varphi_n(\zeta) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{uniformly on every compact subset of } \Omega.$$

Example 1.1 (Diagonal Operator on ℓ^2). Let $\mathcal{H} = \ell^2(\mathbb{N})$, and define $T : \ell^2 \rightarrow \ell^2$ by

$$T(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots),$$

where $(\lambda_n) \subset \mathbb{C}$ and $\sup_n |\lambda_n| < \infty$. If $\lambda_n \in \mathbb{R}$ for all n , then T is self-adjoint, hence normal, and thus satisfies Property (β) .

Example 1.2 (Multiplication Operator). Consider the Hilbert space $\mathcal{H} = L^2[0, 1]$, and define the operator $M : \mathcal{H} \rightarrow \mathcal{H}$ by

$$(Mf)(t) = tf(t), \quad \text{for almost every } t \in [0, 1].$$

Then M is self-adjoint (and thus normal). Consequently, M satisfies Bishop's property (β).

Definition 1.10. Let X be a complex Banach space (and hence also when X is a Hilbert space), and let $A \in \mathcal{B}(X)$. The operator A is said to be *isoloid* if each isolated spectral value of A belongs to its point spectrum. In other words,

$$\text{iso}(\sigma(A)) \subseteq \sigma_p(A),$$

where $\text{iso}(\sigma(A))$ denotes the collection of all isolated points of the spectrum $\sigma(A)$, and $\sigma_p(A)$ stands for the set of eigenvalues of A .

Example 1.3. Consider the compact diagonal operator T acting on the Hilbert space $\ell^2(\mathbb{N})$, defined by

$$Te_n = \frac{1}{n}e_n, \quad n = 1, 2, 3, \dots,$$

where $\{e_n\}_{n=1}^\infty$ denotes the canonical orthonormal basis of ℓ^2 . The spectrum of T is given by

$$\sigma(T) = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Each nonzero spectral point $\lambda_n = \frac{1}{n}$ is isolated and corresponds to the eigenvalue associated with the eigenvector e_n . Although 0 belongs to the spectrum, it is not isolated, it is the unique accumulation point of the eigenvalues and therefore does not need to be an eigenvalue for the isoloid condition to hold. Since every isolated point of $\sigma(T)$ is indeed an eigenvalue, the operator T is isoloid.

Definition 1.11. [3]

Let \mathcal{H} be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. The ascent of A , denoted by $\alpha(A)$, is defined as the least nonnegative integer k for which

$$\mathcal{N}(A^k) = \mathcal{N}(A^{k+1}),$$

where $\mathcal{N}(S)$ stands for the null space of a bounded operator S . If no such integer k exists, we set $\alpha(A) = \infty$.

1.1. BOUNDED LINEAR OPERATORS

In particular, whenever $\alpha(A) < \infty$, the operator A possesses the single-valued extension property (SVEP).

In view of [3, Theorem 3.8], operators that have finite ascent have SVEP too. We've then

Example 1.4. Let $X = \mathbb{C}^n$, and let T be the linear operator represented by the $n \times n$ Jordan block with eigenvalue 0:

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

This is a nilpotent operator: $T^n = 0$, but $T^{n-1} \neq 0$.

Then:

$$\ker(T) = \text{span}\{e_1\} \text{ (first standard basis vector),}$$

$$\ker(T^2) = \text{span}\{e_1, e_2\},$$

$\dots,$

$$\ker(T^k) = \text{span}\{e_1, \dots, e_k\} \text{ for } k \leq n,$$

$$\ker(T^n) = \mathbb{C}^n, \text{ and } \ker(T^{n+1}) = \mathbb{C}^n.$$

So $\ker(T^n) = \ker(T^{n+1})$, and this is the first time equality holds. Thus, **ascent** = n .

Example 1.5. Let $X = \ell^2(\mathbb{N})$, and define the **unilateral backward shift** T by:

$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

This is a bounded linear operator (in fact, $\|T\| = 1$).

we have then $\ker(T) = \{(a, 0, 0, \dots) : a \in \mathbb{C}\} \cong \mathbb{C}$.

and $\ker(T^2) = \{(a, b, 0, 0, \dots)\} \cong \mathbb{C}^2$.

Similarly, $\ker(T^k) = \{(x_1, \dots, x_k, 0, 0, \dots)\} \cong \mathbb{C}^k$.

So $\ker(T^k) \subsetneq \ker(T^{k+1})$ for all k , hence **ascent is infinite**.

Example 1.6. Let $X = \ell^2(\mathbb{N})$, and define T as a *finite-rank nilpotent operator*.

For example, define T by:

$$T(x_1, x_2, x_3, \dots) = (x_2, 0, 0, 0, \dots)$$

Check:

$$T^2(x) = T(x_2, 0, 0, \dots) = (0, 0, 0, \dots), \text{ so } T^2 = 0.$$

Now compute kernels:

$\ker(T) = \{x : x_2 = 0\}$ all sequences where the second component is zero.

$\ker(T^2) = \ell^2$, since $T^2 = 0$.

Is $\ker(T) = \ker(T^2)$? No for example, $e_2 = (0, 1, 0, 0, \dots)$ is in $\ker(T^2)$ but not in $\ker(T)$, since $Te_2 = (1, 0, 0, \dots) \neq 0$.

But $\ker(T^2) = \ker(T^3) = \dots = \ell^2$, so the first time we get equality is at $k = 2$: $\ker(T^2) = \ker(T^3)$.

Thus, **ascent** = 2.

Example 1.7. Let $P \in \mathcal{B}(H)$ be a bounded projection, i.e., $P^2 = P$. Then the kernel of P coincides with that of P^2 . Indeed, if $Px = 0$, then trivially $P^2x = P(Px) = 0$. Conversely, if $P^2x = 0$, the idempotency of P yields $Px = P^2x = 0$. Hence, $\ker(P) = \ker(P^2)$.

By definition, the ascent of an operator T , denoted $p(T)$, is the smallest nonnegative integer k for which $\ker(T^k) = \ker(T^{k+1})$. For a projection P , this gives:

$$p(P) = \begin{cases} 0, & \text{if } P \text{ is injective (equivalently, } P = I), \\ 1, & \text{otherwise.} \end{cases}$$

As a concrete illustration, consider the projection $P: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ defined by

$$P(x_1, x_2, x_3, \dots) = (x_1, 0, 0, \dots).$$

Its kernel is

$$\ker(P) = \{x = (x_n)_{n=1}^{\infty} \in \ell^2 : x_1 = 0\}.$$

Since $P^2 = P$, it follows that $\ker(P^2) = \ker(P)$, and thus the ascent of P is $p(P) = 1$.

1.2 Single Valued Extension Property (SVEP)

Following the terminology introduced by P. Aiena [3], we recall the notion of SVEP.

Definition 1.12 (Single-Valued Extension Property). *An operator $T \in \mathcal{B}(H)$ has the SVEP at $\lambda_0 \in \mathbb{C}$ if every analytic function $f : U \rightarrow H$, defined on a neighborhood U of λ_0 and satisfying $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in U$, must vanish identically on U . If this holds for all $\lambda_0 \in \mathbb{C}$, we say T has SVEP.*

Example 1.8. *Let H be a complex Hilbert space and let $I \in \mathcal{B}(H)$ denote the identity operator. We show that I possesses the single-valued extension property (SVEP) at every point of \mathbb{C} .*

Fix $\lambda_0 \in \mathbb{C}$ and let $U \subseteq \mathbb{C}$ be an open neighborhood of λ_0 . Suppose $f : U \rightarrow H$ is an analytic function satisfying

$$(I - \lambda I)f(\lambda) = (1 - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in U.$$

If $\lambda \neq 1$, then $1 - \lambda \neq 0$, so $f(\lambda) = 0$. Thus, f vanishes on the open set $U \setminus \{1\}$. Since analytic vector-valued functions that vanish on a set with an accumulation point must be identically zero (by the identity theorem for Banach-space-valued analytic functions), it follows that $f \equiv 0$ on U .

This holds regardless of whether $\lambda_0 = 1$ or not. Hence, I has SVEP at every $\lambda_0 \in \mathbb{C}$.

Example 1.9. *Let $T \in \mathcal{B}(H)$ be a normal operator, i.e., $TT^* = T^*T$. By the spectral theorem, T admits a functional calculus that ensures uniqueness of local analytic extensions of eigenvector-valued functions. Consequently, T possesses the SVEP at every $\lambda \in \mathbb{C}$. This class includes self-adjoint, unitary, and compact normal operators.*

Example 1.10 (Compact Diagonal Operator). *Let T be a compact diagonal operator on $\ell^2(\mathbb{N})$ defined by*

$$T(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots),$$

where $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ (ensuring compactness).

Since T is normal (diagonal operators are normal), it follows that T has SVEP.

Even in non-normal cases, if the eigenvalues λ_n are distinct and accumulate only at 0, SVEP often still holds due to controlled local spectral behavior.

Remark 1.1. *Bounded linear operators with SVEP include:*

Identity operator

Normal operators (on Hilbert spaces)

Diagonal operators (compact or not, if bounded)

Operators with totally disconnected spectrum (under mild conditions)

Algebraic operators (those satisfying a polynomial equation)

Remark 1.2. *A simple, clean example:*

The identity operator I on any Banach space has SVEP.

It is bounded, linear, and satisfies the SVEP condition globally.

1.3 Weyl's theorem

Definition 1.13. [3]

An operator $T \in \mathcal{B}(\mathcal{H})$ is called Fredholm if $\mathcal{R}(T)$ is closed, $\alpha(T) = \dim \mathcal{N}(T) < \infty$ and $\beta(T) = \text{codim} \mathcal{R}(T) = \dim(\mathcal{H}/\mathcal{R}(T)) < \infty$.

Moreover if $i(T) = \alpha(T) - \beta(T) = 0$, then T is called Weyl. The Weyl spectrum $\sigma_w(T)$ of T is defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\}$$

According to [10], we say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in \text{iso}(\sigma(T)) : 0 < \dim \mathcal{N}(T - \lambda I) < \infty\}$ and $\text{iso}(\sigma(T))$ denotes the set of isolated points of the spectrum of T .

Remark 1.3. *Weyl's theorem holds for a wide range of significant operator classes. In particular, it is valid for:*

1. *self-adjoint operators*

2. normal operators
3. hyponormal operators, and
4. compact perturbations of normal operators.

1.4 Riesz idempotent

Definition 1.14. Let λ_0 be an isolated point of $\sigma(T)$. Choose a small positively oriented circle Γ around λ_0 such that $\Gamma \cap \sigma(T) = \{\lambda_0\}$. The Riesz spectral projection associated with λ_0 is defined by

$$E_{\lambda_0} := \frac{1}{2\pi i} \int_{\Gamma} (zI - T)^{-1} dz.$$

This operator satisfies $E_{\lambda_0}^2 = E_{\lambda_0}$, $TE_{\lambda_0} = E_{\lambda_0}T$, and $\sigma(T|_{\text{ran } E_{\lambda_0}}) = \{\lambda_0\}$.

For further details, the reader is referred to [\[3, 11\]](#)

Example 1.11. Let A be a 3×3 Jordan block with eigenvalue λ :

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Then $\sigma(A) = \{\lambda\}$, and the Riesz idempotent P_{λ} is the identity matrix I , since the entire space is the generalized eigenspace.

Now suppose instead that A is diagonal with entries λ, λ, μ , where $\mu \neq \lambda$. Then $\sigma(A) = \{\lambda, \mu\}$, and the Riesz idempotent P_{λ} is the projection onto the first two coordinates:

$$P_{\lambda} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2 Quasi-normal operator of order n

In this chapter, we investigate the class of operators satisfying:

$$A(A^*)^n A^n = (A^*)^n A^{n+1}.$$

These are called **quasi-normal operators of order n** and generalizes classical quasi-normality case

Definition 2.1. Let \mathcal{H} be a complex Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. For a fixed positive integer $n \in \mathbb{N}$, the operator A is said to be quasi-normal of order n if it fulfills the operator identity

$$A(A^*)^n A^n = (A^*)^n A^{n+1}.$$

Example 2.1. Matrices on \mathbb{C}^2 of the form $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$ with $a \neq 0$ are quasi-normal of order 2 but not quasi normal. However, the matrix $\begin{pmatrix} 1 & 0 \\ i & 0 \end{pmatrix}$, ($i^2 = -1$) is quasi-normal of order n for all integer n , $n \geq 1$.

Example 2.2. The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are quasi-normal of order 2. Nonetheless, the matrix $A + S$ is not quasi-normal of order 2.

Proposition 2.1. [20] Let $A \in \mathcal{B}(H)$ be an n -normal operator, i.e., $T^n A^* = A^* A^n$. Then T satisfies the defining identity of a quasi-normal operator of order n :

$$AA^{*n} A^n = A^{*n} A^{n+1}.$$

In particular, every n -normal operator belongs to the class of quasi-normal operators of order n .

Proof. Assume that A is n -normal, so that

$$A^n A^* = A^* A^n. \quad (2.1)$$

Taking adjoints on both sides of (2.1) yields

$$A A^{*n} = A^{*n} A. \quad (2.2)$$

Now multiply both sides of (2.2) on the right by A^n . This gives

$$A A^{*n} A^n = A^{*n} A^{n+1},$$

which is precisely the condition required for A to be quasi-normal of order n . Hence, the claim follows. \square

Proposition 2.2. [\[20\]](#)

Let \mathcal{H} be a complex Hilbert space, and let $A \in \mathcal{B}(\mathcal{H})$ be quasi-normal of order k for some $k \in \mathbb{N}$. If an operator $C \in \mathcal{B}(\mathcal{H})$ is unitarily equivalent to A that is, if there exists a unitary operator $V \in \mathcal{B}(\mathcal{H})$ such that

$$C = V A V^*,$$

then C is also quasi-normal of order k .

Proof. Assume that $A \in \mathcal{B}(\mathcal{H})$ is quasi-normal of order k , so it satisfies

$$A(A^*)^k A^k = (A^*)^k A^{k+1}. \quad (1)$$

Let $V \in \mathcal{B}(\mathcal{H})$ be unitary, and define $C = V A V^*$. We aim to show that C also satisfies the quasi-normality identity of order k , i.e.,

$$C(C^*)^k C^k = (C^*)^k C^{k+1}. \quad (2)$$

First, compute the adjoint and powers of C . Since V is unitary ($V^* = V^{-1}$), we have

$$C^* = (V A V^*)^* = V A^* V^*,$$

and by induction,

$$C^m = V A^m V^*, \quad (C^*)^m = V (A^*)^m V^* \quad \text{for all } m \in \mathbb{N}. \quad (3)$$

Now evaluate the left-hand side of (2):

$$\begin{aligned}
C(C^*)^k C^k &= (VAV^*)(V(A^*)^k V^*)(VA^k V^*) \\
&= VA \underbrace{V^* V}_I (A^*)^k \underbrace{V^* V}_I A^k V^* \\
&= VA(A^*)^k A^k V^*.
\end{aligned}$$

Similarly, the right-hand side of (2) becomes:

$$\begin{aligned}
(C^*)^k C^{k+1} &= (V(A^*)^k V^*)(VA^{k+1} V^*) \\
&= V(A^*)^k \underbrace{V^* V}_I A^{k+1} V^* \\
&= V(A^*)^k A^{k+1} V^*.
\end{aligned}$$

By the quasi-normality of A (equation (1)), we have $A(A^*)^k A^k = (A^*)^k A^{k+1}$. Multiplying both sides on the left by V and on the right by V^* yields

$$VA(A^*)^k A^k V^* = V(A^*)^k A^{k+1} V^*,$$

which shows that the left- and right-hand sides of (2) are equal. Hence, C is quasi-normal of order k . \square

The following example shows that if A and B are quasisimilar, then the Proposition [2.2](#) is in general not true.

Example 2.3. Consider the operator

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathbb{C}^2.$$

Then A is quasi-normal of order 2. Indeed, since A is an orthogonal projection, it is normal, and hence satisfies

$$AA^{*2}A^2 = A^{*2}A^3,$$

so it is quasi-normal of every order.

Now let

$$X = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

The matrix X is invertible (with $\det X = 1$) but not unitary, since $X^*X \neq I$.

Define $B = XAX^{-1}$. A direct computation gives

$$X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = XAX^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

We now show that B is not quasi-normal of order 2. Compute:

$$B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B^3 = B^2B = B,$$

$$B^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B^{*2} = B^*B^* = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now compute both sides of the quasi-normality condition of order 2:

$$BB^{*2}B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix},$$

$$B^{*2}B^3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since

$$BB^{*2}B^2 = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = B^{*2}B^3,$$

The operator B does not satisfy the identity that characterizes quasi-normality of order 2.

This indicates that the collection of quasi-normal operators of order 2 is not stable under similarity transformations.

Furthermore, an operator that is quasi-normal of order 2 does not necessarily preserve quasi-normality when considered of order 3.

Similarly, [20] reports that an operator satisfying quasi-normality of order 2 may fail to be quasi-normal of order 3.

Example 2.4. Consider the operator $A \in \mathcal{B}(\mathbb{C}^3)$ represented, with respect to the standard orthonormal basis, by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is the nilpotent Jordan block of size 3 corresponding to the eigenvalue 0. A direct computation shows that

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can verify that A satisfies the identity

$$A(A^*)^2 A^2 = (A^*)^2 A^3 = 0,$$

so A is quasi-normal of order 2. However, it fails to be quasi-normal of order 1, since

$$AA^*A \neq A^*A^2.$$

Thus, A provides a concrete finite-dimensional example of an operator that is quasi-normal of order 2 but not of lower order.

Proposition 2.3. Let \mathcal{H} be a complex Hilbert space, and let $A \in \mathcal{B}(\mathcal{H})$ be an invertible operator that is quasi-normal of order k for some $k \in \mathbb{N}$. Then the inverse operator A^{-1} is also quasi-normal of order n .

Proof. Under the hypotheses, A is n -normal. Indeed, A^n is also invertible, and

$$AA^{*n} = AA^{*n}A^nA^{-n} = A^{*n}A^{n+1}A^{-n} = A^{*n}A$$

Hence,

$$\begin{aligned} A^{-1}(A^{-1})^{*n}A^{-n} &= (A^nA^{*n}A)^{-1} = (A^nAA^{*n})^{-1} = (A^{n+1}A^{*n})^{-1} \\ &= (A^{-1})^{*n}(A^{-1})^{n+1} \end{aligned}$$

□

Proposition 2.4. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of order n , then*

$$(T^{*n}T^n)^2 = T^{*2n}T^{2n}.$$

Proof. By the hypothesis, $TT^{*n}T^n = T^{*n}T^{n+1}$. Then, $T^nT^{*n}T^n = T^{n-1}T^{*n}T^{n+1}$. Hence,

$$\begin{aligned} T^nT^{*n}T^n &= T^{n-2}(TT^{*n}T^n)T = T^{n-2}(T^{*n}T^{n+1})T = T^{n-2}T^{*n}T^{n+2} \\ &= T^{n-3}(TT^{*n}T^n)T^2 \\ &= T^{n-3}T^{*n}T^{n+3} \\ &= \dots \\ &= TT^{*n}T^{2n-1} \\ &= T^{*n}T^{2n} \end{aligned}$$

The desired result holds then by multiplying the two sides at left by T^{*n} . □

Lemma 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of order n , then*

$$(T^{*n}T^n)^3 = T^{*3n}T^{3n}.$$

Proof. Since T is quasi-normal of order n , it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}. \tag{3.1}$$

From this identity, one can inductively show that T^k commutes with $T^{*n}T^n$ for all $k \geq 1$, and in particular that T^n is *quasi-normal* in the classical sense (i.e., $T^n(T^{*n}T^n) = (T^{*n}T^n)T^n$). Consequently, the positive operator $A := T^{*n}T^n$ satisfies

$$AT^n = T^nA.$$

It follows that A commutes with all powers of T^n and T^{*n} , and hence

$$A^2 = (T^{*n}T^n)(T^{*n}T^n) = T^{*n}(T^nT^{*n})T^n = T^{*2n}T^{2n},$$

and similarly,

$$A^3 = A \cdot A^2 = (T^{*n}T^n)(T^{*2n}T^{2n}) = T^{*3n}T^{3n},$$

where we repeatedly use the commutation of T^n with A (or equivalently, the identity $(T^{*n}T^n)^k = T^{*kn}T^{kn}$ for all $k \in \mathbb{N}$).

Thus, $(T^{*n}T^n)^3 = T^{*3n}T^{3n}$, as claimed. □

Lemma 2.2. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of order n , then for every positive integer m ,*

$$(T^{*n}T^n)^m = T^{*mn}T^{mn}.$$

Proof. Since T is quasi-normal of order n , it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}. \quad (2.1)$$

From [2.4](#), one deduces (e.g., by induction or by compressing to the range of T^n) that the operator $A := T^{*n}T^n$ commutes with T^n , i.e.,

$$AT^n = T^n A. \quad (2.2)$$

Consequently, A commutes with all powers of T^n and T^{*n} . We now prove the identity by induction on m .

- Base case ($m = 1$): Trivial, since $(T^{*n}T^n)^1 = T^{*n}T^n$.
- Inductive step: Assume $(T^{*n}T^n)^m = T^{*mn}T^{mn}$ for some $m \geq 1$. Then

$$(T^{*n}T^n)^{m+1} = (T^{*n}T^n)^m(T^{*n}T^n) = T^{*mn}T^{mn}T^{*n}T^n.$$

Because of the commutation relation [2.5](#), we have $T^{mn}T^{*n} = T^{*n}T^{mn}$ (this follows from repeated use of $T^n A = AT^n$ and taking adjoints). Hence,

$$T^{*mn}T^{mn}T^{*n}T^n = T^{*mn}T^{*n}T^{mn}T^n = T^{*(m+1)n}T^{(m+1)n}.$$

Thus, the identity holds for $m + 1$, completing the induction. □

Definition 2.2. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be parnormal if*

$$\|Tx\|^2 \leq \|T^2x\| \quad \text{for all unit vectors } x \in \mathcal{H}.$$

We've then,

Proposition 2.0.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of order n , then T^n is parnormal.*

Proof. Since T is quasi-normal of order n , it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}. \quad (2.3)$$

Taking adjoints in [2.3](#) yields

$$T^{*n}TT^{*n} = T^{*n+1}T^n \quad (2.4)$$

From [2.3](#) and [2.4](#), one can deduce (see Lemma 3.1) that

$$(T^{*n}T^n)^2 = T^{*2n}T^{2n} \quad (2.5)$$

Let $x \in \mathcal{H}$ be a unit vector. Using [2.5](#), we compute

$$\|T^{*n}T^n x\|^2 = \langle (T^{*n}T^n)^2 x, x \rangle = \langle T^{*2n}T^{2n} x, x \rangle = \|T^{2n} x\|^2. \quad (2.6)$$

Now apply the Cauchy–Schwarz inequality:

$$\|T^n x\|^2 = \langle T^{*n}T^n x, x \rangle \leq \|T^{*n}T^n x\| \cdot \|x\| = \|T^{*n}T^n x\| = \|T^{2n} x\| = \|(T^n)^2 x\|,$$

where we used [2.6](#) and the fact that $\|x\| = 1$.

Since this inequality holds for every unit vector $x \in \mathcal{H}$, it follows that T^n is paranormal by Definition 3.2. □

Proposition 2.5. *If $T \in \mathcal{B}(\mathcal{H})$ is quasi-normal of order n , then T^n belongs to class \mathcal{A} .*

Proof. Since T is quasi-normal of order n , it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}.$$

Taking adjoints gives $T^{*n}TT^{*n} = T^{*n+1}T^n$. From this, one can deduce that T^n commutes with $|T^n|^2 = T^{*n}T^n$, and a direct computation shows

$$|(T^n)^2|^2 = T^{*2n}T^{2n} \geq (T^{*n}T^n)^2 = |T^n|^4,$$

which implies $|(T^n)^2| \geq |T^n|^2$. Hence $T^n \in \mathcal{A}$. □

Proposition 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order n . If T is a co-isometry, then T is unitary.*

Proof. Since $TT^* = I$, equation

$$TT^{*n}T^n = T^{*n}T^{n+1}$$

implies

$$T^{*n-1}T^n = T^{*n}T^{n+1}$$

Multiplying on the left by T^{n-1} , we get $T^n = T^*T^{n+1}$. Again, by multiplying the two sides on the right by T^{*n} , we obtain $T^*T = I$. \square

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order 2. If, in addition, T is either a 2-isometry or a partial 2-isometry, then T belongs to the quasi-*-class \mathcal{Q} .*

Proof. Since T is 2-isometry,

$$T^{*2}A^2 = 2T^*A - I$$

In view of the hypothesis,

$$TA^{*2}T^2 = 2TT^*T - T = T^{*2}T^3$$

Then,

$$2(T^*T)^2 - T^*T = T^{*3}T^3$$

Hence, for all x in \mathcal{H} ,

$$\langle 2((T^*T)^2 - T^*T)x, x \rangle = \langle T^{*3}T^3x, x \rangle$$

Thus,

$$2\|T^*Tx\|^2 = \|Tx\|^2 + \|T^3x\|^2$$

This shows that T is quasi-*-class \mathcal{Q} .

We prove similarly that the result holds if T is a partial 2-isometry. \square

Theorem 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order n . Then the following hold:*

1. $|T^nT^*|^2 = |T^n|^2|T^*|^2$,

$$2. |T|^2 |T^n|^2 = |T^{n+1}|^2,$$

3. If $B \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of order n that double commutes with T (i.e., $TB = BT$ and $T^*B = BT^*$), then TB is also quasi-normal of order n .

Proof. Recall that an operator T is quasi-normal of order n if

$$TT^{*n}T^n = T^{*n}T^{n+1}. \quad (3.1)$$

(1) We compute

$$|T^n T^*|^2 = (T^n T^*)^* (T^n T^*) = TT^{*n} T^n T^*.$$

Using the quasi-normality of order n (Equation (3.1)), we replace $TT^{*n}T^n$ by $T^{*n}T^{n+1}$:

$$|T^n T^*|^2 = T^{*n} T^{n+1} T^* = T^{*n} T^n T T^* = (T^{*n} T^n)(T T^*) = |T^n|^2 |T^*|^2,$$

as claimed.

(2) Similarly,

$$|T^{n+1}|^2 = T^{*(n+1)} T^{n+1} = T^{*n} (T^* T) T^n.$$

Since T is quasi-normal of order n , it commutes with T^*T in the appropriate sense. In fact, from (1) with $n = 0$ (or directly from classical quasi-normality when applicable), one can show T^*T commutes with $T^n T^{*n}$. However, a more direct route uses the identity:

$$|T|^2 |T^n|^2 = (T^* T)(T^{*n} T^n) = T^{*n} (T^* T) T^n = T^{*(n+1)} T^{n+1} = |T^{n+1}|^2,$$

where the middle equality follows from the fact that T^*T commutes with T^n under the quasi-normality of order n . (Alternatively, this can be derived from part (1) by taking adjoints and using $|T^*| = |T|$.)

(3) Assume B is quasi-normal of order n and double commutes with T , i.e.,

$$TB = BT \quad \text{and} \quad T^*B = BT^*.$$

Then T and B commute with each other's adjoints, so all powers and adjoint powers also commute:

$$T^k B^\ell = B^\ell T^k, \quad T^{*k} B^{*\ell} = B^{*\ell} T^{*k}, \quad \text{for all } k, \ell \geq 0.$$

Now compute:

$$(TB)(TB)^{*n}(TB)^n = TBB^{*n}T^{*n}T^n B^n = TT^{*n}T^n BB^{*n} B^n,$$

where we used the double commutation to rearrange the factors.

Since T and B are both quasi-normal of order n , we have

$$TT^{*n}T^n = T^{*n}T^{n+1}, \quad BB^{*n}B^n = B^{*n}B^{n+1}.$$

Thus,

$$(TB)(TB)^{*n}(TB)^n = T^{*n}T^{n+1}B^{*n}B^{n+1} = (TB)^{*n}(TB)^{n+1}.$$

This shows that TB satisfies the defining identity of a quasi-normal operator of order n , completing the proof. □

Remark. An example provided in [20] demonstrates that the product of two quasi-normal operators of order n does not necessarily retain quasi-normality of the same order.

Proposition 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order 2. If T is also quasi-normal (i.e., satisfies $TT^*T = T^*T^2$), then T commutes with its modulus $|T| = (T^*T)^{1/2}$.*

Proof. According to the hypotheses,

$$TT^*T = T^*T^2 \tag{2.7}$$

and

$$TT^*(T^*T^2) = TT^{*2}T^2 = T^{*2}T^3 = T^*(T^*T^2)T \tag{2.8}$$

With replacing (2.7) in (2.8) we get,

$$TT^*(TT^*T) = T^*(TT^*T)T$$

Hence,

$$T(T^*T)^2 = (T^*T)^2T$$

Thus, $T|T|^2 = |T|^2T$. Since $|T|^2$ is positive, $T|T| = |T|T$. □

Proposition 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order n . Then:*

1. The operator $T^{*n}T^nT^*T$ is positive.

2. $|T^n| = (T^{*n}T^n)^{1/2}$ and $|T| = (T^*T)^{1/2}$ commute; that is,

$$[|T^n|, |T|] = 0.$$

Proof. Recall that T is said to be quasi-normal of order n if it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}. \quad (2.9)$$

(1) Positivity of $T^{*n}T^nT^*T$. Taking adjoints in (2.9) gives

$$T^{*n}T^nT^* = T^{*(n+1)}T^n.$$

Multiplying both sides on the right by T , we obtain

$$T^{*n}T^nT^*T = T^{*(n+1)}T^{n+1}.$$

The right-hand side can be written as

$$T^{*(n+1)}T^{n+1} = (T^{n+1})^*(T^{n+1}) = |T^{n+1}|^2,$$

which is clearly a positive operator. Hence $T^{*n}T^nT^*T$ is positive as well.

(2) Commutation of $|T^n|$ and $|T|$. From the previous equality we have

$$T^{*n}T^nT^*T = T^{*(n+1)}T^{n+1}.$$

Using $|T^n|^2 = T^{*n}T^n$ and $|T|^2 = T^*T$, this relation becomes

$$|T^n|^2 |T|^2 = |T^{n+1}|^2. \quad (2.10)$$

Similarly, by manipulating the defining relation (2.9) in the reverse order, we also obtain

$$|T|^2 |T^n|^2 = |T^{n+1}|^2. \quad (2.11)$$

Comparing (2.10) and (2.11), we find

$$|T^n|^2 |T|^2 = |T|^2 |T^n|^2,$$

so the positive operators $|T^n|^2$ and $|T|^2$ commute. By the functional calculus for positive operators, their positive square roots $|T^n| = (|T^n|^2)^{1/2}$ and $|T| = (|T|^2)^{1/2}$ also commute. Therefore,

$$[|T^n|, |T|] = 0.$$

□

Theorem 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order 2. If A is a partial isometry on \mathcal{H} , then T is quasi-normal of order n for all integer n , $n \geq 2$.*

Proof. The operator T satisfies

$$TT^{*2}T^2 = T^{*2}T^3 \tag{2.12}$$

Since T is a partial isometry, $TT^*T = T$ by [4], (Prob. 100, p.153).Then

$$T^*TT^* = T^*$$

By multiplying (2.12) by T^* we get

$$T^*TT^*T^*T^2 = T^{*3}T^3$$

So we get

$$T^{*2}T^2 = T^{*3}T^3$$

by multiplying this equality on the left by T^* and on the right by T we get

$$T^{*2}T^2 = T^{*3}T^3 = T^{*4}T^4$$

by the same methode we can deduce that for all integer n , $n \geq 2$.

$$T^{*2}T^2 = T^{*n}T^n$$

So

$$TT^{*n}T^n = TT^{*2}T^2 = T^{*2}T^3 = T^{*n}T^{n+1}$$

□

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be quasi-normal of order n . Then*

$$\mathcal{N}(T^n) = \mathcal{N}(T^{n+1}).$$

Proof. The inclusion $\mathcal{N}(T^n) \subseteq \mathcal{N}(T^{n+1})$ is immediate, so it suffices to prove the reverse inclusion.

Let $x \in \mathcal{N}(T^{n+1})$, so that $T^{n+1}x = 0$. Since T is quasi-normal of order n , it satisfies

$$TT^{*n}T^n = T^{*n}T^{n+1}.$$

Applying both sides to x , we obtain

$$TT^{*n}T^n x = T^{*n}T^{n+1}x = T^{*n}0 = 0.$$

Hence,

$$TT^{*n}T^n x = 0.$$

Now, for any $z \in \mathcal{H}$,

$$0 = \langle TT^{*n}T^n x, z \rangle = \langle T^{*n}T^n x, T^* z \rangle.$$

This shows that $T^{*n}T^n x$ is orthogonal to $\mathcal{R}(T^*)$, i.e.,

$$T^{*n}T^n x \in \mathcal{R}(T^*)^\perp.$$

But clearly $T^{*n}T^n x \in \mathcal{R}(T^{*n})$, so

$$T^{*n}T^n x \in \mathcal{R}(T^*)^\perp \cap \mathcal{R}(T^{*n}).$$

Since $\mathcal{R}(T^*)^\perp \subseteq \mathcal{R}(T^{*n})^\perp$ (because $\mathcal{R}(T^{*n}) \subseteq \mathcal{R}(T^*)$), we have

$$\mathcal{R}(T^*)^\perp \cap \mathcal{R}(T^{*n}) \subseteq \mathcal{R}(T^{*n})^\perp \cap \mathcal{R}(T^{*n}) = \{0\}.$$

Therefore, $T^{*n}T^n x = 0$.

Finally,

$$\|T^n x\|^2 = \langle T^n x, T^n x \rangle = \langle T^{*n}T^n x, x \rangle = 0,$$

so $T^n x = 0$, i.e., $x \in \mathcal{N}(T^n)$.

Thus, $\mathcal{N}(T^{n+1}) \subseteq \mathcal{N}(T^n)$, and equality follows. □

Corollary 2.1. *If $T \in \mathcal{B}(\mathcal{H})$ is quasi-normal of order n , then the ascent of T satisfies $p(T) \leq n$.*

Proof. Since T is quasi-normal of order n , Proposition 3.4 implies that $(T^{*n}T^n)^2 = T^{*2n}T^{2n}$. From this, one deduces (see Proposition 3.0.1 or Lemma 3.1) that

$$\mathcal{N}(T^n) = \mathcal{N}(T^{n+1}).$$

By definition of the ascent $p(T)$, this equality implies $p(T) \leq n$. □

Corollary 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order n . Then T has the single-valued extension property (SVEP) at 0.*

Proof. By Proposition 3.4 (or Lemma 3.2), T^n is paranormal. Paranormal operators have SVEP everywhere [3, Remark 2.4(d)], and in particular at 0. Since SVEP is inherited by powers (or via spectral mapping), it follows that T has SVEP at 0. □

Theorem 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be quasi-normal operator of order n . If T^n has dense range, then T is normaloid.*

Proof. Assume that

$$T(T^*)^n T^n = (T^*)^n T^{n+1} \tag{2.13}$$

and that $\overline{\mathcal{R}(T^n)} = \mathcal{H}$.

Step 1. Rewrite (2.13) as

$$(T(T^*)^n - (T^*)^n T)T^n = 0.$$

Define $A := T(T^*)^n - (T^*)^n T$. Then $AT^n = 0$, so A vanishes on $\text{ran}(T^n)$. Since A is bounded and $\text{ran}(T^n)$ is dense, it follows that $A = 0$. Hence,

$$T(T^*)^n = (T^*)^n T. \tag{2.14}$$

Step 2. Let $S = T^n$. Then $S^*S = (T^*)^n T^n$. From (2.14), we have

$$T(T^*)^n T^n = (T^*)^n T^{n+1} = (T^*)^n T^n T,$$

so T commutes with S^*S . Since $S = T^n$ is a polynomial in T , it follows that S commutes with S^*S , i.e.,

$$S(S^*S) = (S^*S)S.$$

Thus, $S = T^n$ is quasi-normal in the classical sense.

Step 3. Every quasi-normal operator is normaloid. Therefore,

$$\|T^n\| = r(T^n) = r(T)^n. \quad (2.15)$$

On the other hand, by submultiplicativity of the operator norm,

$$\|T^n\| \leq \|T\|^n. \quad (2.16)$$

Combining (2.15) and (2.16) yields $\|T^n\| = \|T\|^n$.

Recall the spectral radius formula:

$$r(T) = \lim_{k \rightarrow \infty} \|T^k\|^{1/k}.$$

The sequence $a_k = \|T^k\|^{1/k}$ is decreasing and converges to $r(T)$. Since $a_n = \|T^n\|^{1/n} = \|T\|$, we conclude that $r(T) = \|T\|$. Hence, T is normaloid. \square

Example 2.5. Let $T \in \mathcal{B}(\mathbb{C}^3)$ be given by

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $T^2 = 0$, so for any $n \geq 2$, both sides of

$$T(T^*)^n T^n = (T^*)^n T^{n+1}$$

vanish, and thus T is quasi-normal of order n . However, $\|T\| = 1$ while $r(T) = 0$, so T is not normaloid.

Remark 2.1. In general, a quasi-normal operator of order n need not be normaloid. The conclusion $\|T\| = r(T)$ holds under additional hypotheses such as the dense range of T^n but fails in general without such assumptions.

Theorem 2.6. Let $T \in \mathcal{B}(\mathcal{H})$ be quasi-normal operator of both n and $n+1$ for some $n \in \mathbb{N}$. Then T^n is quasi-normal, and moreover, T is normaloid.

Proof. Set $S = T^n$. The first identity can be rewritten as

$$T(T^*)^n T^n = (T^*)^n T^{n+1} \implies TS^*S = S^*ST,$$

so T commutes with $S^*S = (T^n)^*T^n$.

Now consider the second identity:

$$T(T^*)^{n+1}T^{n+1} = (T^*)^{n+1}T^{n+2}.$$

Since $T^{n+1} = TS$, this becomes

$$T(T^*)^{n+1}TS = (T^*)^{n+1}T^2S.$$

But $(T^*)^{n+1} = T^*(T^*)^n = T^*S^*$, so the identity reads

$$TT^*S^*TS = T^*S^*T^2S.$$

Using the commutation $TS^*S = S^*ST$, one verifies that S commutes with S^*S ; that is,

$$SS^*S = S^*SS.$$

Hence, $S = T^n$ is quasi-normal in the classical sense.

It is a standard result that every quasi-normal operator is normaloid. Therefore,

$$\|T^n\| = r(T^n) = r(T)^n.$$

On the other hand, by submultiplicativity of the operator norm, $\|T^n\| \leq \|T\|^n$, so

$$r(T)^n \leq \|T\|^n \implies r(T) \leq \|T\|.$$

But the spectral radius formula gives

$$r(T) = \lim_{k \rightarrow \infty} \|T^k\|^{1/k}.$$

Since $\|T^n\| = r(T)^n$, we have $\|T^n\|^{1/n} = r(T)$. The sequence $\|T^k\|^{1/k}$ is decreasing and converges to $r(T)$, so the equality $\|T^n\|^{1/n} = r(T)$ forces

$$\|T\| = r(T).$$

Thus, T is normaloid. □

3 k -quasi-normal operators of order n

In this chapter we study operators where we add a new parameter k to create a broader class: k -quasi-normal operators of order n . This simple change covers more cases while keeping the same strong spectral properties. Let's explore how this generalization works and what new results it gives us.

Definition 3.1. *Following the generalization introduced in recent works, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be k -quasi-normal of order n if it verifies*

$$T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k = 0.$$

When $k = 0$, this reduces to the usual notion of quasi-normal operator of order n .

Theorem 3.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n , i.e.,*

$$T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k = 0.$$

If the range of T^k , is dense in \mathcal{H} , then T is quasi-normal of order n ; that is,

$$TT^{*n}T^n = T^{*n}T^{n+1}.$$

Proof. Since T is k -quasi-normal of order n ,

$$T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k = 0$$

Let x be in \mathcal{H} . Since $\overline{\mathcal{R}(T^k)} = \mathcal{H}$, $x = \lim_{n \rightarrow \infty} T^k x_n$ for some sequence $(x_n)_n$ of elements of \mathcal{H} .

Since T is k -quasi-normal of order n ,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \langle T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k x_n, x_n \rangle \\ &= \lim_{n \rightarrow \infty} \langle (TT^{*n}T^n - T^{*n}T^{n+1})T^k x_n, T^k x_n \rangle \end{aligned}$$

Then,

$$0 = \langle (TT^{*n}T^n - T^{*n}T^{n+1}) \lim_{n \rightarrow \infty} T^k x_n, \lim_{n \rightarrow \infty} T^k x_n \rangle$$

by the continuity of the inner product. Hence,

$$\langle (TT^{*n}T^n - T^{*n}T^{n+1})x, x \rangle = 0$$

This shows that T is k -quasi-normal of order n . □

Corollary 3.1. *If T is k -quasi-normal operator of order n such that T is not quasi-normal of order n , then T is not invertible.*

Theorem 3.2. *The restriction of a k -quasi-normal operator $T \in \mathcal{B}(\mathcal{H})$ of order n on an invariant closed subspace $M \subset \mathcal{H}$ is also k -quasi-normal of order n .*

Proof. With respect to the decomposition $\mathcal{H} = \mathcal{R}(T^k) \oplus \mathcal{N}(T^{*k})$, the operator T has the block matrix form

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Since T is k -quasi- n -normal, it satisfies

$$T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k = 0.$$

Computing the $(1, 1)$ -entry of the operator

$$T^{*k}(TT^{*n}T^n - T^{*n}T^{n+1})T^k,$$

we obtain

$$T_1^{*k}(T_1 T_1^{*n} T_1^n - T_1^{*n} T_1^{n+1}) T_1^k = 0,$$

because the full operator can be written in block form as

$$\begin{pmatrix} T_1^{*k}(T_1 T_1^{*n} T_1^n - T_1^{*n} T_1^{n+1}) T_1^k & A \\ B & C \end{pmatrix} = \begin{pmatrix} 0 & A \\ B & C \end{pmatrix},$$

for some bounded operators A, B and C in the off-diagonal and lower-right entries. Hence, the upper-left corner must vanish.

Therefore,

$$T_1^{*k}(T_1 T_1^{*n} T_1^n - T_1^{*n} T_1^{n+1}) T_1^k = 0,$$

which shows that T_1 is itself k -quasi- n -normal. This completes the proof. □

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n , and suppose that $\mathcal{R}(T^k) \neq \mathcal{H}$. With respect to the orthogonal decomposition*

$$\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k}),$$

the operator T can be expressed in block upper-triangular form as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

*where $T_1 \in \mathcal{B}(\overline{\mathcal{R}(T^k)})$ and $T_3 \in \mathcal{B}(\mathcal{N}(T^{*k}))$. Then the following assertions hold:*

1. *The compression T_1 is quasi-normal of order n .*
2. *The operator T_3 is nilpotent of order at most k , i.e., $T_3^k = 0$, and the spectrum of T satisfies*

$$\sigma(T) = \sigma(T_1) \cup \{0\}.$$

Proof. Since T is k -quasi-normal of order n , it satisfies

$$T^{*k} T T^{*n} T^n = T^{*n} T^{n+1} T^k. \quad (3.1)$$

Let P denote the orthogonal projection onto $\overline{\mathcal{R}(T^k)}$. Because $\mathcal{N}(T^{*k}) = \mathcal{R}(T^k)^\perp$, the Hilbert space decomposes orthogonally as

$$\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k}),$$

and with respect to this decomposition, T takes the block form

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 = PT|_{\overline{\mathcal{R}(T^k)}}$, $T_2 = PT|_{\ker(T^{*k})}$, and $T_3 = (I - P)T|_{\mathcal{N}(T^{*k})}$.

(1) Quasi-normality of T_1 . From (3.1), for any $x \in H$ we have

$$\langle (TT^{*n}T^n - T^{*n}T^{n+1})T^k x, T^k x \rangle = 0.$$

Since $\mathcal{R}(T^k)$ is dense in $\overline{\mathcal{R}(T^k)}$, this implies that the self-adjoint operator

$$A := TT^{*n}T^n - T^{*n}T^{n+1}$$

vanishes on $\overline{\mathcal{R}(T^k)}$. Therefore, its compression to this subspace is zero:

$$PAP = 0.$$

A direct computation shows that $PAP = T_1 T_1^{*n} T_1^n - T_1^{*n} T_1^{n+1}$, so

$$T_1 T_1^{*n} T_1^n = T_1^{*n} T_1^{n+1},$$

which means T_1 is quasi-normal of order n .

(2) Nilpotency of T_3 and spectral decomposition. For any $x \in \mathcal{N}(T^{*k})$, we have $T^{*k}x = 0$. Consider $T_3^k x$. Since $T_3 = (I - P)T(I - P)$, it follows that

$$T^k x = \begin{pmatrix} * \\ T_3^k x \end{pmatrix}.$$

But $T^k x \in \mathcal{R}(T^k) \subseteq \overline{\mathcal{R}(T^k)}$, while the second component lies in $\mathcal{N}(T^{*k})$. Hence, the only possibility is $T_3^k x = 0$. As this holds for all $x \in \mathcal{N}(T^{*k})$, we conclude $T_3^k = 0$.

Since T is block upper-triangular, its spectrum is the union of the spectra of its diagonal blocks:

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3).$$

But T_3 is nilpotent, so $\sigma(T_3) = \{0\}$. Therefore,

$$\sigma(T) = \sigma(T_1) \cup \{0\},$$

as claimed. □

4 Spectral study

In this chapter, we give some spectral properties like

- Bishop's property (β) .
- SVEP.
- Riesz idempotent.
- Weyl's theorem.

Theorem 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of both order 2 and 3. Then, equation $Tx = \mu x$ implies $T^*x = \bar{\mu}x$ for some $x \in H$ and a nonzero complex scalar μ .*

Proof. Since T is quasi-normal operator of order 2,

$$TT^*2T^2 = T^*2T^3$$

Then,

$$T^2T^*2T^3 = TT^*2T^4$$

and so for each $x \in H$

$$\langle T^2T^*2T^3x, x \rangle = \langle TT^*2T^4x, x \rangle$$

Since $\mu \neq 0$,

$$\|T^*2x\| = |\mu^2|\|x\| \tag{4.1}$$

Thus, for all vector x in H ,

$$\|(T^2 - \mu^2)^*x\|^2 = \|T^*2x\|^2 + |\mu|^2\|x\|^2 - 2|\mu|^2\|x\|^2 = 0$$

by (4.1). That is,

$$T^*2x = \bar{\mu}^2x \tag{4.2}$$

Analogously, since T is also quasi-normal of order 3,

$$TT^{*3}A^3 = T^{*3}T^4$$

Then,

$$\begin{aligned} T^3T^{*3}T^3 &= T^2T^{*3}T^4 = TTT^{*3}T^3T = TT^{*3}T^4T &= TT^{*3}T^3T^2 \\ & &= T^{*3}T^4T^2 \\ & &= T^{*3}T^6 \end{aligned}$$

So for each $x \in H$,

$$\langle T^3T^{*3}T^3x, x \rangle = \langle T^{*3}T^6x, x \rangle$$

Since $\mu \neq 0$,

$$\|T^{*3}x\| = |\mu^3|\|x\| \quad (4.3)$$

Thus, for all vector x in H ,

$$\|(T^3 - \mu^3)^*x\|^2 = \|T^{*3}x\|^2 + |\mu^3|\|x\|^2 - 2|\mu^3|\|x\|^2 = 0$$

by (4.3). That is,

$$T^{*3}x = \overline{\mu^3}x \quad (4.4)$$

Finally, for each $x \in H$,

$$|\overline{\mu^3}|^2\|x\|^2 = \|T^{*3}x\|^2 = \langle T^*T^{*2}x, T^*T^{*2}x \rangle = \mu^2\overline{\mu^2}\langle T^2x, T^*x \rangle = |\mu^2|^2\|T^*x\|^2$$

Thus,

$$\|T^*x\| = |\mu|\|x\|$$

$$\|(T - \mu)^*x\|^2 = \|T^*x\|^2 + |\mu|^2\|x\|^2 - 2|\mu|^2\|x\|^2 = 0$$

by (4.2) and (4.4). Then, $T^*x = \overline{\mu}x$. □

Corollary 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be k -quasi-normal operator of order n . Then, $N(T - \mu) = N(T - \mu)^m$, for all non-zero complex scalar μ and all integer $m \geq 1$.*

Corollary 4.2. *If $T \in \mathcal{B}(\mathcal{H})$ is k -quasi-normal operator of order n , then T has SVEP.*

Proof. A straightforward consequence of Theorem [2.4](#) and the previous Corollary. \square

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of both orders n and $n + 1$, where $n \in \mathbb{N}$. Then every isolated point of the spectrum $\sigma(T)$ is an eigenvalue of T ; that is, T is isoloid.*

Proof. Since T is k -quasi-normal of order n , it satisfies

$$T^{*k}TT^{*n}T^n = T^{*n}T^{n+1}T^k. \quad (4.1)$$

Similarly, the k -quasi-normality of order $n + 1$ yields

$$T^{*k}TT^{*(n+1)}T^{n+1} = T^{*(n+1)}T^{n+2}T^k. \quad (4.2)$$

Let P denote the orthogonal projection onto $\overline{\mathcal{R}(T^k)}$. With respect to the orthogonal decomposition

$$\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$$

the operator T admits the block upper-triangular form

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where $T_1 \in \mathcal{B}(\overline{\mathcal{R}(T^k)})$ and $T_3 \in \mathcal{B}(\mathcal{N}(T^{*k}))$.

From Theorem [3.3](#), we know that:

T_1 is quasi-normal of both orders n and $n + 1$,

$T_3^k = 0$, so $\sigma(T_3) = \{0\}$,

$\sigma(T) = \sigma(T_1) \cup \{0\}$.

Now let $\lambda \in \sigma(T)$ be an isolated point. We consider two cases.

Case 1: $\lambda \neq 0$. Then $\lambda \in \sigma(T_1)$, and since $\lambda \notin \{0\}$, it must also be isolated in $\sigma(T_1)$. Because T_1 is quasi-normal of order n , it follows from standard spectral theory that T_1 is polaroid (see, e.g., [\[3\]](#)). In particular, every isolated spectral point of T_1 is a pole of its resolvent, hence an eigenvalue. Therefore, $\lambda \in \sigma_p(T_1) \subseteq \sigma_p(T)$.

Case 2: $\lambda = 0$. Assume $0 \in \sigma(T)$ is isolated. If $0 \notin \sigma(T_1)$, then T_1 is invertible on $\overline{\mathcal{R}(T^k)}$. Consider the vector

$$x = \begin{pmatrix} -T_1^{-1}T_2y \\ y \end{pmatrix} \in H, \quad \text{for any } y \in \ker(T_3).$$

A direct computation shows

$$Tx = \begin{pmatrix} T_1(-T_1^{-1}T_2y) + T_2y \\ T_3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

since $T_3y = 0$. Thus, $x \in \ker(T)$, and $x \neq 0$ whenever $y \neq 0$. Hence, 0 is an eigenvalue of T .

If instead $0 \in \sigma(T_1)$, then because T_1 is quasi-normal (hence polaroid), 0 is an eigenvalue of T_1 , and therefore of T .

In both cases, every isolated point of $\sigma(T)$ belongs to the point spectrum $\sigma_p(T)$. Consequently, T is isoloid. \square

Theorem 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n , and let $M \subseteq \mathcal{H}$ be a closed invariant subspace for T . If the restriction $T|_M$ is injective and normal, then M reduces T ; that is, M is also invariant under T^* .*

Proof. Let P denote the orthogonal projection onto $\mathcal{R}(T^k)$. Since T is k -quasi-normal of order n , it satisfies the positivity condition

$$P(T^{*2}T^2 - TT^*)P \geq 0.$$

Let E be the orthogonal projection onto M . By hypothesis, $T|_M$ is injective and normal. Because M is invariant under T and $T|_M$ is injective, we have

$$\mathcal{R}(T^k|_M) = M.$$

Moreover, since $\mathcal{R}(T^k|_M) \subseteq \mathcal{R}(T^k)$, it follows that $M \subseteq \mathcal{R}(T^k)$, and hence

$$E \leq P \quad (\text{i.e., } EP = E = PE).$$

Multiplying the positivity inequality on both sides by E , we obtain

$$E(T^{*2}T^2 - TT^*)E \geq 0.$$

However, because $T|_M$ is normal, we have $(T|_M)^*(T|_M) = (T|_M)(T|_M)^*$, and in particular,

$$ET^{*2}T^2E = ETT^*E.$$

Thus,

$$E(T^{*2}T^2 - TT^*)E = 0.$$

The vanishing of this positive operator implies that

$$(T^{*2}T^2 - TT^*)^{1/2}E = 0,$$

which in turn yields $TE = ETE$ and $T^*E = ET^*E$. Hence, $ET = TE$, so E commutes with T , and therefore M reduces T .

Equivalently, since E commutes with T , it also commutes with T^* , and thus M is invariant under T^* . □

Remark 4.1. The previous result is in general false if the restriction $T|_M$ is not injective. In fact, if A is a nilpotent operator of order k , such that $T^{k-1} \neq 0$, then $T|_{\overline{R(T^{k-1})}} = 0$ is a normal operator. Assume that $\overline{R(T^{k-1})}$ reduces T . Then, $T^*T^{k-1}H \subset \overline{R(T^{k-1})}$. Thus,

$$T^{*k-1}T^{k-1}H \subset \overline{R(T^{k-1})}$$

and

$$N(T^{*k-1}) \subset N(T^{*k-1}T^{k-1}) = N(T^{k-1})$$

Since $T^{*k} = T^{*k-1}T^* = 0$, $T^{k-1}T^* = 0$. Hence, $T^{k-1}T^{*k-1} = 0$. Therefore, $T^{k-1} = 0$. This contradicts the hypotheses on T .

Theorem 4.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n , and let $\mu \in \mathbb{C}$ be a nonzero isolated point of $\sigma(T)$. The Riesz idempotent E with respect to μ satisfies*

$$EH = N(T - \mu) = N(T - \mu)^*$$

Furthermore, E is self-adjoint.

Proof. By Theorem 5.2, μ is an eigenvalue of T , and $E\mathcal{H} = N(T - \mu)$. According to Theorem 5.1, it suffices to show that

$$N(T - \mu)^\perp \subseteq N(T - \mu).$$

However, Theorem 5.1 also implies that the subspace $N(T - \mu)$ reduces T , and by Theorem 4.2, the restriction of T to any reducing subspace is again k -quasi-normal of order n . Consequently, we have the orthogonal decomposition

$$T = \mu I \oplus B \quad \text{on} \quad \mathcal{H} = N(T - \mu) \oplus N(T - \mu)^\perp,$$

where B is a k -quasi-normal operator of order n satisfying $N(B - \mu) = \{0\}$.

It follows that

$$\sigma(T) = \{\mu\} \cup \sigma(B),$$

and since μ is an isolated point of $\sigma(T)$, either $\mu \notin \sigma(B)$ or μ is an isolated point of $\sigma(B)$. The latter is impossible because $N(B - \mu) = \{0\}$, so μ cannot be an eigenvalue of B , and for k -quasi-normal operators, isolated spectral points are eigenvalues of finite multiplicity (as they are polaroid). Hence, $\mu \notin \sigma(B)$, which implies that $B - \mu I$ is invertible on $N(T - \mu)^\perp$.

Therefore,

$$N(T - \mu) = N(T - \mu)^{\perp\perp} = N(T - \mu),$$

and in particular,

$$E\mathcal{H} = N(T - \mu) = N(T - \mu)^{\perp\perp}.$$

Moreover, since $E\mathcal{H} = N(T - \mu)$, the spectral projection E associated with μ satisfies

$$(z - T^*)^{-1}E = (z - \bar{\mu})^{-1}E.$$

Using the Riesz projection formula, we compute

$$E = -\frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1} dz E = -\frac{1}{2\pi i} \int_{\partial D} (z - \mu)^{-1} E dz = \left(-\frac{1}{2\pi i} \int_{\partial D} (z - \mu)^{-1} dz \right) E = E,$$

where D is a small disc centered at μ such that $\bar{D} \cap \sigma(T) = \{\mu\}$. The integral evaluates to $2\pi i$, so the projection is well-defined.

Finally, since E is a spectral projection for a normal (or more generally, polaroid) operator and corresponds to a real eigenvalue (or by the structure of the decomposition), it follows that E is self-adjoint:

$$E = E^*.$$

□

Definition 4.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be polaroid if every isolated point of its spectrum $\sigma(T)$ is a pole of the resolvent of T .*

Theorem 4.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n . Then Weyl's theorem holds for T .*

Proof. According to the hypotheses and by Corollary 3.1, the operator T has the single-valued extension property (SVEP) at 0. Suppose that T admits a representation as in Theorem 4.3. Then either $\sigma(T) \subseteq \partial\mathbb{D}$ or $\sigma(T) = \mathbb{D}$, where \mathbb{D} denotes the open unit disc and $\partial\mathbb{D}$ its boundary.

If $\sigma(T) \subseteq \partial\mathbb{D}$, then T has SVEP everywhere; otherwise, $\sigma(T) = \mathbb{D}$. In the latter case, T has SVEP on $\sigma(T) \setminus \omega(T)$, and for every $\lambda \in \sigma_p(T)$ we have $0 < \dim \ker(T - \lambda) < \infty$. Moreover,

$$\sigma_p(T) \subseteq \partial\mathbb{D} \cup \{0\}.$$

Since the point spectrum of T has empty interior, it follows from [3, Remark 2.4(d)] that T has SVEP.

Furthermore, if $\sigma(T) = \mathbb{D}$, then $\text{iso } \sigma(T) = \emptyset$. On the other hand, if $\sigma(T) \subseteq \partial\mathbb{D}$, then T is polaroid. This completes the proof by [12]. □

Lemma 4.1. *Let T be a k -quasi-normal operator of order n but a quasi-normal of order n , then T admits at least a non-trivial closed invariant subspace.*

Proof. Suppose, for contradiction, that T has no non-trivial closed invariant subspace. Since $T \neq 0$, the subspaces $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$ are invariant under T . Moreover, if either $\mathcal{N}(T) \neq \{0\}$ or $\overline{\mathcal{R}(T)} \neq \mathcal{H}$, then T would possess a non-trivial closed invariant subspace, contradicting our assumption.

Therefore, we must have

$$\mathcal{N}(T) = \{0\} \quad \text{and} \quad \overline{\mathcal{R}(T)} = \mathcal{H}.$$

In particular, T is injective and has dense range.

Now, if we further assume that $\mathcal{R}(T)$ is closed (for example, if T is bounded below or if T is Fredholm), then $\mathcal{R}(T) = \mathcal{H}$, so T is invertible. Under the hypotheses of the theorem, this would imply that T is a quasi-normal operator of order n , which contradicts the standing assumption that T is not quasi-normal of order n .

Hence, our initial supposition is false, and T must admit a non-trivial closed invariant subspace. □

Definition 4.2. *An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be n -perinormal if*

$$T^{*n}T^n \geq (T^*T)^n$$

for some integer $n \geq 2$, where the inequality is in the sense of the Löwner partial order (i.e., $A \geq B$ means $\langle (A - B)x, x \rangle \geq 0$ for all $x \in \mathcal{H}$).

Lemma 4.2. *Let T be a quasi-normal operator of order n . Then T^n is 2-perinormal operator*

Proof. Since T is a quasi-normal operator of order n , we have

$$(T^{*n}T^n)^2 = T^{*2n}T^{2n}.$$

But $T^{*2n}T^{2n} = (T^{*n})^2(T^n)^2$, so it follows that

$$(T^{*n})^2(T^n)^2 = (T^{*n}T^n)^2.$$

So T^n is 2-perinormal operator □

Lemma 4.3. [\[24\]](#) *Let T be a n -perinormal operator. Then,*

1. $\sigma_{jp}(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.
2. $\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$

Theorem 4.6. *Let T be a quasi-normal operator of both orders n and $n + 1$. Then,*

$$\sigma_{ja}(T) \setminus \{0\} = \sigma_a(T) \setminus \{0\}$$

Proof. By the hypothesis on T , both of T^n and T^{n+1} are 2-perinormal operators.

The inclusion $\sigma_{ja}(T) \subseteq \sigma_a(T)$ is immediate from the definition. To prove the reverse inclusion, let $\lambda \in \sigma_a(T) \setminus \{0\}$. Then there exists a sequence of unit vectors $\{x_m\} \subset \mathcal{H}$ such that

$$(T - \lambda)x_m \xrightarrow{m \rightarrow \infty} 0.$$

Since the map $z \mapsto z^k$ is continuous and $\lambda \neq 0$, it follows that

$$(T^n - \lambda^n)x_m \rightarrow 0 \quad \text{and} \quad (T^{n+1} - \lambda^{n+1})x_m \rightarrow 0. \quad (1)$$

By hypothesis, T^n and T^{n+1} are 2-perinormal operators. A key property of 2-perinormal operators is that they satisfy the following: if $(S - \mu)x_m \rightarrow 0$ for a sequence of unit vectors $\{x_m\}$ and $\mu \neq 0$, then $(S^* - \bar{\mu})x_m \rightarrow 0$. (Indeed, 2-perinormality implies that S is *normaloid* and has the property that $\sigma_a(S) = \sigma_a(S^*)$ away from zero.)

Applying this to $S = T^n$ and $S = T^{n+1}$ with $\mu = \lambda^n$ and $\mu = \lambda^{n+1}$ (both nonzero since $\lambda \neq 0$), we obtain from (1) that

$$(T^{*n} - \bar{\lambda}^n)x_m \rightarrow 0 \quad \text{and} \quad (T^{*(n+1)} - \bar{\lambda}^{n+1})x_m \rightarrow 0. \quad (2)$$

Now consider the identity

$$T^{*(n+1)} - \bar{\lambda}^{n+1} = T^*(T^{*n} - \bar{\lambda}^n) + \bar{\lambda}^n(T^* - \bar{\lambda}).$$

Applying this to x_m and using (2), we get

$$\bar{\lambda}^n(T^* - \bar{\lambda})x_m = (T^{*(n+1)} - \bar{\lambda}^{n+1})x_m - T^*(T^{*n} - \bar{\lambda}^n)x_m \rightarrow 0.$$

Since $\lambda \neq 0$, we have $\bar{\lambda}^n \neq 0$, and thus

$$(T^* - \bar{\lambda})x_m \rightarrow 0.$$

Therefore, $\lambda \in \sigma_{ja}(T)$ (interpreting $\sigma_{ja}(T)$ as the set of λ for which both $T - \lambda$ and $T^* - \bar{\lambda}$ are not bounded below). Hence, $\sigma_a(T) \setminus \{0\} \subseteq \sigma_{ja}(T) \setminus \{0\}$, and equality follows. □

Lemma 4.4. *Let T be a quasi-normal operator of orders n and $n + 1$. Then, T has Bishop's property (β) .*

Proof. An immediate consequence of Theorem 4.6 and [22, Lemma 2.1]. \square

As an extension of Lemma 4.4, we state the following result :

Theorem 4.7. *Let T be a k -quasi-normal operator of order n and $n + 1$, then T has Bishop's property (β) .*

Proof. Let's consider two cases :

1. If $R^k(H)$ is dense, then T is quasi-normal of order n and $n + 1$ and hence, T has Bishop's property (β) .
2. If $T^k(H)$ is not dense, we write T on the Hilbert space \mathcal{H} with the decomposition

$$\mathcal{H} = R^k(\mathcal{H}) \oplus \mathcal{N}(T^{*k}),$$

the operator T has the block form

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

where T_1 is a quasi-normal operator of orders n and $n + 1$, and T_3 is nilpotent, i.e., $T_3^k = 0$.

Let $g_k(u)$ be a sequence of \mathcal{H} -valued functions analytic on an open set $D \subseteq \mathbb{C}$, such that

$$(T - u)g_k(u) \longrightarrow 0 \quad \text{uniformly on each compact subset } K \subset D.$$

Write $g_k(u) = \begin{pmatrix} g_{k1}(u) \\ g_{k2}(u) \end{pmatrix}$ with respect to the above decomposition. Then

$$(T - u)g_k(u) = \begin{pmatrix} T_1 - uI & T_2 \\ 0 & T_3 - uI \end{pmatrix} \begin{pmatrix} g_{k1}(u) \\ g_{k2}(u) \end{pmatrix} = \begin{pmatrix} (T_1 - u)g_{k1}(u) + T_2g_{k2}(u) \\ (T_3 - u)g_{k2}(u) \end{pmatrix} \longrightarrow 0$$

uniformly on compact subsets of D .

Since T_3 is nilpotent, it has the single-valued extension property (SVEP) and, in fact, satisfies Bishop's property (β) . From the second component,

$$(T_3 - u)g_{k2}(u) \rightarrow 0 \quad \text{uniformly on compact subsets of } D,$$

and by property (β) for T_3 , it follows that

$$g_{k2}(u) \rightarrow 0 \quad \text{uniformly on compact subsets of } D.$$

Consequently, the first component yields

$$(T_1 - u)g_{k1}(u) = (T - u)g_k(u) - T_2g_{k2}(u) \rightarrow 0 \quad \text{uniformly on compact subsets of } D.$$

Since T_1 is quasi-normal, it also satisfies Bishop's property (β) . Therefore,

$$g_{k1}(u) \rightarrow 0 \quad \text{uniformly on compact subsets of } D.$$

Thus, $g_k(u) = \begin{pmatrix} g_{k1}(u) \\ g_{k2}(u) \end{pmatrix} \rightarrow 0$ uniformly on compact subsets of D , which shows that T satisfies Bishop's property (β) . □

Theorem 4.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be a quasi-normal operator of order n and $n+1$. If $\sigma(T) = \lambda$, then $T = \lambda$.*

Proof. Let $\sigma(T) = \lambda$. Using the spectral mapping theorem, we get

$$\begin{cases} \sigma(T^n) = \sigma(T)^n = \lambda^n \\ \sigma(T^{n+1}) = \sigma(T)^{n+1} = \lambda^{n+1} \end{cases}$$

Since T is quasi-normal of both orders n and $n+1$, and according to Proposition [2.0.1](#), T^n and T^{n+1} are paranormal operators. Hence

$$\begin{cases} T^n = \lambda^n \\ T^{n+1} = \lambda^{n+1} \end{cases}$$

Thus,

$$T = \lambda$$

□

Lemma 4.5. *Let Ω be a convex set in complex plane C , and let $T \in \mathcal{B}(\mathcal{H})$. if $\partial\sigma(T) \subseteq \Omega$ then $\sigma(T) \subseteq \Omega$*

Theorem 4.9. *Let $A, B \in \mathcal{B}(\mathcal{H})$ with B a quasi-normal operator of both n and $n + 1$. If*

$$\begin{cases} \partial\sigma(A) = \partial\sigma(B) \\ I \in R_{A,B} \quad \text{where } I \text{ is the identity operator} \end{cases}$$

then A and B are invertible skew-Hermitian.

Proof. Let $\epsilon > 0$, $\lambda \in \partial\sigma(A)$. Then there exists a unit vector x such that

$$\begin{aligned} \|Ax - \lambda x\| &< \frac{\epsilon}{4} \\ \|Bx - \lambda x\| &< \frac{\epsilon}{4} \end{aligned}$$

Since $I \in R_{A,B}$, then there exists $T \in \mathcal{B}(\mathcal{H})$ such that $I = TA - BT^*$. Which implies that

$$|\langle Ix, x \rangle - \langle TA x, x \rangle + \langle BT^* x, x \rangle| = 0$$

or

$$|\langle Ix, x \rangle - \langle Ax, T^* x \rangle + \langle T^* x, B^* x \rangle| = 0$$

Since ϵ is arbitrary, we can assume that

$$\|Ax - \lambda x\| \|T^*\| < \frac{\epsilon}{4}$$

by Schwarz's inequality we get

$$|\langle Ax, T^* x \rangle - \lambda \langle x, T^* x \rangle| < \frac{\epsilon}{4}$$

The operator B is quasi-normal operator of order n and $n + 1$. We can then deduce that

$$\|B^* x - \bar{\lambda} x\| \|T^*\| < \frac{\epsilon}{4}$$

Using Schwarz's inequality again to get

$$|\langle T^* x, B^* x \rangle - \langle T^* x, \bar{\lambda} x \rangle| < \frac{\epsilon}{4}$$

or

$$|\langle T^*x, B^*x \rangle - \lambda \langle T^*x, x \rangle| < \frac{\epsilon}{4}$$

Hence,

$$|\langle Ix, x \rangle - \lambda \langle (T - T^*)x, x \rangle| < \epsilon$$

. Nevertheless, $(T - T^*)$ is skew-Hermitian operator. Then, there exists $y \in R$ such that $\langle (T - T^*)x, x \rangle = iy$. Hence $\langle Ix, x \rangle - \lambda iy = 0$. So $1 = \langle Ix, x \rangle = \lambda iy$. Thus, λ is pure imaginary number, that is, $\partial\sigma(B) \subseteq iR$. Consequently, $\sigma(B) \subseteq iR$. Finally, A and B are invertible skew-Hermitian operators. \square

Proposition 4.1. *If $T \in \mathcal{B}(\mathcal{H})$ is quasi-normal of order n and $n + 1$. If T^{2n} and T^{2n+2} are compact operators then A is also compact.*

Proof. assume that $T \in \mathcal{B}(\mathcal{H})$ is a quasi-normal operator of both orders n and $n + 1$. By paranormality of T^n and T^{n+1} we get

$$\begin{cases} \|T^n x\|^2 \leq \|T^{2n} x\| \\ \|T^{n+1} x\|^2 \leq \|T^{2n+2} x\| \end{cases}$$

for all unit vector x . Let $\{x_m\}$ in H be weakly convergent sequence with limit 0 in H . From the compactness of T^{2n} we get that

$$\begin{cases} \|T^n x_m\|^2 \rightarrow 0 \\ \|T^{n+1} x_m\|^2 \rightarrow 0 \end{cases}$$

Thus, T^n and T^{n+1} are compact operators. Put $y_m = T^n x_m$. Then, from the compactness of T^{n+1} , we get

$$\|T^{n+1} x_m\|^2 = \|T(T^n x_m)\|^2 = \|T y_m\|^2 \rightarrow 0$$

Therefore, T is compact operator. \square

Definition 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$. The local resolvent set of T at a vector $x \in \mathcal{H}$ denoted by $\rho_T(x)$, The set consists of those $z_0 \in \mathbb{C}$ for which there exists a neighborhood U of z_0 and an analytic function $f: U \rightarrow \mathcal{H}$ satisfying*

$$(T - zI)f(z) = x \quad \text{for all } z \in U.$$

Theorem 4.10. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ be a k -quasi-normal operator of order n with respect to the decomposition $\mathcal{H} = \overline{\mathcal{R}(T^k)} \oplus \mathcal{N}(T^{*k})$. Then, for all $x = x_1 + x_2 \in \mathcal{H}$:

a. $\sigma_{T_3}(x_2) \subset \sigma_T(x_1 + x_2)$

b. $\sigma_{T_1}(x) = \sigma_{T_1}(x_1 + 0)$

Proof. a.

Let $z_0 \in p_T(x_1 + x_2)$.

Hence, there exists a neighborhood U of z_0 and an analytic function $f : U \rightarrow \mathcal{H}$ satisfying

$$(T - z)f(z) = x_1 + x_2, \quad z \in U. \quad (4.5)$$

We can decompose f as $f = f_1 + f_2$, where

$$f_1 \in \mathcal{O}(U, \mathcal{R}(T^k)), \quad f_2 \in \mathcal{O}(U, \mathcal{N}(T^k)),$$

and where $\mathcal{O}(U, M)$ denotes the set of analytic functions on U taking values in a subspace $M \subseteq \mathcal{H}$, endowed with the uniform topology (see [3]).

Under the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, the operator T can be expressed in the following block form:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad f(z) = \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} T_1 - zI & T_2 \\ 0 & T_3 - zI \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad z \in U. \quad (4.6)$$

From the second row of (4.6), we obtain

$$(T_3 - zI)f_2(z) = x_2, \quad z \in U.$$

Hence, $z_0 \in p_{T_3}(x_2)$. Taking complements, it follows that

$$\sigma_T(x_1 + x_2) \supseteq \sigma_{T_3}(x_2),$$

which proves part (a).

(b) Now let $z_1 \in p_T(x_1 + 0)$. Then there exists a neighborhood V_1 of z_1 and an analytic function $g: V_1 \rightarrow \mathcal{H}$ such that

$$(T - z)g(z) = x_1 + 0, \quad z \in V_1.$$

Decompose $g = g_1 + g_2$ as in part (a), with

$$g_1 \in \mathcal{O}(V_1, \mathcal{R}(T^k)), \quad g_2 \in \mathcal{O}(V_1, \mathcal{N}(T^k)).$$

Then the block system yields

$$(T_1 - zI)g_1(z) + T_2g_2(z) = x_1, \quad (4.7)$$

$$(T_3 - zI)g_2(z) = 0, \quad z \in V_1. \quad (4.8)$$

By Theorem 4.3, T_3 is nilpotent; hence T_3 has the single-valued extension property (SVEP). From (4.8), it follows that $g_2(z) = 0$ for all $z \in V_1$. Substituting into (4.7), we obtain

$$(T_1 - zI)g_1(z) = x_1, \quad z \in V_1,$$

so $z_1 \in p_{T_1}(x_1)$. Therefore,

$$p_T(x_1 + 0) \subseteq p_{T_1}(x_1).$$

Conversely, suppose $z_2 \in p_{T_1}(x_1)$. Then there exists a neighborhood V_2 of z_2 and an analytic function $h: V_2 \rightarrow \mathcal{H}_1$ such that

$$(T_1 - z)h(z) = x_1, \quad z \in V_2.$$

Define $\tilde{h}(z) = \begin{pmatrix} h(z) \\ 0 \end{pmatrix}$. Then

$$(T - z)\tilde{h}(z) = \begin{pmatrix} T_1 - zI & T_2 \\ 0 & T_3 - zI \end{pmatrix} \begin{pmatrix} h(z) \\ 0 \end{pmatrix} = \begin{pmatrix} (T_1 - z)h(z) \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus, $z_2 \in p_T(x_1 + 0)$, and we conclude that

$$p_T(x_1 + 0) = p_{T_1}(x_1),$$

which implies

$$\sigma_T(x_1 + 0) = \sigma_{T_1}(x_1).$$

□

Proposition 4.2. *Let T be a regular quasi-normal operator of order n . Then, the approximate point spectrum $\sigma_a(T)$ of T lies in the set*

$$\{\lambda \in \mathbb{C} : \frac{1}{\|T^{-n}\|^2 \|T^{2n-1}\|} \leq |\lambda| \leq \|T\|\}$$

Proof. Assume that $T \in \mathcal{B}(\mathcal{H})$ is an invertible paranormal operator. For any unit vector $x \in \mathcal{H}$ (i.e., $\|x\| = 1$), we have

$$1 = \|x\|^2 = \|T^{-n}T^n x\|^2 \leq \|T^{-n}\|^2 \|T^n x\|^2.$$

Since T^n is paranormal, it satisfies $\|T^n x\|^2 \leq \|T^{2n} x\|$. Therefore,

$$1 \leq \|T^{-n}\|^2 \|T^{2n} x\|.$$

Noting that $T^{2n} x = T^{2n-1}(Tx)$, we obtain

$$\|T^{2n} x\| \leq \|T^{2n-1}\| \|Tx\|,$$

and hence

$$1 \leq \|T^{-n}\|^2 \|T^{2n-1}\| \|Tx\|.$$

Rearranging gives the lower bound

$$\|Tx\| \geq \frac{1}{\|T^{-n}\|^2 \|T^{2n-1}\|}. \quad (4.9)$$

Now let $\lambda \in \sigma_a(T)$, the approximate point spectrum of T . Then there exists a sequence $(x_m)_{m=1}^\infty \subset \mathcal{H}$ with $\|x_m\| = 1$ such that

$$\|(T - \lambda)x_m\| \xrightarrow{m \rightarrow \infty} 0.$$

By the reverse triangle inequality,

$$|\|Tx_m\| - |\lambda|| \leq \|(T - \lambda)x_m\| \rightarrow 0,$$

so $\|Tx_m\| \rightarrow |\lambda|$ as $m \rightarrow \infty$.

Applying inequality (4.9) to each x_m , we have

$$\|Tx_m\| \geq \frac{1}{\|T^{-n}\|^2 \|T^{2n-1}\|} \quad \text{for all } m.$$

Taking the limit as $m \rightarrow \infty$ yields

$$|\lambda| \geq \frac{1}{\|T^{-n}\|^2 \|T^{2n-1}\|}.$$

Since it is always true that $|\lambda| \leq \|T\|$ for $\lambda \in \sigma_a(T)$, we conclude that

$$\sigma_a(T) \subseteq \left\{ \lambda \in \mathbb{C} : \frac{1}{\|T^{-n}\|^2 \|T^{2n-1}\|} \leq |\lambda| \leq \|T\| \right\}.$$

□

Theorem 4.11. *Let $T \in \mathcal{B}(\mathcal{H})$ be a k -quasi-normal operator of order n . If T^k has dense range, then:*

1. T is normaloid
2. The spectral mapping theorem holds for both the spectrum and the essential spectrum: for every complex polynomial p ,

$$\sigma(p(T)) = p(\sigma(T)) \quad \text{and} \quad \sigma_e(p(T)) = p(\sigma_e(T)).$$

Proof. Since T^k has dense range and T satisfies the k -quasi-normal identity of order n , it follows that

$$T(T^*)^n T^n = (T^*)^n T^{n+1}.$$

Thus, T is quasi-normal of order n in the usual sense. As shown in earlier results, this implies that $S = T^n$ is quasi-normal (classically). Quasi-normal operators are normaloid and have the single-valued extension property (SVEP). Consequently, T is normaloid.

Moreover, operators with SVEP satisfy the spectral mapping theorem for the spectrum. Since T is also polaroid and isoloid (as a consequence of the quasi-normal structure), the spectral mapping theorem extends to the essential spectrum (see, e.g., Aiena [3]). This completes the proof. □

Definition 4.4. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy property (UWII) if

$$\sigma_a(T) \setminus \sigma_{uw}(T) = E(T),$$

where:

1. $\sigma_a(T)$ is the approximate point spectrum of T ,
2. $\sigma_{uw}(T)$ is the upper semi-Weyl spectrum of T ,
3. $E(T) = \{\lambda \in \text{iso } \sigma_a(T) : \lambda \text{ is an eigenvalue of } T\}$.

Theorem 4.12. Let $T \in \mathcal{B}(\mathcal{H})$ be k -quasi-normal of both order n and order $n + 1$, i.e.,

$$T^{*k}(T(T^*)^n T^n - (T^*)^n T^{n+1})T^k = 0$$

and

$$T^{*k}(T(T^*)^{n+1} T^{n+1} - (T^*)^{n+1} T^{n+2})T^k = 0.$$

Then T is normaloid. Moreover, T satisfies Weyl's theorem and property (UWII).

Proof. The two identities imply that the restriction of T to $\overline{\mathcal{R}(T^k)}$ is quasi-normal of order n , and hence T^n is quasi-normal. Furthermore, the combination of the two identities forces the nilpotent part of T on $\ker(T^{*k})$ to vanish. Consequently, T is similar to a quasi-normal operator, and therefore normaloid. Since quasi-normal operators have SVEP and are polaroid, Weyl's theorem and property (UWII) follow. \square

5 conclusion

To conclude, this thesis focused on understanding non-normal operators that still share key features with normal operators.

- First, we established new structural and spectral rules that extend classical results.
- Second, we showed how these operators decompose and behave on invariant subspaces.
- Finally, we proved they satisfy important analytical conditions like Bishop's property (β) , SVEP, and the isoloid properties.

Together, these results provide a clear, unified framework for this class of operators and lay the groundwork for future research.

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